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A Leray-type regularization for the isentropic Euler equations

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Abstract
We consider a Leray-type regularization for the isentropic Euler equations for a γ-law gas, and we investigate the existence of smooth solutions for the regularized system. The technique we use is the weakly nonlinear geometrical optics (WNGO) asymptotic theory. The WNGO theory applied to our system of equations predicts shock formation in finite time for $\gamma \neq 1$ and suggests existence of global smooth solutions for $\gamma = 1$. We also perform numerical computations and show that the WNGO predictions are correct.

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1. Introduction

In this paper, we consider the following system of equations:

$$\rho_t + u\rho_x + \rho v_x = 0, \tag{1a}$$

$$v_t + uv_x + \frac{p_x}{\rho} = 0, \tag{1b}$$

$$v = u - \alpha^2 u_{xx}, \tag{1c}$$

where α is a positive parameter. When $\alpha = 0$ (i.e. $v = u$) the system represents the 1D isentropic compressible Euler equations. Here, ρ denotes the mass density, v the velocity and $p = p(\rho)$ the pressure of the gas. We assume that $p(\rho)$ is given by γ law

$$p = \kappa\rho^\gamma, \tag{2}$$

where $\kappa > 0$ and $\gamma > 0$ are constants.

The goal of this work is to investigate whether or not the system (1*a*)–(1*c*) regularizes the 1D isentropic compressible Euler equations. The first aspect one needs to investigate is the global existence of smooth solutions for system (1*a*)–(1*c*). Provided the well-posedness for system (1*a*)–(1*c*) is established, the next question that could be addressed is whether the solutions of (1*a*)–(1*c*) converge in some sense, as $\alpha \to 0$, to solutions of the 1D compressible Euler equations. The results of this paper deal with the first aspect only, leaving the second question for future work.

The idea of using a system like (1*a*)–(1*c*) in attempting to regularize the compressible Euler equations goes back to Leray [Ler34]. Working in the context of the incompressible Navier–Stokes equations, Leray first proposed replacing the nonlinear term $(v \cdot \nabla)v$ with a term $(u \cdot \nabla)v$. Here $u = K^{\epsilon} * v$ for some smoothing kernel K^{ϵ}. Leray's program consisted of proving existence of solutions for his modified equations and then showing that these solutions converge, as $\epsilon \downarrow 0$, to weak solutions of Navier–Stokes—see [Ler34] for details. More recently, the Leray model has been used as a subgrid scale model of 3D turbulence—see [CHOT05]. We also mention that Leray-type ideas were recently used to regularize the Burgers equation (see [MZM06, BF06]). In fact, the Leray regularization of the Burgers equation will play a central role in the subsequent analysis.

We borrow these ideas and use them for compressible fluids. Equations (1*a*) and (1*b*) are obtained from the compressible Euler equations by replacing the convective velocity v with a smoothed version of it, u, where $u = \mathcal{H}^{-1}v$. Here, $\mathcal{H}^{-1}$ represents the inverse of the Helmholtz operator

$$\mathcal{H} = \mathrm{Id} - \alpha^2 \partial_{xx}. \tag{3}$$

Using Green's function of $\mathcal{H}$, we have an explicit formula for u in terms of v:

$$u(x, t) = \frac{1}{2\alpha} \int_{-\infty}^{\infty} \exp(-|x - y|/\alpha) v(y, t)\, \mathrm{d}y. \tag{4}$$

We will see ((7) below) that the Leray system (1*a*)–(1*c*) agrees with compressible Euler on the linear level. Only nonlinear wave steepening is different. The compressible Euler equations are genuinely nonlinear [Lax73] for any $\gamma > 0$, and therefore small-amplitude solutions develop shocks in finite time [Joh74]. The wave steepening mechanism of the Leray system (1*a*)–(1*c*) is different. Asymptotics and numerics show that for $\gamma > 1$ a shock forms in finite time, while for $\gamma = 1$ the slope grows exponentially in time but remains finite at any time.

We use the asymptotic method weakly nonlinear geometric optics (WNGO) [HK83, MR84] to predict shock formation or lack thereof in the Leray system. Many studies have shown that WNGO correctly predicts shock formation in borderline situations. For example [AHPR93] it correctly predicts breakdown of radial disturbances for compressible Euler in 3D but not in 4D. In our case, the predictions of WNGO are in detailed quantitative agreement with numerical computations.

The WNGO treatment of system (1*a*)–(1*c*) is presented in section 2. Section 3 contains the numerical results as compared with the WNGO predictions.

The conclusions that can be drawn from our study of the Leray system (1*a*)–(1*c*) are as follows.

1. For $\gamma \neq 1$ the system (1*a*)–(1*c*) fails to have global smooth solutions and therefore does not regularize the equations for γ-law gas dynamics. Both WNGO theory and the numerics show that the blow-up rate for solutions of (1*a*)–(1*c*) is slower than that for solutions of the Euler equations.

2. For $\gamma = 1$ the system does not develop shocks in the first order terms of the expansion. Both WNGO theory and the numerics suggest that the slope of the amplitude increases exponentially fast but does not become infinite in finite time.

2. Weakly nonlinear geometrical optics

We take a small perturbation of a constant solution of (1*a*)–(1*c*) and then study the resulting system for the perturbation using the WNGO approach [HK83, MR84].

We can write (1*a*) and (1*b*) as

$$\begin{pmatrix} \rho \\ v \end{pmatrix}_t + \begin{pmatrix} u & \rho \\ \kappa\gamma\rho^{\gamma-2} & u \end{pmatrix} \begin{pmatrix} \rho \\ v \end{pmatrix}_x = 0, \tag{5}$$

where u and v are related by (1*c*). Now consider $(\rho_0, 0)$ a constant solution of (1*a*)–(1*c*), take $\rho = \rho_0 + \rho'$, $v = 0 + v'$ (and correspondingly, $u = 0 + u'$) and plug these expressions into (5). After expanding the nonlinear terms in Taylor series around $(\rho_0, 0)$ and truncating the series at the second order we obtain

$$\begin{pmatrix} \rho' \\ v' \end{pmatrix}_t + \begin{pmatrix} 0 & \rho_0 \\ \kappa\gamma{\rho_0}^{\gamma-2} & 0 \end{pmatrix} \begin{pmatrix} \rho' \\ v' \end{pmatrix}_x + \begin{pmatrix} u' & \rho' \\ \kappa\gamma(\gamma-2){\rho_0}^{\gamma-3}\rho' & u' \end{pmatrix} \begin{pmatrix} \rho' \\ v' \end{pmatrix}_x = 0. \tag{6}$$

To simplify notation, we delete the primes in (6) and rename (ρ', v') as (ρ, v). Hence, we will study

$$\begin{pmatrix} \rho \\ v \end{pmatrix}_t + A \begin{pmatrix} \rho \\ v \end{pmatrix}_x + B \begin{pmatrix} \rho \\ v \end{pmatrix}_x = 0, \tag{7}$$

and the matrices A and B are given by

$$A = \begin{pmatrix} 0 & \rho_0 \\ \kappa\gamma{\rho_0}^{\gamma-2} & 0 \end{pmatrix},$$

$$B(\rho, v) = \begin{pmatrix} u & \rho \\ \varphi(\rho_0)\rho & u \end{pmatrix},$$

with

$$\varphi(\rho_0) = \kappa\gamma(\gamma-2){\rho_0}^{\gamma-3}.$$

The linearization matrix A in (7) is the same as it would have been for the isentropic compressible Euler equations [CF76]. Therefore the eigenvalues $\lambda_1 = -c_0$ and $\lambda_2 = c_0$ (with $c_0 = \sqrt{p'(\rho_0)} = \sqrt{\kappa\gamma{\rho_0}^{\gamma-1}}$) and the corresponding eigenvectors $\boldsymbol{r}_1 = (\rho_0, -c_0)^{\mathrm{T}}$ and $\boldsymbol{r}_2 = (\rho_0, c_0)^{\mathrm{T}}$ are the same as for the compressible Euler equations [Smo83]. The leading order nonlinearity is represented by B. Compressible Euler differs by having v instead of u on the diagonal of B, which leads to finite time breakdown (in Euler) for all small-amplitude localized or periodic initial data.

For the present one dimensional situation, WNGO is little more than the method of multiple scales applied to estimating the slowly changing shape of a wave under the influence of weak nonlinearity [KC96]. If we neglect the nonlinear term in (7), the general solution is a superposition of left and right-moving waves: $(\rho, v)^{\mathrm{T}} = G_1(x + c_0 t)\boldsymbol{r}_1 + G_2(x - c_0 t)\boldsymbol{r}_2$. With $B \neq 0$, waves with amplitude ϵ will change shape on a time scale of order $1/\epsilon$. We take this into account by adding dependence on a slow time variable $\tau = \epsilon t$. The dependence of the leading order term on the slow variable is determined by the requirement that the first correction be bounded for times of order $1/\epsilon$. The WNGO ansatz is

$$\begin{pmatrix}\rho\\ v\end{pmatrix} = \epsilon g_1(x+c_0t,\epsilon t)\boldsymbol{r}_1 + \epsilon g_2(x-c_0t,\epsilon t)\boldsymbol{r}_2 + \epsilon^2 m_1(x,t,\epsilon t)\boldsymbol{r}_1 + \epsilon^2 m_2(x,t,\epsilon t)\boldsymbol{r}_2 + O(\epsilon^3). \tag{8}$$

We denote the characteristic variables $\xi_1 = x + c_0 t$ and $\xi_2 = x - c_0 t$. After plugging the ansatz (8) into (7), the order $O(\epsilon)$ term gives an identity,

$$(c_0\boldsymbol{r}_1 + A\boldsymbol{r}_1)g_{1,\xi_1}(\xi_1,\tau) + (-c_0\boldsymbol{r}_2 + A\boldsymbol{r}_2)g_{2,\xi_2}(\xi_2,\tau) = 0.$$

Here and in what follows, we use comma subscripts to denote differentiation. At the order $O(\epsilon^2)$ we obtain

$$\begin{aligned} &g_{1,\tau}(\xi_1,\tau)\boldsymbol{r}_1 + g_{2,\tau}(\xi_2,\tau)\boldsymbol{r}_2 + m_{1,t}\boldsymbol{r}_1 + m_{2,t}\boldsymbol{r}_2 + m_{1,x}A\boldsymbol{r}_1 + m_{2,x}A\boldsymbol{r}_2 \\ &\quad + \begin{pmatrix} -c_0 f_1 + c_0 f_2 & \rho_0 g_1 + \rho_0 g_2 \\ \varphi(\rho_0)(\rho_0 g_1 + \rho_0 g_2) & -c_0 f_1 + c_0 f_2 \end{pmatrix} \left(g_{1,\xi_1}(\xi_1,\tau)\boldsymbol{r}_1 + g_{2,\xi_2}(\xi_1,\tau)\boldsymbol{r}_2\right) = 0, \end{aligned} \tag{9}$$

where f_1, g_1 and f_2, g_2 are related as u and v in (1*c*), i.e. $g_i = \mathcal{H} f_i$, $i = 1, 2$ with $\mathcal{H}$ as in (3). We project equation (9) on $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, respectively. In order to do this we need to express the last term in the LHS of (9) in this basis. Thus we perform the multiplication and obtain

$$\begin{pmatrix} g_{1,\xi_1}\rho_0 c_0(-f_1 + f_2 - g_1 - g_2) + g_{2,\xi_2}\rho_0 c_0(-f_1 + f_2 + g_1 + g_2) \\ g_{1,\xi_1}[\varphi(\rho_0){\rho_0}^2(g_1+g_2) - {c_0}^2(-f_1+f_2)] + g_{2,\xi_2}[\varphi(\rho_0){\rho_0}^2(g_1+g_2) + {c_0}^2(-f_1+f_2)] \end{pmatrix}.$$

This expression can be simplified. Note that

$$\begin{aligned} \varphi(\rho_0){\rho_0}^2 &= \kappa\gamma(\gamma-2){\rho_0}^{\gamma-1} \\ &= (\gamma-2){c_0}^2. \end{aligned}$$

Hence, the last term in the LHS of (9) reads

$$c_0\begin{pmatrix} \rho_0[g_{1,\xi_1}(-f_1 + f_2 - g_1 - g_2) + g_{2,\xi_2}(-f_1 + f_2 + g_1 + g_2)] \\ c_0[g_{1,\xi_1}((\gamma-2)(g_1+g_2) + f_1 - f_2)) + g_{2,\xi_2}((\gamma-2)(g_1+g_2) - f_1 + f_2))] \end{pmatrix}.$$

Write this resulting vector in the $\{\boldsymbol{r}_1, \boldsymbol{r}_2\}$ basis, i.e. write it as

$$a\boldsymbol{r}_1 + b\boldsymbol{r}_2.$$

After some algebra, we obtain

$$a = -\frac{c_0}{2}\left[g_{1,\xi_1}(2f_1 - 2f_2 + (\gamma-1)(g_1+g_2)) + g_{2,\xi_2}(\gamma-3)(g_1+g_2)\right]$$

and

$$b = \frac{c_0}{2}\left[g_{1,\xi_1}(\gamma-3)(g_1+g_2) + g_{2,\xi_2}((\gamma-1)(g_1+g_2) - 2f_1 + 2f_2)\right].$$

Now, by projecting (9) on the first eigenvector $\boldsymbol{r}_1$ we get

$$-(m_{1,t} - c_0 m_{1,x}) = R_1(x,t,\tau), \tag{10}$$

where

$$\begin{aligned} R_1(x,t,\tau) = g_{1,\tau}(x+c_0t,\tau) - \frac{c_0}{2}[g_{1,\xi_1}(2f_1 - 2f_2 + (\gamma-1)(g_1+g_2)) \\ + g_{2,\xi_2}(\gamma-3)(g_1+g_2)]. \end{aligned} \tag{11}$$

We remind the reader that in this equation, g_1 and f_1 are functions of $\xi_1 = x + c_0 t$ and $\tau = \epsilon t$, while g_2 and f_2 are functions of $\xi_2 = x - c_0 t$ and τ.

The general solution to (10) is

$$m_1(x,t,\tau) = h_1(x+c_0t) - \int_0^t R_1(x + c_0(t-s), s, \tau)\,\mathrm{d}s,$$

with h_1 an arbitrary function. Using (11), we evaluate the last integral as follows:

$$\begin{aligned}
&\int_0^t R_1(x+c_0(t-s),s,\tau)\,\mathrm{d}s \\
&\quad = t\Big(g_{1,\tau}(\xi_1,\tau) - c_0 f_1(\xi_1,\tau)g_{1,\xi_1}(\xi_1,\tau) - \frac{c_0}{2}(\gamma-1)g_1(\xi_1,\tau)g_{1,\xi_1}(\xi_1,\tau)\Big) \\
&\quad + c_0 g_{1,\xi_1}(\xi_1,\tau)\int_0^t f_2(x+c_0t-2c_0s,\tau)\,\mathrm{d}s - \frac{c_0}{2}(\gamma-1)g_{1,\xi_1}(\xi_1,\tau) \\
&\quad \times \int_0^t g_2(x+c_0t-2c_0s,\tau)\,\mathrm{d}s \\
&\quad -\frac{c_0}{2}(\gamma-3)g_1(\xi_1,\tau)\underbrace{\int_0^t g_{2,\xi_2}(x+c_0t-2c_0s,\tau)\,\mathrm{d}s}_{I} \\
&\quad -\frac{c_0}{2}(\gamma-3)\underbrace{\int_0^t (g_2g_{2,\xi_2})(x+c_0t-2c_0s,\tau)\,\mathrm{d}s}_{II}\,.
\end{aligned}\tag{12}$$

Following [MR84], we make the following assumptions on g_1 and g_2:

1. g_i and g_{i,ξ_i} are bounded functions of ξ_i, where $i = 1, 2$.
2. The averages

$$\bar{g}_i(\tau) = \lim_{L\to\infty}\frac{1}{2L}\int_{-L}^{L} g_i(\xi,\tau)\,\mathrm{d}\xi, \qquad 1 \leqslant i \leqslant 2,$$

exist and

$$\frac{1}{L}\int_0^L g_i(\xi+\eta,\tau)\,\mathrm{d}\xi = \bar{g}_i(\tau) + o(1) \qquad \text{as} \quad L\to\pm\infty,$$

uniformly in $-\infty < \eta < \infty$.

The m_i, $i = 1, 2$, are assumed to be smooth functions of their arguments, bounded in x and having at most sublinear growth in t as $t \pm \infty$. This assumption makes the perturbation expansion (8) formally valid for times t of order at least ϵ^{-1}.

Integrals I and II from (12) are easily seen to be integrals of derivatives of, respectively, g_2 and $(g_2)^2$. Evaluating these integrals and using assumption 1, one finds that I and II yield terms that are uniformly bounded in time. Hence,

$$\begin{aligned}
&\int_0^t R_1(x+c_0(t-s),s,\tau)\,\mathrm{d}s = t\Big(g_{1,\tau}(\xi_1,\tau) - c_0 f_1(\xi_1,\tau)g_{1,\xi_1}(\xi_1,\tau) \\
&\quad -\frac{c_0}{2}(\gamma-1)g_1(\xi_1,\tau)g_{1,\xi_1}(\xi_1,\tau) + c_0 g_{1,\xi_1}(\xi_1,\tau)\underbrace{\frac{1}{-2c_0t}\int_0^{-2c_0t} f_2(x+c_0t+z,\tau)\,\mathrm{d}z}_{\text{III}} \\
&\quad -\frac{c_0}{2}(\gamma-1)g_{1,\xi_1}(\xi_1,\tau)\underbrace{\frac{1}{-2c_0t}\int_0^{-2c_0t} g_2(x+c_0t+z,\tau)\,dz}_{\text{IV}}\Big) + \text{bounded terms.}
\end{aligned}\tag{13}$$

We apply assumption 2 to integrals III and IV from (13) and replace these integrals by barred quantities. Now, to suppress the linear growth of m_1 in time, we require

$$g_{1,\tau} - c_0 f_1 g_{1,\xi_1} - \frac{c_0}{2}(\gamma-1)g_1g_{1,\xi_1} + c_0\bar{f}_2(\tau)g_{1,\xi_1} - \frac{c_0}{2}(\gamma-1)\bar{g}_2(\tau)g_{1,\xi_1} = 0. \tag{14}$$

After projecting (9) on the second eigenvector $\boldsymbol{r}_2$ and carrying out similar computations we obtain the following equation for g_2:

$$g_{2,\tau} + c_0 f_2 g_{2,\xi_2} + \frac{c_0}{2}(\gamma - 1)g_2 g_{2,\xi_2} - c_0 \bar{f}_1(\tau) g_{2,\xi_2} + \frac{c_0}{2}(\gamma - 1)\bar{g}_1(\tau) g_{2,\xi_2} = 0. \tag{15}$$

Recall that equations (14) and (15) should be coupled with

$$g_i = \mathcal{H} f_i, \qquad i = 1, 2, \tag{16}$$

with $\mathcal{H}$ as in (3). Note that

$$f_1 \partial_{\xi_1} g_1 = f_1 \partial_{\xi_1}(f_1 - \alpha^2 \partial_{\xi_1}^2 f_1)$$
$$= \partial_{\xi_1}\left(\tfrac{1}{2}\partial_{\xi_1} f_1 - \alpha^2 f_1 \partial_{\xi_1}^2 f_1 + \tfrac{1}{2}\alpha^2 (\partial_{\xi_1} f_1)^2\right),$$

and hence, equation (14) can be written in conservation law form:

$$\partial_\tau g_1 - c_0 \partial_{\xi_1}\Big(\tfrac{1}{2}\partial_{\xi_1} f_1 - \alpha^2 f_1 \partial_{\xi_1}^2 f_1 + \tfrac{1}{2}\alpha^2(\partial_{\xi_1} f_1)^2 + \tfrac{1}{4}(\gamma - 1) g_1^2 - \bar{f}_2(\tau) g_1$$
$$+ \tfrac{1}{2}(\gamma - 1)\bar{g}_2(\tau) g_1\Big) = 0. \tag{17}$$

From (17) we infer

$$\partial_\tau \bar{g}_1 = 0, \qquad \text{i.e.} \quad \bar{g}_1 \text{is a constant.}$$

It then follows that $\bar{f}_1$ is a constant. Similarly, $\bar{g}_2$, $\bar{f}_2$ are constants as well.

Analysis. Based on the previous observation, the barred coefficients in equations (14) and (15) are constant and thus the analysis of the two equations is greatly simplified. We will consider the equation for the right-moving wave only, i.e. (15) coupled with (16) for $i = 2$. The results we derive clearly apply to g_1 as well.

Case $\gamma = 1$. For $\gamma = 1$ the system (15), (16) reduces to

$$g_{2,\tau} + c_0 f_2 g_{2,\xi_2} - c_0 \bar{f}_1 g_{2,\xi_2} = 0, \tag{18}$$

where

$$g_2 = \mathcal{H} f_2.$$

Note that up to a change of variables, this is precisely the Leray-type regularization of the Burgers equation that was first proposed in [MZM06] and later analysed in [BF06]. In [BF06] it is shown that the Cauchy problem for (18) is well-posed for all $\alpha > 0$: a classical solution $g_2^\alpha(\xi, \tau)$ to (18) exists globally in time, given initial data in $W^{2,1}(\mathbb{R})$. Therefore, at first order in the asymptotic expansion, there are no shocks that develop in finite time and this suggests that (1) has global smooth solutions. Motivated in part by this suggestion, the work of [Fet07] has recently established that (1*a*)–(1*c*) (with p given by the γ law (2) with $\gamma = 1$) does in fact have global smooth solutions, validating the WNGO prediction.

The negative infimum $-\inf_\xi v_\xi(\xi, \tau)$ increases exponentially in time. This can be seen from the following argument. Ignore the translation term and consider the long time behaviour for the equation[4]

$$v_t + u v_x = 0, \tag{19}$$

where $v = u - \alpha^2 u_{xx}$. The long time behaviour for (18) should be similar to that for (19).

Define

$$s(t) = v_x(\zeta(t), t),$$

[4] In what follows, we abuse notation and use (x, t) instead of (ξ, τ).

where $\zeta(t)$ satisfies

$$v_x(\zeta(t), t) = \inf_x v_x(x, t).$$

Hence, $v_{xx}(\zeta(t), t) = 0$. Now differentiate (19) with respect to x once and evaluate the resulting equation at $x = \zeta(t)$ to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} s(t) = -u_x(\zeta(t), t) s(t).$$

We argue that we can replace $u_x(\zeta(t), t)$ by a constant in the previous equation and yet recover the correct behaviour of $s(t)$ for long times. The reason is that the infimum of u_x does not blow up and saturates at some value of order $O(\alpha^{-1})$. One can see this by differentiating (4) and using the fact that v remains uniformly bounded for all times.

Therefore, we examine instead

$$\frac{\mathrm{d}}{\mathrm{d}t} s(t) = C s(t),$$

where $C > 0$ is a constant. A trivial integration yields

$$s(t) = s(0)\mathrm{e}^{Ct}, \tag{20}$$

which justifies the assertion above.

The numerical results presented in section 3 confirm the WNGO predictions and the validity of (20).

Case $\gamma \neq 1$. For $\gamma \neq 1$ equation (15) has a Burgers-like nonlinear term, $\frac{c_0}{2}(\gamma - 1) g_2 g_{2,\xi_2}$ that produces finite time blow-up in the first derivative g_{2,ξ_2}, as long as the initial slope $g_{2,\xi_2}(0)$ is negative for at least one point. Therefore, the original system (1*a*)–(1*c*) exhibits finite time shock formation as well.

We will briefly investigate the finite time blow-up in the equation for g_2. For the Burgers equation, $v_t + v v_x = 0$, the infimum $\inf_x v_x(x, t)$ blows up like $O(1/(t - T))$, where T is the time where the shock forms. Now consider the equation

$$v_t + u v_x + v v_x = 0, \tag{21}$$

where $v = u - \alpha^2 u_{xx}$. The shock formation for (15) should be of the same type as that for (21).

We use the same notations for $s(t)$ and $\zeta(t)$ as above. By differentiating (21) with respect to x once and evaluating the resulting equation at $x = \zeta(t)$ one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t} s(t) = -u_x(\zeta(t), t) s(t) - s(t)^2.$$

An argument similar to the one used in the $\gamma = 1$ case enables us to replace $u_x(\zeta(t), t)$ by a constant. Hence, we examine

$$\frac{\mathrm{d}}{\mathrm{d}t} s(t) = C s(t) - s(t)^2,$$

where $C > 0$ is a constant. An integration yields

$$\frac{C}{s(t)} = 1 - \mathrm{e}^{-C(t-T)},$$

where T is a constant of integration which also represents the blow-up time. We Taylor expand the exponential term in the right-hand side around $t = T$ and ignore the terms of third order and higher. After some trivial algebra we obtain

$$\frac{1}{s(t)} \approx (t - T) - \frac{1}{2} C (t - T)^2. \tag{22}$$

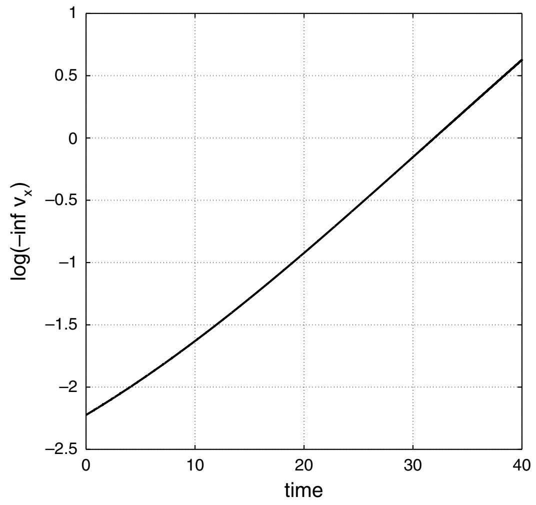

Figure 1. Near-linear dependence of $\log[-\inf_x v_x(x,t)]$ on t, for solution of (1*a*)–(1*c*) with $\gamma = 1$ and one-wave initial data (23).

The quadratic correction means that the blow-up time for (21) is greater than that for the Burgers equation. The term uv_x slows down the break-up of the solution.

To conclude, for $\gamma = 1$, the WNGO analysis suggests existence of smooth solutions for the original system (1*a*)–(1*c*). The negative infimum of the slope v_x is expected to increase exponentially in time. For $\gamma \neq 1$, the WNGO analysis predicts shock formation and the approximate behaviour of $\inf_x v_x(x,t)$ near the shock is given by (22). In section 3 we confirm numerically all these predictions.

3. Numerics

In this section we present numerical results that validate the WNGO perturbation theory developed in section 2.

We solve numerically the system (1*a*)–(1*c*) on the interval [0, 1] with periodic boundary conditions using a pseudospectral method and $N = 16384$ grid points. For all computations we use the γ law (2) with $\kappa = 0.4$, and either $\gamma = 1$ or $\gamma = 1.4$.

One-wave initial data. Consider initial data that is a perturbation of the constant state $(\rho_0, 0)^{\mathrm{T}}$ in the direction of the second eigendirection $\boldsymbol{r}_2 = (\rho_0, c_0)^{\mathrm{T}}$:

$$\begin{pmatrix}\rho\\ v\end{pmatrix}(x,0) = \begin{pmatrix}\rho_0\\ 0\end{pmatrix} + \epsilon \exp\left(-((x-0.3)/0.05)^2\right)\begin{pmatrix}\rho_0\\ c_0\end{pmatrix}, \tag{23}$$

with $\epsilon = 0.01$, $\rho_0 = 0.5$ and $c_0 = \sqrt{\kappa\gamma{\rho_0}^{\gamma-1}}$. We consider two values for γ:

1. $\gamma = 1$. We solve for ρ and v and examine the quantity

$$\log\left[-\inf_x v_x(x,t)\right]. \tag{24}$$

Figure 1 indicates that (24) is very close to a straight line, especially for t large.

Supposing that $\log[-\inf_x v_x(x,t)] = a_1 t + a_0$ for positive constants a_1 and a_0, we find that

$$\inf_x v_x(x,t) = -\exp(a_1 t + a_0).$$

This confirms the WNGO results for $\gamma = 1$ (see (20)), which hold that v steepens exponentially fast, but for any finite t, the slope v_x remains finite.

2. $\gamma = 1.4$. Again solving for ρ and v, we examine the quantity

$$q_1(t) = \frac{1}{\inf_x v_x(x,t)}. \tag{25}$$

A plot of $q_1(t)$ versus t is presented in figure 2. From the plot, it is clear that $q_1(t)$ is concave and that, roughly speaking, it depends quadratically on t, consistent with the WNGO results—see (22). To make a closer comparison with WNGO theory, we solve

$$v_\tau + c_0 u v_x + c_0 \frac{\gamma-1}{2} v v_x = 0, \tag{26a}$$

$$v(x,0) = \exp(-((x-0.3)/0.05)^2). \tag{26b}$$

This scalar PDE is simply (15), where we have ignored translation terms given by constant multiples of v_x. Note that the initial data in (26a) and (26b) is chosen to be the function $v(x,0)$ from (23) rescaled by a factor of $(\epsilon c_0)^{-1}$. Having computed numerically the solution $v(x,\tau)$ of (26a) and (26b), we examine the quantity

$$q_2(\tau) = \frac{1}{\epsilon c_0 \inf_x v_x(x,\tau)}. \tag{27}$$

Recall that we also have an algebraic expression (see (22)) for the blow-up behaviour of solutions of (21). Analogously, one can derive such a formula for solutions of (26a) and obtain

$$\frac{1}{\inf_x v_x(x,\tau)} \approx \frac{c_0}{2}(\gamma-1)(\tau-\epsilon T) - \frac{c_0^2}{4}(\gamma-1)C(\tau-\epsilon T)^2,$$

where C is a constant of integration of order $1/\alpha$ and ϵT is the blow-up time. Here we rescaled the blow-up time T by ϵ to account for the slow time variable $\tau = \epsilon t$ present in (26a).

To compare this expression with $q_1(t)$, we must rescale by $(\epsilon c_0)^{-1}$, just as we did for (27). This gives the following algebraic approximation for the blow-up:

$$q_3(\tau) = \frac{1}{2\epsilon}(\gamma-1)(\tau-\epsilon T) - \frac{c_0}{4\epsilon}(\gamma-1)C(\tau-\epsilon T)^2. \tag{28}$$

A nonlinear least-squares optimizer is used to find parameters C and T that minimize the L^2 distance between $q_3(\epsilon t)$ from (28) and $q_1(t)$ from (25). To two decimal places, we find that $C = 18.77$ and $T = 20.11$.

Now we compare the three functions q_1, q_2 and q_3 that were obtained from the numerical results for the original system (1a)–(1c), the numerics for the scalar PDE (26a) yielded by WNGO and the algebraic expression for the blow-up in (26a), respectively. The three curves are plotted in figure 2.

To quantify the differences between the three curves, we compute the relative L^2 errors between $q_1(t)$ and, respectively, $q_2(\epsilon t)$ and $q_3(\epsilon t)$. We find that

$$\frac{\|q_1(t) - q_2(\epsilon t)\|_{L^2}}{\|q_1(t)\|_{L^2}} = 9.11 \times 10^{-3}$$

and

$$\frac{\|q_1(t) - q_3(\epsilon t)\|_{L^2}}{\|q_1(t)\|_{L^2}} = 1.35 \times 10^{-3}.$$

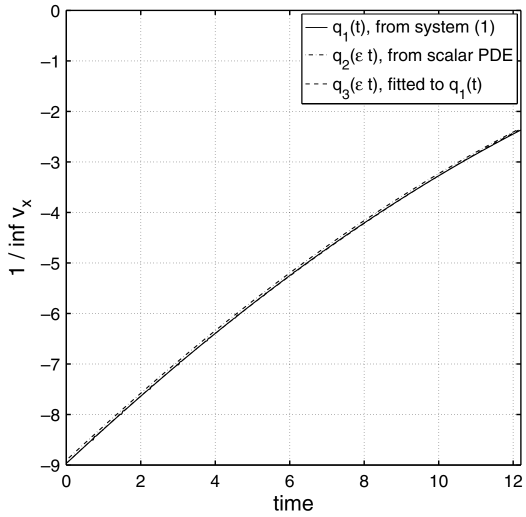

Figure 2. Close agreement of $1/\inf v_x$ for a solution of (1*a*)–(1*c*) with $\gamma = 1.4$, a solution of the scalar PDE (26*a*) and the algebraic expression (28) for the blow-up in (26*a*). We used the one-wave initial data (23) for (1*a*)–(1*c*). The corresponding initial data for (26*a*) is given by (26*b*).

These numbers are consistent with the fact that in figure 2, the three curves are nearly indistinguishable. The only discernable difference one would expect from the L^2 errors is that $q_2(\epsilon t)$ should be about 10 times further away from $q_1(t)$ than $q_3(\epsilon t)$ is from $q_1(t)$. This can be seen by examining figure 2 closely.

Overall, figure 2 shows excellent agreement between numerics and WNGO theory.

Two-wave initial data. Next we take initial data that has components in both eigendirections:

$$\begin{pmatrix}\rho\\ v\end{pmatrix}(x,0) = \begin{pmatrix}\rho_0\\ 0\end{pmatrix} + \epsilon\begin{pmatrix}\exp(-((x-0.4)/0.05)^2)\\ \exp(-((x-0.3)/0.05)^2)\end{pmatrix}, \qquad (29)$$

with $\epsilon = 0.01$ and $\rho_0 = 0.5$. We again consider two values of γ, and construct plots just as in the one-wave case.

1. $\gamma = 1$.
   Because we chose the initial data to have components in both eigendirections, the time evolution of the solution features wave interactions. In fact, each interaction corresponds to a spike in the plot shown in figure 3, where again we have plotted $\log[-\inf_x v_x(x,t)]$ as in (24). Having explained the reason for the spikes, we may focus on the overall behaviour of the underlying curve, which compares quite well with the curve shown in figure 1.
   This time, the underlying behaviour appears to consist of a long-period, small-amplitude oscillation about a straight line. That is to say, in the two-wave case, the numerics suggest that

   $$\inf_x v_x(x,t) = -\exp(a_1 t + a_0 + \text{small-amplitude, slow, periodic function of } t).$$

   Once again this supports the WNGO findings that v_x does not blow up in finite time, but that v does steepen exponentially fast.
2. $\gamma = 1.4$.
   The spikes present in figure 4 are again due to wave interactions, which were not present in the one-wave case shown in figure 2. Behind these spikes is a curve indicating that in

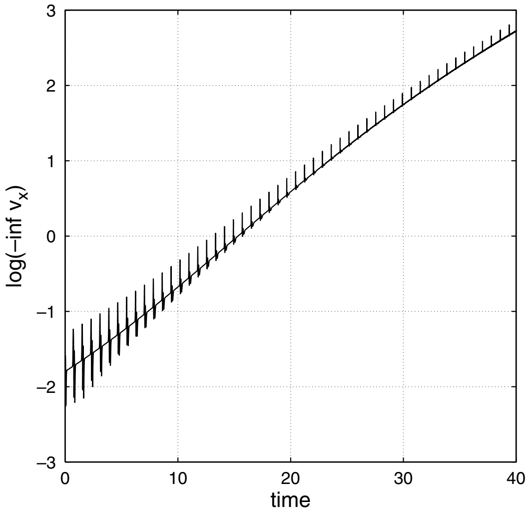

Figure 3. $\log[-\inf_x v_x(x,t)]$ versus t for (1a)–(1c) with $\gamma = 1.0$ and two-wave initial data (29).

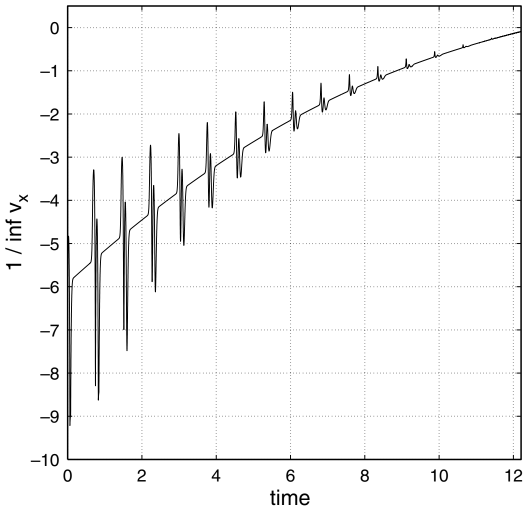

Figure 4. $1/\inf v_x(x,t)$ versus t for (1a)–(1c) with $\gamma = 1.4$ and two-wave initial data (29).

the two-wave case, $1/\inf v_x$ has the same concavity and near-quadratic dependence on t that we saw in the one-wave case. This indicates a finite time blow-up, again confirming the predictions of WNGO theory.

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