An alternative energy bound derivation for a generalized Hasegawa-Mima equation

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Abstract

We present an alternative derivation of the H^1-boundedness of solutions to a generalized Hasegawa-Mima equation, first investigated by Grauer [4]. We apply a Lyapunov function technique similar to the one used for constructing energy bounds for the Kuramoto-Sivashinsky equation. Different from Grauer [4], who uses this technique in a Fourier space approach, we employ the physical space construction of the Lyapunov function, as developed in [1]. Our approach has the advantage that it is more transparent in what concerns the estimates and the dominant terms that are being retained. A key tool of the present work, which replaces the algebraic manipulations on the Fourier coefficients from the other approach, is a Hardy-Rellich type inequality.

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1 Introduction

We consider the following extension of the two-dimensional Hasegawa-Mima equation derived in [8] and later analyzed by Grauer [4]:

$$\partial_t(1-\Delta)u = -ku_y - \alpha u_{yy} + \{u, \Delta u\} + \beta\{u, u_y\} + \Delta^3 u. \tag{1}$$

Here, k, α and β are positive constants and $\{\cdot,\cdot\}$ denotes the Poisson bracket

$$\{f, g\} = (\partial_x f)(\partial_y g) - (\partial_y f)(\partial_x g).$$

We require that $u(x, y, t)$ is periodic in y with period L_y and antisymmetric and periodic in x with period $2L_x$ for all times. Note that the equation is invariant under the change $u(x, y, t) \to -u(-x, y, t)$.

The Hasegawa-Mima equation [6],

$$\partial_t(1-\Delta)u = -ku_y + \{u, \Delta u\}, \tag{2}$$

models the dynamics of electrostatic drift waves, here described by the electrostatic potential flutuations u. Drift wave turbulence is of particular interest to plasma confinement in tokamaks

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and controlled nuclear fusion research [7]. The Hasegawa-Mima equation is similar in structure with the 2D incompressible fluid equations, written in terms of the stream-function. In fact, the same equation was derived in a geophysical context by Charney to describe the dynamics of Rossby waves [10]. Among other similarities with the 2D fluid equations, (2) has an inverse energy cascade from small to large scales and forms dipole vortices and other coherent structures [7].

Liang *et al.* [8] extend the original model (2) in several ways. First, they consider the electrons to be non-adiabatic, and introduce the so called $E \times B$ drift nonlinearity, described in (1) by the term $\beta\{u, u_y\}$. The $E \times B$ drift transfers energy non-locally from large to small scales (as in $3D$ turbulence) and therefore competes with the polarization drift nonlinearity $\{u, \Delta u\}$ originally considered by Hasegawa and Mima. Second, Liang *et al.* consider the destabilizing term $-\alpha u_{yy}$ due to trapped particle instabilities. Finally, they introduce an energy sink. Similar to Grauer [4], we take a 6th order damping term, $\Delta^3 u$.

The Hasegawa-Mima equation conserves the H^1-energy, as can be easily checked by multiplying (2) by u and integrating over the spatial domain, with use of the periodic boundary conditions. Hence, using energy methods, existence and uniqueness results for Hasegawa-Mima equation can be established [5]. The *generalized* Hasegawa-Mima equation (1) does not conserve (or dissipate) the H^1-energy, due to the destabilizing anti-diffusive term $-\alpha u_{yy}$. Indeed, after multiplying (1) by u and integrating with the respect to space over $[-L_x, L_x] \times [0, L_y]$, we get

$$\partial_t \frac{1}{2} \int \left(u^2 + (\nabla u)^2\right) dxdy = \alpha \int (u_y)^2\, dxdy - - \int (\nabla \Delta v)^2\, dxdy. \tag{3}$$

Clearly, the right-hand-side of (3) is not sign-definite and no conclusive statement regarding the energy behavior can be immediately drawn. Grauer [4] makes an analogy with the Kuramoto-Sivashinsky (KS) equation,

$$u_t = -uu_x - u_{xx} - u_{xxxx}, \tag{4}$$

a model for certain hydrodynamic problems, in particular the propagation of flame fronts [11]. Indeed, multiply (4) and integrate over space, using the periodic boundary conditions, to get an energy evolution that has no definite sign in its right-hand-side:

$$\partial_t \frac{1}{2} \int u^2 dx = \int (u_x)^2 dx - \int (u_{xx})^2 dx.$$

The spectrum of the linear operator in (4) has a number of unstable modes that grows linearly with the size of the period. Upon reaching large amplitudes however, the linear instabilities are compensated by the nonlinearities and the high-order dissipative term, through an intricate stabilization mechanism. Numerical experiments confirm this stabilizing scenario.

Various methods have been developed to show energy bounds for the KS equation. In particular, the Lyapunov function technique, that has seen constant improvements over the years [9, 2, 3, 1], uses a suitable background flow to absorb the instability. Grauer [4] applies this technique to show H^1-energy bounds for solutions of the generalized Hasegawa-Mima equation (1). He uses the approach developed in [2], where the L^2-energy bound is shown through calculations done entirely in the Fourier space. The estimates in [4] then reduce to various (clever) algebraic manipulations to bound the contribution from the destabilizing or sign-indefinite terms by the Fourier coefficients generated by the 6th order dissipation. We think that this approach obstructs to some extent the intuition behind the various terms that are involved in the calculation, in particular the way they get subdominated by higher order, sign definite quantities.

In the present article we offer an alternative derivation of the H^1-energy bound for solutions of (1) by following the physical space approach developed by Bronski and Gambill [1] to show sharper L^2-bounds for the KS equation. The construction of the background flow potential is made explicitly and all estimates are done in the physical space, for a better transparency in what concerns the dominant versus the subdominant terms. The main result is stated in Theorem 1. Its proof is presented toward the end of the article, after a series of technical lemmas have been stated and the analogy with the KS equation has been properly addressed. The key ingredient used by [1], and not present in the Fourier-based methods, is a Hardy-Rellich type inequality (see Lemma 2) that allows to bound the quadratic form from below.

2 Energy estimate

Lyapunov function. This paper follows the Lyapunov function construction used for the Kuramoto-Sivashinsky (KS) equation and detailed in numerous papers [9, 2, 3]. The main idea is to establish that the H^1-norm of $u(x, y, t) - \phi(x)$ is a Lyapunov functional for some suitably chosen function ϕ. To this purpose we write

$$u(x, y, t) = v(x, y, t) + \phi(x), \tag{5}$$

where $\phi(x)$ is an antisymmetric function. Equation (1) can be written as

$$\begin{aligned}\partial_t(1 - \Delta)v = &-kv_y - \alpha v_{yy} + \{v, \Delta v\} + \beta\{v, v_y\} + \Delta^3 v \\ &+ \phi_x \Delta v_y - v_y \phi_{xxx} + \beta \phi_x v_{yy} + \Delta^3 \phi.\end{aligned}$$

We multiply the equation by v, integrate over the domain $\Omega = [-L_x, L_x] \times [0, L_y]$ and integrate by parts to obtain

$$\begin{aligned}\partial_t \frac{1}{2} \int \left(v^2 + (\nabla v)^2\right) dxdy = \alpha \int (v_y)^2\, dxdy - \int (\nabla \Delta v)^2\, dxdy \\ - \int \phi_x v_y v_{xx}\, dxdy - \beta \int \phi_x (v_y)^2\, dxdy - \int \phi_{xxx} v_{xxx}\, dxdy. \end{aligned} \tag{6}$$

The main result of this paper is the following energy estimate, whose proof is deferred to the end of the section, after various technical results are stated and proved.

Theorem 1. *Consider a smooth, antisymmetric in x, solution u of the generalized Hasegawa-Mima equation (1) on Ω, with periodic boundary conditions. Then, there exists a suitable function ϕ such that*

$$\partial_t \int \left(v^2 + (\nabla v)^2\right) dxdy \leq -\lambda \int \left(v^2 + (\nabla v)^2\right) dxdy + M, \tag{7}$$

where $\lambda > 0$ depends on L_x and $M > 0$ depends on ϕ. Here, v and ϕ are related to u through the background flow decomposition (5). Further, it follows that

$$\limsup_{t \to \infty} \|v\|^2_{H^1} \leq \frac{M}{\lambda}, \tag{8}$$

and one can conclude the existence of an attracting ball in H^1 of radius $\sqrt{\frac{M}{\lambda}}$.

Similarities to the KS equation. The integral $-\int (\nabla\Delta v)^2\, dxdy$ contains the only terms in the right-hand-side of (6) that are negative sign-definite. Expand this expression to get

$$-\int (\nabla\Delta v)^2\, dxdy = -\int (v_{xxx}^2 + 3v_{xxy}^2 + 3v_{xyy}^2 + v_{yyy}^2)\, dxdy.$$

Commute the mixed partial derivatives in one of the v_{xxy}^2 terms and group it with the term that multiplies β in (6), to obtain the following quadratic form for v_y:

$$-\langle v_y, K v_y\rangle = \int \left(v_{yxx}^2 + \beta\phi_x v_y^2\right)\, dxdy.$$

A similar quadratic form appears in the study of the KS equation, where it serves as the main tool used to show energy bounds for KS using a Lyapunov function argument [2]. The strategy in to show coercivity of the quadratic form, which in our context reads

$$\int \left(v_{yxx}^2 + \beta\phi_x v_y^2\right)\, dxdy \geq A \int v_y^2\, dxdy, \tag{9}$$

for a constant A independent of L_x and L_y. Provided A can be chosen large enough (in particular, $A > \alpha$), the term that multiplies α in (6) is subdominant with respect to $-A\int v_y^2\, dxdy$.

The last term in (11) can be estimated easily as

$$\left|\int \phi_{xxx} v_{xxx} dxdy\right| \leq \frac{1}{2}\int v_{xxx}^2 dx + \frac{1}{2}\int \phi_{xxx}^2. \tag{10}$$

Therefore, provided the quadratic form K is coercive with coercivity constant A, we can derive from the previous considerations,

$$\partial_t \frac{1}{2}\int \left(v^2 + (\nabla v)^2\right) dxdy \leq -\frac{1}{2}\int \left(v_{xxx}^2 + 4v_{xxy}^2\right)\, dxdy + (\alpha - A)\int v_y^2\, dxdy$$
$$- \int \phi_x v_y v_{xx}\, dxdy + \frac{1}{2}\int \phi_{xxx}^2\, dxdy. \tag{11}$$

We first address the coercivity of the quadratic form, by following closely the results from [1]. The main remaining issue is to estimate the term $\int \phi_x v_y v_{xx}\, dxdy$. We make use of the explicit construction of the potential ϕ_x in the physical space from [1], that we summarize below.

Potential construction from [1]. We assume that the function ϕ_x takes the following form

$$\beta\phi_x = \gamma L_x^{c_2 - c_1 - 1} + L_x^{c_2} q(L_x^{c_1} x), \qquad \gamma, c_1, c_2 > 0, \tag{12}$$

where γ is a constant and q is a compactly supported smooth function. The potential comprises a positive constant, $\gamma L_x^{c_2-c_1-1}$, and a very localized component of width $O(L_x^{-c_1})$ and amplitude $O(L_x^{c_2})$. More specifically,

$$q(z) = \frac{Q(z)}{z^2}, \tag{13}$$

where Q is an even function, defined for $z \geq 0$, by

$$Q(z) = \begin{cases} -q_0 f(\frac{z}{\delta}) & z \in (0, \delta) \\ -q_0 & z \in (\delta, \frac{a}{2} - \delta) \\ -q_0 + (q_0 + q_1) f(\frac{z - \frac{a}{2} + \delta}{\delta}) & z \in (\frac{a}{2} - \delta, \frac{a}{2}) \\ q_1 & z \in (\frac{a}{2}, a) \\ q_1 f(1 + \frac{a-z}{\delta}) & z \in (a, a+\delta) \\ 0 & z \in (a+\delta, \infty). \end{cases} \tag{14}$$

Here, $f(z)$ is a $C^\infty(0,1)$, non-decreasing function satisfying

$$\lim_{z\to 0} f(z) = 0, \qquad \lim_{z\to 0} f^{(k)}(z) = 0 \quad \text{for } k \geq 1,$$
$$\lim_{z\to 1} f(z) = 1, \qquad \lim_{z\to 1} f^{(k)}(z) = 0 \quad \text{for } k \geq 1,$$

and q_0, q_1, a and δ are positive constants. Note that $supp(q) \subset (-a-\delta, a+\delta)$, where *supp* denotes the function support.

Lemma 1. *Provided the positive constants q_0, q_1, a and δ satisfy*

$$q_0 a^2 < 1,$$
$$q_1 > \frac{q_0}{1 - a^2 q_0},$$
$$a + \delta < L_x^{1+c_1},$$

the potential functions q and Q defined in (13) and (14) have the following properties [1]:

(i) $q \in C_0^\infty$,

(ii) $\int q(z)\, dz \leq -A$ (for this condition to hold, δ has to satisfy an additional smallness assumption),

(iii) $\int_{-L_x^{1+c_1}}^{L_x^{1+c_1}} \frac{1}{2} v_z^2 + Q v^2 \, dz \geq 0$,

for any constant A and all $v \in H^1(-L_x^{1+c_1}, L_x^{1+c_1})$.

One of the main tools used in the proof of the energy bound for the KS equation from [1] is the following Hardy-Rellich type inequality, which we state as a lemma:

Lemma 2. *Suppose that $w \in C^3(-a,a)$ with $w(0) = 0$. Then, if $v(z) = w(z)/z$, we have the inequality:*

$$\int_{-a}^{a} \frac{1}{2} w_{zz}^2 \, dz \geq \int_{-a}^{a} v_z^2 \, dz.$$

For the proofs of Lemmas 1 and 2 we refer to [1]. The following result provides the coercivity of the quadratic form used in deriving (9).

Lemma 3. *There exists a potential ϕ defined by (12) such that*

$$\int_{-L_x}^{L_x} \left(w_{xx}^2 + \beta \phi_x w^2 \right) \, dx \geq A \int_{-L_x}^{L_x} w^2 \, dx, \tag{15}$$

for all $w \in C^3(-L_x, L_x)$ with $w(0) = 0$, where A represents an arbitrary fixed constant.

Proof. Using (12) and the change of variable $z = L^{c_1}x$, we get

$$\int_{-L_x}^{L_x} \left(w_{xx}^2 + \beta\phi_x w^2\right)\,dx = L_x^{3c_1}\int_{-L_x^{1+c_1}}^{L_x^{1+c_1}} \left(w_{zz}^2 + L_x^{c_2-4c_1}q(z)w^2\right)\,dz + \gamma L_x^{c_2-c_1-1}\int_{-L_x}^{L_x} w^2\,dx$$

To satisfy (15) it would be sufficient that

$$\gamma L_x^{c_2-c_1-1} \geq A, \tag{16}$$

and

$$\int_{-L_x^{1+c_1}}^{L_x^{1+c_1}} \left(w_{zz}^2 + L_x^{c_2-4c_1}q(z)w^2\right)\,dz \geq 0. \tag{17}$$

Case $L_x > 1$. This is the case considered in [1] and all the other works regarding optimal energy bounds for the KS equation. More specifically, it is the scaling of the energy with the period L_x as $L_x \to \infty$ that is of interest in those studies. To have (16) hold uniformly in $L_x > 1$ we need c_1 and c_2 to satisfy

$$c_2 - c_1 - 1 \geq 0. \tag{18}$$

Similarly, for (17) to hold uniformly in $L_x > 1$ we need

$$c_2 - 4c_1 \leq 0. \tag{19}$$

Regarding the latter condition, one can argue it by contradiction. Suppose that $c_2 - 4c_1 > 0$ and take a function w whose support is contained in a region where $q < 0$. Then conclude that (17) cannot hold uniformly in $L_x > 1$.

Due to periodic boundary conditions, the potential $\beta\phi_x$ has zero mean on $(-L_x, L_x)$ and hence, from (12) we infer that

$$2\gamma = -\int q(z)\,dz. \tag{20}$$

Using Lemma 1, property (ii), we conclude that the constant γ in the expression of the potential (12) can be made arbitrarily large. This proves (16), provided (18) holds.

To show (17), we use (19), Lemma 2 and Lemma 1 (property (iii)) to get

$$\begin{aligned}\int_{-L_x^{1+c_1}}^{L_x^{1+c_1}} \left(w_{zz}^2 + L_x^{c_2-4c_1}q(z)w^2\right)\,dz &\geq L_x^{c_2-4c_1}\int_{-L_x^{1+c_1}}^{L_x^{1+c_1}} \left(w_{zz}^2 + q(z)w^2\right)\,dz \\ &\geq L_x^{c_2-4c_1}\int_{-L_x^{1+c_1}}^{L_x^{1+c_1}} \left(\frac{1}{2}v_z^2 + \underbrace{z^2q(z)}_{Q(z)}v^2\right)dz \\ &\geq 0.\end{aligned}$$

Case $0 < L_x < 1$. This case is not of interest for energy bounds for the KS equation, but the procedure from [1] used for the case $L_x > 1$ above, applies as well. Based on similar considerations used to derive (18) and (19), we note that for (16) and (17) to hold uniformly for $L_x < 1$, c_1 and c_2 must necessarily satisfy

$$c_2 - c_1 - 1 \leq 0,$$
$$c_2 - 4c_1 \geq 0.$$

Considerations very similar to those used in the previous case follow easily, completing the proof. □

Estimate of $\int \phi_x v_y v_{xx}\, dxdy$. We state and prove the following lemma:

Lemma 4. *For appropriate choices of the constants c_1 and c_2 that enter the equation (12) for the potential, we have*

$$\left| \int \phi_x v_y v_{xx}\, dxdy \right| \le \frac{1}{4} \int \left(v_{xxx}^2 + v_{yxx}^2 \right) dxdy + \frac{1}{16} \int v_y^2 dxdy.$$

Proof. Use (12) to get:

$$\int \phi_x v_y v_{xx}\, dxdy = \frac{\gamma}{\beta} L_x^{c_2 - c_1 - 1} \int v_y v_{xx}\, dxdy + \frac{1}{\beta} L_x^{c_2} \int q(L_x^{c_1} x) v_y v_{xx}\, dxdy. \tag{21}$$

Periodic (in y) boundary conditions and integration by parts yield the first term in the right-hand-side of (21), zero:

$$\begin{aligned} \int v_y v_{xx}\, dxdy &= -\frac{1}{2} \int \partial_y (v_x)^2\, dxdy \\ &= 0. \end{aligned}$$

Now use $supp(q) \subset (-a-\delta, a+\delta)$ to estimate the remainder term as follows:

$$\begin{aligned} L_x^{c_2} \left| \int_0^{L_y} \int_{-L_x}^{L_x} q(L_x^{c_1} x) v_y v_{xx}\, dxdy \right| &= L_x^{c_2} \left| \int_0^{L_y} \int_{-L_x^{-c_1}(a+\delta)}^{L_x^{-c_1}(a+\delta)} q(L_x^{c_1} x) v_y v_{xx}\, dxdy \right| \\ &\le \sup |q(x)| L_x^{c_2} \int_0^{L_y} \int_{-L_x^{-c_1}(a+\delta)}^{L_x^{-c_1}(a+\delta)} |v_y||v_{xx}|\, dxdy \\ &\le \frac{1}{2} \sup |q(x)| L_x^{c_2} \left[\int_0^{L_y} \int_{-L_x^{-c_1}(a+\delta)}^{L_x^{-c_1}(a+\delta)} v_y^2\, dxdy + \int_0^{L^y} \int_{-L_x^{-c_1}(a+\delta)}^{L_x^{-c_1}(a+\delta)} v_{xx}^2\, dxdy \right]. \end{aligned} \tag{22}$$

Since v is an antisymmetric function of x, so are v_y and v_{xx}. Hence, v_y and v_{xx} have zero mean on a symmetric x-interval, and by Wirtinger's inequality (a special one-dimensional case of Poincaré's inequality) in the x variable we have

$$\begin{aligned} \int_{-L_x^{-c_1}(a+\delta)}^{L_x^{-c_1}(a+\delta)} v_y^2\, dx &\le \frac{1}{\pi^2} L_x^{-2c_1} (a+\delta)^2 \int_{-L_x^{-c_1}(a+\delta)}^{L_x^{-c_1}(a+\delta)} v_{yx}^2\, dx \\ &\le \frac{1}{\pi^2} L_x^{-2c_1} (a+\delta)^2 \int_{-L_x}^{L_x} v_{yx}^2\, dx, \end{aligned}$$

where for the second inequality we used $a + \delta < L_x^{1+c_1}$ (see the hypotheses on the parameters stated in Lemma 1. Similarly,

$$\int_{-L_x^{-c_1}(a+\delta)}^{L_x^{-c_1}(a+\delta)} v_{xx}^2 dx \le \frac{1}{\pi^2} L_x^{-2c_1} (a+\delta)^2 \int_{-L_x}^{L_x} v_{xxx}^2\, dx.$$

By combining the last two estimates with (21) and (22) we obtain

$$\left| \int \phi_x v_y v_{xx}\, dxdy \right| \le \frac{1}{2\pi^2 \beta} \sup |q(x)| L_x^{c_2 - 2c_1} (a+\delta)^2 \left[\int_0^{L_y} \int_{-L_x}^{L_x} v_{yx}^2 dxdy + \int_0^{L_y} \int_{-L_x}^{L_x} v_{xxx}^2 dxdy \right]. \tag{23}$$

The inequality $k^2 \le k^4 + \frac{1}{4}$ for all k implies, using the Fourier transform, that

$$\int_0^{L_y}\int_{-L_x}^{L_x} v_{yx}^2 dxdy \le \int_0^{L_y}\int_{-L_x}^{L_x} v_{yxx}^2 dxdy + \frac{1}{4}\int_0^{L_y}\int_{-L_x}^{L_x} v_y^2 dxdy.$$

Then, the lemma follows from (23) and the previous estimate, provided we choose the positive constants c_1 and c_2 such that

$$\frac{1}{2\pi^2\beta}\sup|q(x)|L_x^{c_2-2c_1}(a+\delta)^2 \le \frac{1}{4}. \tag{24}$$

If $L_x > 1$, then we need

$$c_2 - 2c_1 \le \log_{L_x}\left(\frac{\pi^2\beta}{2\sup|q(x)|(a+\delta)^2}\right). \tag{25}$$

Combining (25) with the previous requirements (18) and (19), we infer that a necessary condition for c_1 is

$$c_1 \ge \max\left\{\frac{1}{3}, 1 - \log_{L_x}\left(\frac{\pi^2\beta}{2\sup|q(x)|(a+\delta)^2}\right)\right\} \tag{26}$$

For $0 < L_x < 1$, the inequality (25) is reversed and combined with the previous constraints on c_1 and c_2, it yields the necessary condition

$$c_1 \le \min\left\{\frac{1}{3}, 1 - \log_{L_x}\left(\frac{\pi^2\beta}{2\sup|q(x)|(a+\delta)^2}\right)\right\}.$$

□

Proof of Theorem 1.

Proof. The proof follows easily from the previous results. In (6), use the coercivity property stated in Lemma 3 for $w = v_y$, the estimate from Lemma 4 and (10) to get

$$\partial_t \frac{1}{2}\int\left(v^2 + (\nabla v)^2\right) dxdy \le -\frac{1}{4}\int\left(v_{xxx}^2 + v_{xxy}^2\right) dxdy + \left(\frac{1}{16} + \alpha - A\right)\int v_y^2 \, dxdy + \frac{1}{2}\int \phi_{xxx}^2 \, dxdy. \tag{27}$$

Recall that v is antisymmetric in x, in particular of mean zero on $(-L_x, L_x)$. Apply the Wirtinger's inequality in the x variable to get

$$\int v^2 \, dxdy \le \frac{L_x^2}{\pi^2}\int v_x^2 \, dxdy.$$

Furthermore, due to periodic boundary conditions, v_x and v_{xx} are also mean zero,

$$\int v_x^2 \, dxdy \le \frac{L_x^4}{\pi^4}\int v_{xxx}^2 \, dxdy.$$

Similarly, v_y and v_{yx} are mean zero in the x variable and we have

$$\int v_y^2 \, dxdy \le \frac{L_x^4}{\pi^4}\int v_{xxy}^2 \, dxdy.$$

The last three Poincaré-type inequalities can be used for the first term in the right-hand-side of (27) to generate the H^1-norm of v, as needed in (7). Since according to Lemma 3, we can choose the constant A such that $A > \frac{1}{16} + \alpha$, we have from (27),

$$\partial_t \int \left(v^2 + (\nabla v)^2\right) dxdy \leq -\frac{1}{4} \min\left\{\frac{\pi^6}{L_x^6}, \frac{\pi^4}{L_x^4}\right\} \int \left(v^2 + (\nabla v)^2\right) dxdy + \int \phi_{xxx}^2 \, dxdy.$$

Choosing

$$\lambda = \frac{1}{4} \min\left\{\frac{\pi^6}{L_x^6}, \frac{\pi^4}{L_x^4}\right\},$$

and

$$M = \int \phi_{xxx}^2 \, dxdy,$$

proves the theorem. □

Large period regime. This is the regime of interest for the KS equation, since the number of the unstable modes of the linearization about the zero state, increases linearly with the period. In this context, the primary goal is to find sharp L^2-bounds on the size of the attracting ball, i.e., find the optimal (minimum) exponent p, such that $\limsup_{t\to\infty} \|u\|_2 = O(L^p)$, where L is the period. As far as the modified Hasegawa-Mima equation (1) is concerned, its linearization about the zero-state does not seem to indicate any particular interest in the limit $L_x \to \infty$. We will discuss briefly however, the regime of large L_x and L_y, mainly to compare the outcomes of the two approaches, the physical space construction used in the present paper and the Fourier space method used in [4].

Using (12), we immediately infer that

$$\int \phi_{xxx}^2 dxdy = O\left(L_y L_x^{2c_2+3c_1}\right), \tag{28}$$

For large L_x, $\lambda = \frac{\pi^6}{4} L_x^{-6}$ and hence, from (8),

$$\limsup_{t\to\infty} \|v\|_{H^1}^2 \leq O\left(L_y L_x^{2c_2+3c_1+6}\right). \tag{29}$$

We have to minimize the exponent $2c_2 + 3c_1 + 6$ subject to the constraints (18), (19) and (25). Inspect (26) for large L_x and take $c_1 = 1 - \log_{L_x} C$, where $C = \pi^2\beta/(2\sup|q(x)|(a+\delta)^2)$. Also, choose optimally $c_2 = c_1 + 1$ to find from (29), that

$$\limsup_{t\to\infty} \|v\|_{H^1}^2 \leq O\left(L_y L_x^{13}\right).$$

Interestingly, the Fourier method used by Grauer [4] gives the same exponents in the large period regime.

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