Memorable Back-To-Back Hands

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I'm sure we've all heard, more than once, someone utter a comment about being dealt back-to-back hands. Examples are the identical two cards ("I had the nine of clubs and seven of diamonds last hand"), or back-to-back pocket aces. Let's examine the probabilities of these two particular situations occurring during a session of n hold'em hands.

The total number of different hold'em hands a player may be dealt is $C(52, 2) = 1, 326$. Thus, the total number of sequences of n possible hands is 1,326 raised to the n-th power. This follows because there are 1,326 choices for each hand in the sequence. It is now easy to count the number of sequences without two successive identical hands. The first hand may be anything; that is, there are 1,326 choices for the first hand. Once the first hand is chosen, the second hand may be any of 1,325. Once the second hand is chosen, the third hand may be any of 1,325.

We continue in this way and see that the number of sequences of length n with no two successive hands the same is 1,326 multiplied by 1,325 raised to the $(n - 1)$-th power. We then divide this by the total number of sequences to get the probability there are no successive identical hands among the n hands. Subtracting this number from 1 then gives the probability of back-to-back identical hands being dealt at least once in a session consisting of n hands. The following table contains the latter probabilities for a few values of n. Readers may be surprised that a session of only 200 hands already gives one a reasonable chance of seeing back-to-back identical hands (13.94 percent to be precise).

n	2	10	50	150	200
probability	.00075	.00677	.03629	.10632	.1394

Back-to-back Identical Hold'Em Hands

We now consider back-to-back pocket aces. Let $c(n)$ denote the number of ways of dealing n successive hold'em hands to a fixed player so that the player does not receive back-to-back pocket aces. Clearly, $c(1) = 1, 326$ since this is simply the number of ways of dealing a single hand to a player. There are $1, 326^2 = 9, 771, 876$ ways of dealing two successive hands to a player and 36 of them would be back-to-back pocket aces because there are 6 ways to deals aces to a player. Subtraction gives $c(2) = 9, 771, 840$, this is the number without back-to-back pocket aces.

Let $c(n; b)$ be the number of ways of dealing n successive hold'em hands without back-to-back pocket aces so that the n-th hand is not pocket aces. Let $c(n; a)$ be the number of ways of dealing n successive hold'em hands without back-to-back pocket aces so that the n-th hand is pocket aces. It is easy to see that $c(n) = c(n; a) + c(n; b)$. Now if the n-th hand is pocket aces, the preceding hand must not be pocket aces or else we would have back-to-back pocket aces.

Thus, $c(n;a) = 6c(n-1,b)$. On the other hand, if the n-th hand is not pocket aces, then the preceding hand can be anything. Thus, $c(n;b) = 1,320c(n-1)$.

Performing a little algebra leads to $c(n) = 1,320c(n-1) + 7,920c(n-2)$. This is called a recurrence relation and theoretically allows us to calculate any value of $c(n)$ step-by-step. Since we know $c(1)$ and $c(2)$, we can calculate $c(3)$ using the above formula. Once we know $c(3)$, we can calculate $c(4)$. Continuing in this way, we can calculate any particular value of $c(n)$. However, if asked for a particular value, say $c(200)$, you can see that it would be extremely tedious to obtain it. In addition, a long sequence of such calculations is highly prone to error.

Miraculously, there is a clever method to get a direct formula for any particular value of $c(n)$. For details of the derivation, see my website (http://www.math.sfu.ca/~alspach) in the directory entitled "Poker Computations". Once we have the value $c(n)$, we divide by 1,326 raised to the n-th power and subtract from 1 to get the probability that a player is dealt back-to-back pocket aces over a hold'em session of n hands. The next table has some sample values for various n.

n	2	10	50	150	200
probability	.00002	.00018	.000998	.00303	.00405

BACK-TO-BACK POCKET ACES