A Powerful Poker Polynomial

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When I learned that Bib Ladder, my old friend from my Vancouver days, was planning to attend the recently concluded Harvest Poker Classic at Casino Regina, I made arrangements to have lunch with him before the tournaments got underway. We had been talking about thirty minutes at lunch when he brought up my earlier article dealing with counting the number of poker hands when suits are ignored.

Bib enjoyed learning that there is a method to handle hands with any number of cards, but he complained that the method was fussy and required him to keep going back to the article when he tried doing his own examples. He asked me whether there was another way that would be easier to remember. I told him that there is, in fact, a very compact way to encode the information. One still has to do some computation, but what needs to be remembered is much less.

I wrote down the following polynomial for him:

$$\frac{1}{24}(x_1^{52} + 6x_1^{26}x_2^{13} + 8x_1^{13}x_3^{13} + 3x_2^{26} + 6x_4^{13}).$$

I then said, "Bib, this polynomial is a wonderful mathematical object. It encodes all the information about beginning hands, where we are ignoring suits, and yet it is quite compact. It is not difficult to remember were you inclined to do so. This is a good example of mathematical elegance."

"Professor, I can see that it is a simple expression, but I don't offhand see any information about beginning poker hands. For example, I don't see a 169 anywhere in the expression and we all know that is the number of hold'em hands when suits are ignored."

"Aha, Bib, good point! I said the polynomial encodes all the information, but I have not given you the decoding scheme. Encoding the information means that it is there, but we have to learn how to extract it." Here is what I explained for Bib.

For the variable x_1, substitute $1+w$; for the variable x_2, substitute $1+w^2$; for the variable x_3 substitute $1+w^3$; and for the varaiable x_4 substitute $1+w^4$. After you make these substitutions, expand the polynomial and the coefficients may surprise you. This is simply high school algebra, but to clear the dust off what may be mostly forgotten, I shall give the details for finding the coefficient of w^2.

There is one fact from your high school algebra course that we use over and over. If one is expanding $(a+b)^n$, then the coefficient of a^ib^{n-i} is $C(n,i)$, that is, the number of ways of choosing i objects from n objects. So let's use this fact in determining the coefficient of w^2 after making the above substitutions in the given polynomial.

Following the substitution, the first term becomes $(1+w)^{52}$. Using the fact just given, the coefficient of w^2 in this term is $C(52,2) = 1,326$. The second term becomes $6(1+w)^{26}(1+w^2)^{13}$. Expanding each of the two terms separately, allows us to determine that the coefficient of w^2 is $6(C(26,2)+13) = 2,028$.

The third term becomes $8(1+w)^{13}(1+w^3)^{13}$. The coefficient of w^2 from this term is $8C(13,2) = 624$. The fourth term becomes $3(1+w^2)^{26}$. The coefficient of w^2 in this term is 78. The last term does not contribute anything to the coefficient of w^2 after the appropriate substitution.

We sum the contributions towards the coefficient of w^2 and obtain 4,056. We divide by 24 because of the 1/24 in the expression for the polynomial. The result is 169. Thus, the coefficient of w^2 after making the above substitutions is 169. This happens to be the number of hold'em hands where we ignore suits. If we determine other coefficients as we have just done, we find that the first few terms of the polynomial are

$$1 + 13w + 169w^2 + 1,755w^3 + 16,432w^4 + 134,459w^5 + 962,988w^6 + \cdots$$

These are exactly the numbers we obtained from the other approach we took to counting hands, where suits are ignored, with various numbers of cards. So making the substitutions described above into the single polynomial also given above produces an expression where the coefficient of w^m is the number of different hands with m cards ignoring suits. This is a really elegant result on counting card hands.

Of course, Bib Ladder, having the sharp mind he does, noticed quickly that there is no information about beginning seven-card stud hands. The expression just obtained has no distinction made between some cards dealt face up and other cards dealt face down. But I was able to trump the old guy this time. I told him that we can use the same encoding polynomial but now we have to make different substitutions. For x_1 we substitute $1+u+v$; for x_2 we substitute $1+u^2+v^2$; for x_3 we substitute $1+u^3+v^3$; and for x_4 we substitute $1+u^4+v^4$. Upon expanding the polynomial after making these substitutions, it turns out that the coefficient of $u^m v^n$ is the number of hands with m cards face down and n cards face up where we ignore suits.

If one carries out the expansion for this new polynomial, the coefficient of u^2v is just 5,083. That is the number of beginning seven-card stud hands.

This article has presented a single polynomial that can be used to count the number of starting hands for any number of cards, where suits are ignored. The number is obtained by making straightforward substitutions in the polynomial followed by expanding and obtaining the coefficients. Once the polynomial is given, all that is required is high school algebra. Bib asked me how the encoding polynomial is found and I had to tell him that that question took us into something I did not want to attempt to explain.