A LINEAR TIME ALGORITHM FOR THE 3-NEIGHBOUR TRAVELLING SALESMAN PROBLEM ON HALIN GRAPHS AND EXTENSIONS

BRAD WOODS, ABRAHAM PUNNEN, AND TAMON STEPHEN

Abstract. The Quadratic Travelling Salesman Problem (QTSP) is to find a least cost Hamilton cycle in an edge-weighted graph, where costs are defined for all pairs of edges contained in the Hamilton cycle. The problem is shown to be strongly NP-hard on a Halin graph. We also consider a variation of the QTSP, called the k-neighbour TSP (TSP(k)). Two edges e and f, $e \neq f$, are *k-neighbours* on a tour τ if and only if a shortest path (with respect to the number of edges) between e and f along τ and containing both e and f, has exactly k edges, for $k \geq 2$. In (TSP(k)), a fixed nonzero cost is considered for a pair of distinct edges in the cost of a tour τ only when the edges are p-neighbours on τ for $2 \leq p \leq k$. We give a linear time algorithm to solve TSP(k) on a Halin graph for $k = 3$, extending existing algorithms for the cases $k = 1, 2$. Our algorithm can be extended further to solve TSP(k) in polynomial time on a Halin graph with n nodes when $k = O(\log n)$. The possibility of extending our results to some fully reducible class of graphs are also discussed. TSP(k) can be used to model various machine scheduling problems as well as an optimal routing problem for unmanned aerial vehicles (UAVs).

1. Introduction

The Travelling Salesman Problem (TSP) is to find a least cost Hamilton cycle in an edge weighted graph. It is one of the most widely studied combinatorial optimization problems and is well-known to be NP-hard. The TSP model has been used in a wide variety of applications. For details we refer the reader to the well-known books [2, 8, 18, 23, 27] as well as the papers [4, 13, 21, 22].

For some applications, more than linear combinations of distances between consecutive nodes are desirable in formulating an objective function. Consider the problem of determining an optimal routing of an unmanned aerial vehicle (UAV) which has a list of targets at specific locations. This can be modelled as a TSP which requires a tour that minimizes the distance travelled. However, such a model neglects to take into account the physical limitations of the vehicle, such as turn radius or momentum. To illustrate this idea, in Figure 1 we give a Hamilton path, in Figure 2 we give the corresponding flight path, and Figure 3 shows a route which is longer but can be travelled at a greater speed and hence reducing the overall travel time. To model the traversal time, we can introduce penalties for pairs of (not necessarily adjacent) edges to force a smooth curve for its traversal. In this paper we consider a generalization of the TSP which can be used to model similar situations and contains many variations of the TSP, such as the angular-metric TSP [1], Dubins TSP [24] and the TSP [5] as special cases.

Let $G = (V, E)$ be an undirected graph on the node set $V = \{0, 1, \ldots, n-1\}$ with the convention that all indices used hereafter are taken modulo n. For each edge $(i, j) \in E$ a nonnegative cost c_{ij} is given. Let be $\mathcal{F}$ the set of all tours (Hamiltonian cycles) in G and let $\tau = (v_0, v_1, \ldots, v_{n-1}, v_0) \in \mathcal{F}$. The edges

Date: March 21, 2017.
This work was supported by NSERC discovery grants awarded to Abraham P. Punnen and Tamon Stephen.
Brad Woods, Department of Mathematics, Simon Fraser University, British Columbia, Canada. `bdw2@sfu.ca`.
Abraham Punnen, Department of Mathematics, Simon Fraser University, British Columbia, Canada. `apunnen@sfu.ca`.
Tamon Stephen, Department of Mathematics, Simon Fraser University, British Columbia, Canada. `tamon@sfu.ca`.

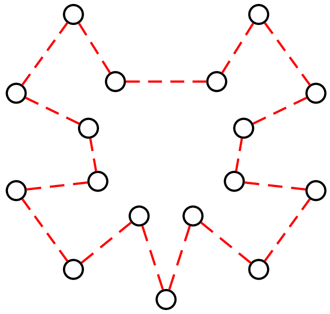

FIGURE 1. Optimal TSP tour with respect to length.

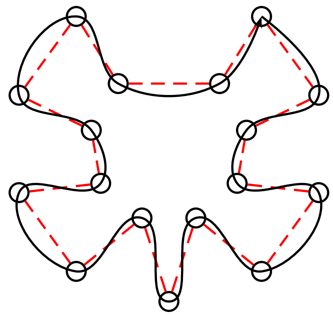

FIGURE 2. Smoothing of the tour.

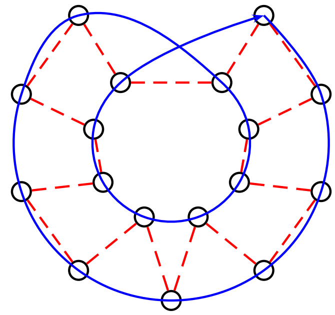

FIGURE 3. A tour which can be travelled more quickly.

$e = (v_i, v_{i+1})$ and $f = (v_j, v_{j+1})$, $e \neq f$, are *k-neighbours* on τ, if and only if a shortest path between e and f on τ containing these edges has exactly k edges, for $k \geq 2$. Here the shortest path refers to the path with the least number of edges, rather than the minimum cost path. Thus e and f are 2-neighbours in τ if and only if they share a common node in τ.

Let $q(e,f)$ be the cost of the pair (e,f) of edges and $\delta(k,\tau) = \{(e,f) : e,f \in \tau \text{ and } e \text{ and } f \text{ are } p\text{-neighbours on } \tau \text{ for some } 2 \leq p \leq k\}$. Assume that $q(e,f) = q(f,e)$ for every pair of edges $e,f \in E$. Then the *k-neighbour TSP* (TSP(k)) is defined as in [31]

$$\begin{aligned} TSP(k): \quad & \text{Minimize} \quad \sum_{(e,f)\in\delta(k,\tau)} q(e,f) + \sum_{e\in\tau} c(e) \\ & \text{Subject to} \quad \tau \in \mathcal{F}. \end{aligned}$$

A closely related problem, the Quadratic TSP (QTSP), is defined as follows:

$$\begin{aligned} QTSP: \quad & \text{Minimize} \quad \sum_{(e,f)\in\tau\otimes\tau} q(e,f) + \sum_{e\in\tau} c(e) \\ & \text{Subject to} \quad \tau \in \mathcal{F}. \end{aligned}$$

where $\tau \otimes \tau = \tau \times \tau \setminus \{(e,e) : e \in \tau\}$. Note:

$$\tau \otimes \tau = \begin{cases} \delta((n+2)/2, \tau) & \text{if } n \text{ is even} \\ \delta((n+1)/2, \tau) & \text{if } n \text{ is odd.} \end{cases}$$

Thus when $k \geq n/2$ (for n even) or $k \geq (n+1)/2$ (for n odd), the k-neighbour TSP reduces to the Quadratic TSP [33]. Define TSP(1) to be the original TSP. Elsewhere in the literature (e.g. [17], [14]), the term Quadratic TSP is sometimes used for what we refer to as TSP(2). That is, quadratic terms are allowed, but only for pairs of edges that share a node.

The bottleneck version of TSP(k) was introduced by Arkin et al. in [3], denoted as the k-neighbour maximum scatter TSP. Jäger and Molitor [19] encountered TSP(2) while studying the Permuted Variable

Length Markov model. Several heuristics are proposed and compared in [16, 19] as well as a branch and bound algorithm for TSP(2) in [16]. A column generation approach to solve TSP(2) is given in [28], lower bounding procedures discussed in [29], and polyhedral results were reported by Fischer and Helmberg [17], Fischer [14], and Fischer and Fischer [15]. The k-neighbour TSP is also related to the k-peripatetic salesman problem [12, 20] and the watchman problem [7]. Algorithms for maximization and minimization versions of TSP(2) was studied by Staněk [30]. To the best of our knowledge, no other works in the literature address TSP(k).

Referring to the UAV example discussed earlier, it is clear that the flight subpaths depend on both the angle and distances between successive nodes. By precalculating these and assigning costs to $q(e, f)$, for $e, f \in E$, we see that QTSP is a natural model for this problem. In fact, the flight paths may be affected by edges further downstream. Thus we can get successively better models by considering TSP(1), TSP(2),. . ., TSP(k) in turn. In practice we expect diminishing returns to take hold quickly and hence TSP(k) with small values of k are of particular interest.

In this paper we show that QTSP is NP-hard even if the costs are restricted to 0-1 values and the underlying graph is Halin. In contrast, TSP and TSP(2) on a Halin graph can be solved in $O(n)$ time [9, 33]. Interestingly, we show that TSP(3) can also be solved on a Halin graph in $O(n)$ time, although as we move from TSP(2) to TSP(3), the problem gets much more complicated. In fact, our approach can be extended to obtain polynomial time algorithms for TSP(k) whenever $k = O(\log n)$. We note that while Halin graphs have treewidth 3, the results on graphs with bounded treewidth (e.g. [6, 11]) cannot easily be extended to optimization problems with quadratic objective functions.

The paper is organized as follows. In Section 2 we introduce some preliminary results and notations for the problem. The complexity result for QTSP on Halin graphs is given in Section 3. An $O(n)$ algorithm to solve TSP(3) on Halin graphs is given in Section 4.1, which can be extended to obtain an $O(n2^{(k-1)/2})$ algorithm for TSP(k). Further extensions of this result to fully reducible classes of graphs are briefly discussed in Section 5.

An earlier version of the NP-completeness results presented here were included as part of the M.Sc. thesis of the first author [31].

2. Notations and definitions

A Halin graph $H = T \cup C$ is obtained by embedding a tree with no nodes of degree two in the plane and connecting the leaf nodes of T in a cycle C so that the resulting graph remains planar. Unless otherwise stated, we always assume that a Halin graph or its subgraphs are given in the planar embedded form. The non-leaf nodes belonging to T are referred to as *tree* or *internal* nodes and the nodes in C are referred to as *cycle* or *outer* nodes of H. A Halin graph with exactly one internal node is called a *wheel*. If H has at least two internal nodes and w is an internal node of T which is adjacent to exactly one other internal node, then w is adjacent to a set of consecutive nodes of C, which we denote by $C(w)$. Note that $|C(w)| \geq 2$. The subgraph of H induced by $\{w\} \cup C(w)$ is referred to as a *fan*, and we call w the centre of the fan. See Figure 4.

Lemma 2.1 (Cornuejols et al. [9]). *Every Halin graph which is not a wheel has at least two fans.*

Let $G = (V, E)$ be a graph and let $S \subseteq V$ be a connected subgraph of G. Let $\varphi(S)$ be the cutset of S, that is, the smallest set of edges whose removal disconnects S from the vertices in $V \setminus S$. Let G/S be the

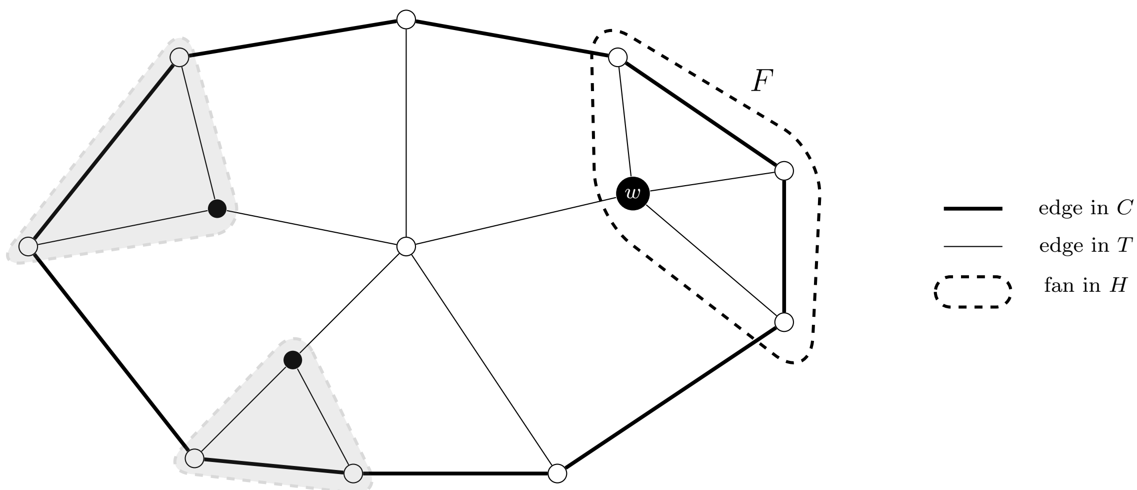

FIGURE 4. A Halin graph H with 3 fans. w is the centre of fan F.

graph obtained by contracting S into a single node, called a 'pseudonode' and is denoted by v_S [9]. The edges in G/S are obtained as follows:

(1) An edge with both ends in S is deleted;
(2) An edge with both ends in $G - S$ remains unchanged;
(3) An edge (v_1, v_2) with $v_1 \in G - S$, $v_2 \in S$ is replaced by the edge (v_1, v_s).

Lemma 2.2 (Cornuejols et al. [9])**.** *If F is a fan in a Halin graph H, then H/F is a Halin graph.*

Note that each time a fan F is contracted using the graph operation H/F, the number of non-leaf nodes of the underlying tree is reduced by one. That is, after at most $\lceil (n-1)/2 \rceil$ fan contractions, a Halin graph will be reduced to a wheel.

Let w be the centre of a fan F, and label the outer nodes in F in the order they appear in C as, $u_1, u_2, \ldots, u_r$ $(r \geq 2)$. Let (j, k, l) be the 3-edge cutset $\varphi(F)$ which disconnect F from G such that j is adjacent to u_1, k is adjacent to w but not adjacent to u_i for any i, $1 \leq i \leq r$, and l is adjacent to u_r (See Figure 5, $r = 4$).

Note that every Hamiltonian cycle τ in H contains exactly two edges of $\{j, k, l\}$. The pair of edges chosen gives us a small number of possibilities for traversing F in a tour τ. For example, if τ uses k and l, it contains the subsequence $w, u_1, u_2, \ldots, u_r$ (call this a *left-traversal* of F), if τ uses j and k it contains the subsequence $u_1, u_2, \ldots u_r, w$ (call this a *right-traversal* of F) and if τ uses j and l, it contains a subsequence of the form $u_1, u_2, \ldots, u_i, w, u_{i+1}, \ldots, u_r$, for some $i \in \{1, 2, \ldots, r-1\}$ as it must detour through the centre of F (call this a *centre-traversal* of F).

3. Complexity of QTSP on Halin graphs

Many optimization problems that are NP-Hard on a general graph are solvable in polynomial time on a Halin graph [9, 25, 26]. In particular, TSP on a Halin graph is solvable in linear time. Unlike this special case, we show that QTSP is strongly NP-hard on Halin graphs. The decision version of QTSP on a Halin graph, denoted by RQTSP, can be stated as follows:

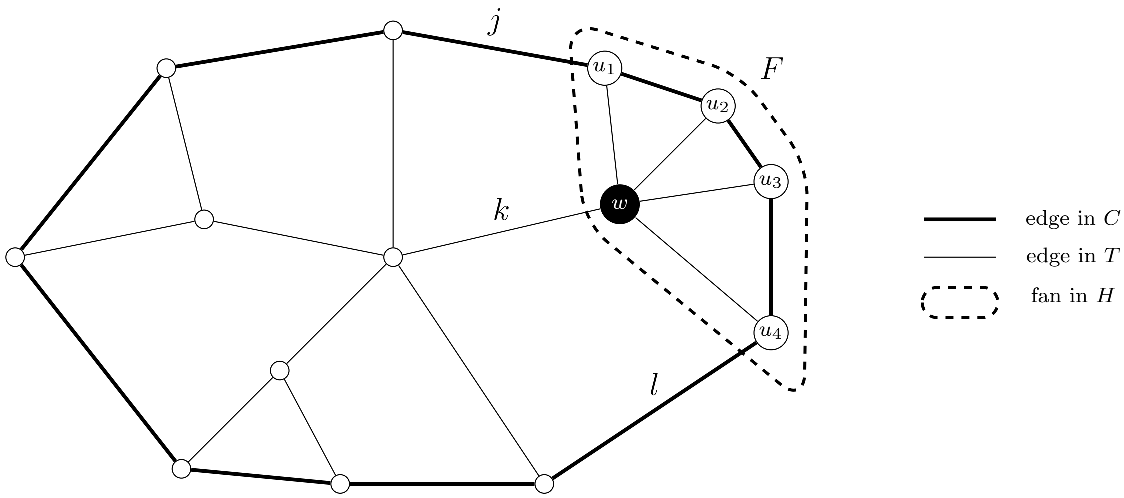

FIGURE 5. A Halin graph H containing fan F. $\{j,k,l\}$ is a 3-edge cutset which disconnects F from H.

"Given a Halin graph H and a constant θ, does there exist a tour τ in H such that $\sum_{e\in\tau} c(e) + \sum_{e,f\in\tau} q(e,f) \leq \theta$?"

Theorem 3.1 (Woods [31]). *RQTSP is NP-complete even if the values $c(e) \in \{0,1\}$ and $q(e,f) \in \{0,1\}$ for $e,f \in H$.*

Proof. RQTSP is clearly in NP. We now show that the 3-SAT problem can be reduced to RQTSP. The 3-SAT problem can be stated as follows: "Given a Boolean formula R in Conjunctive Normal Form (CNF) containing a finite number of clauses $C_1, C_2, \ldots, C_h$ on variables $x_1, x_2, \ldots, x_t$ such that each clause contains exactly three literals ($L_1, \ldots, L_{3h}$ where for each i, $L_i = x_j$ or $L_i = \neg x_j$ for some $1 \leq j \leq t$), does there exists a truth assignment such that R yields a value 'true'?"

From a given instance of 3-SAT, we will construct an instance of RQTSP. The basic building block of our construction is a 4-fan gadget obtained as follows. Embed a star on 5 nodes with center v and two specified nodes ℓ and r on the plane and add a path P from ℓ to r covering each of the pendant nodes so that the resulting graph is planar (see Figure 6). Call this special graph a 4-fan gadget.

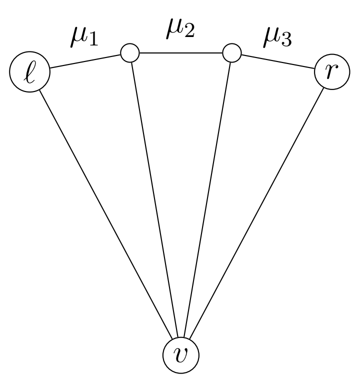

FIGURE 6. 4-fan gadget constructed by embedding a star on 5 nodes in the plane and adding a path.

The nodes on path P of this gadget are called *outer nodes* and edges on P are called *outer edges*. Let μ_1, μ_2, μ_3 be edges with distinct end points in P. Note that any ℓ-r Hamilton path of the gadget must contain all the outer edges except one which is skipped to detour through v. We will refer to an ℓ-r Hamilton path in a 4-fan gadget as a *center-traversal* as before.

We will construct a Halin graph H using one copy of the gadget for each clause and let μ_1, μ_2, μ_3 correspond to literals contained in that clause. We will assign costs to pairs of edges such that every Hamilton cycle with cost 0 must contain a centre-traversal for each clause. To relate a Hamiltonian cycle to a truth assignment, a centre-traversal which does not contain edge μ_i corresponds to an assignment of a *true* value to literal L_i.

Now construct H as follows. For each clause $C_1, \dots, C_h$, create a copy of the 4-fan gadget. The r, ℓ, and v nodes of the 4-fan gadget corresponding to the clause C_i are denoted by r_i, ℓ_i and v_i respectively. Connect the node r_i to the node ℓ_{i+1}, $i = 1, 2, \dots, h-1$. Introduce nodes v_x and v_y and the edges $(\ell_1, v_x), (v_x, v_y), (v_y, r_h)$. Also introduce a new node w and connect it to v_x, v_y and v_i for $i = 1, 2, \dots, h-1$. The resulting graph is the required Halin graph H. See Figure 7.

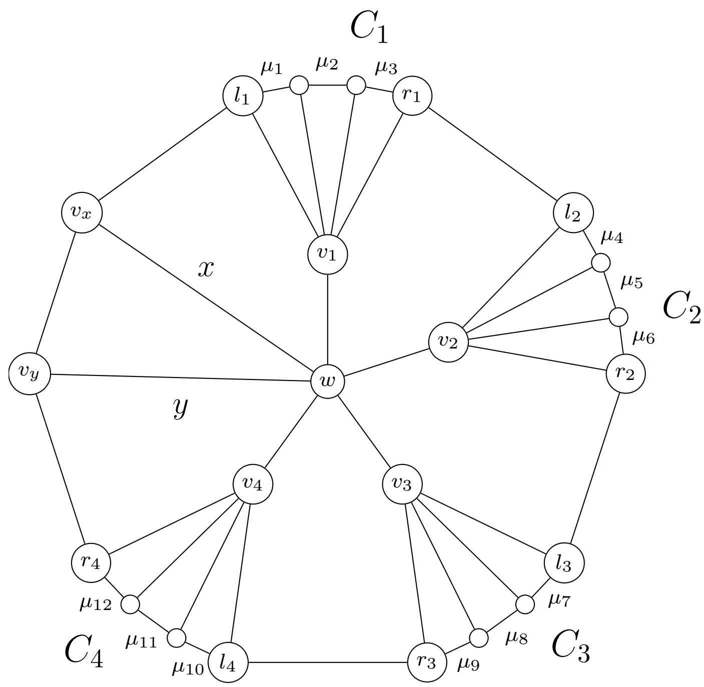

FIGURE 7. Example of the Halin graph constructed from $F = C_1 \wedge C_2 \wedge C_3 \wedge C_4$.

Assign the cost $c(e) = 0$ for every edge in H. Let $x = (v_x, w)$ and $y = (v_y, w)$. Note that every tour which contains edges x and y traverses every gadget using a centre-traversal. For each gadget: assign paired cost $q(e, f) = 1$ for pairs of edges which are neither outer edges nor both adjacent to the same literal edge μ_1, μ_2 or μ_3, and for all other pairs of edges within the gadget assign cost 0. For each variable x_j, $j = 1, \dots, n$, and all literals L_m, L_q $(m \neq q)$ if $x_j = L_m = \neg L_q$, assign cost $q(\mu'_m, \mu'_q) = 1$ where μ'_m and μ'_q are edges connecting μ_m (and μ_q) to the respective 4-fan gadget centre v. All other paired costs are assumed to be 0.

Suppose B is a valid truth assignment. Then in each clause there exists at least one true literal. Consider a tour τ in H which contains the edges x and y and traverses every gadget such that τ detours around exactly one literal edge which corresponds to a literal which is *true* in B. Since the truth assignment is valid, such a τ exists. Clearly τ has cost 0, since no costs are incurred by pairs of edges contained in a single gadget, nor are costs incurred of the form $q(\mu'_a, \mu'_b)$ where $L_a = \neg L_b$. The latter must be true because in any truth assignment, the variable corresponding to L_a, say x_a must be either assigned a value of *true* or *false*. Suppose a cost of 1 is incurred by $q(\mu'_a, \mu'_b)$ and hence $L_a = \neg L_b$. If x_a is *true*, and $x_a = L_a$, then L_b clearly must be *false*, so τ cannot detour to miss both μ'_a and μ'_b. The same contradiction arises, if x_a is *false*. Hence a *yes* instance of 3-SAT can be used to construct a *yes* instance for RQTSP with $\theta = 0$.

Now suppose there is a tour which solves RQTSP with $\theta = 0$. Suppose τ' is such a tour. Clearly it must use edges x and y, and hence must traverse every gadget via a centre-traversal. Such a detour must skip a literal edge in every gadget, otherwise a cost of 1 is incurred. Suppose $D = \{L_1, \ldots, L_s\}$ is the set of literals which are skipped. $L_i \neq \neg L_j$ for any i, j, otherwise a cost of 1 is incurred. This implies that a truth assignment which results in every literal in D being *true* is a valid truth assignment to the variables $x_1, \ldots, x_t$. That is, for each literal edge which is skipped in τ', assign *true* or *false* to the corresponding variable such that the literal evaluates to *true* (if $L_i = x_j$, set $x_j = true$ and if $L_i = \neg x_j$, set $x_j = false$). The truth values for any remaining variables can be assigned arbitrarily. This truth assignment returns *true* for each clause since exactly one literal in each clause is detoured, and evaluates to *true*. Hence this truth assignment is a valid assignment for 3-SAT. □ □

4. Complexity of k-neighbour TSP on Halin graphs

Let G be a planar embedding of a planar graph and e and f are two distinct edges of G. Then e and f are said to be *cofacial* if there exists a face of G which contains both e and f. This may include the *outer face*.

Theorem 4.1. *Let τ be a tour in the planar embedding $H = T \cup C$ of a Halin graph. Then, any two edges adjacent in τ must be cofacial.*

Proof. Suppose the result is not true. Then, there exists a tour τ in H containing two adjacent edges $e = (u, x), f = (x, v)$ such that e and f are not cofacial. From our previous discussion on fan traversals, we can assume that $x \notin C$. Since e and f are not cofacial at x, there exists edges $g = (y, x)$ and $h = (x, z)$ in H such that the clockwise ordering of edges incident on x is of the form $f, \ldots, g, \ldots, e, \ldots, h$ (See Figure 8). Without loss of generality, assume T is rooted at x. Then T has at least four subtrees T_u, T_v, T_y, T_z rooted respectively at u, v, y and z. Since τ is a tour containing the edge e, it must contain a path, say P_1, through the subtree T_u from u to $u_c \in C$. Note that u_c could be the same as u and in this case the subtree T_u is the isolated node u. Similarly, τ must contain a path P_2 in T_v from v to $v_c \in C$ (See Figure 8). Note that $P = P_1 \cup P_2 \cup \{e, f\}$ is a path in τ. Deleting the vertex set $V(P)$ of P and its incident edges from H yields a disconnected graph. Thus, $\tau - V(P)$ must be disconnected, a contradiction. □ □

A preliminary version of this result is given in [31].

A path in the planar embedding $H = T \cup C$ of a Halin graph called a *candidate paths* if its consecutive edges are cofacial. A candidate path with k-edges is called *candidate k-paths*. Note that only candidate paths can be subpaths of a tour but it is possible that there are candidate paths that are not part of a tour.

Corollary 4.2. *Let $H = T \cup C$ be a planar embedding of a Halin graph and e be a specified edge of H. Then, H has at most $k \cdot 2^{k-1}$ candidate k-paths containing e.*

Proof. Consider extending e, if necessary in either directions in H, to a candidate k-path P. For a given end point of a sub path of P, say vertex u, by Theorem 4.1, there are only two possible edges incident on

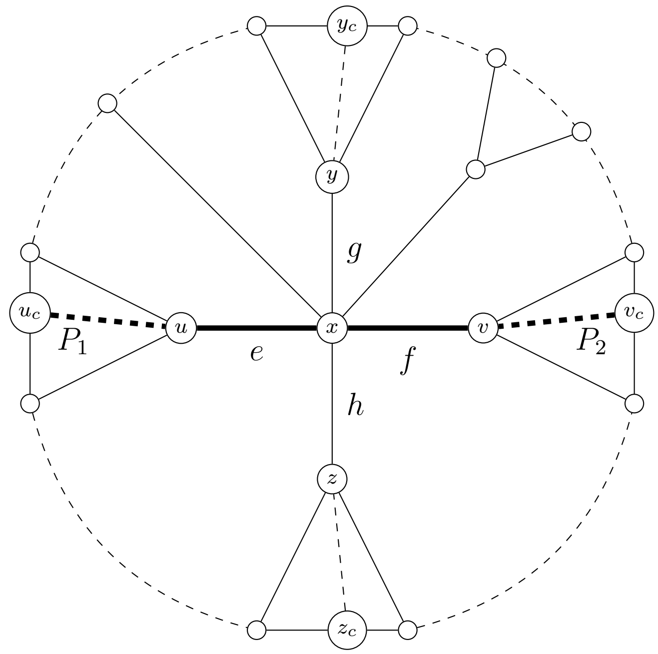

FIGURE 8. A Halin graph H with non-consecutive edges e and f at node x. $H - P$ where $P = P_1 \cup P_2 \cup \{e, f\}$ has two components, hence no τ contains both edges e and f.

u, which may belong to P. Thus, there are at most 2^{k-1} candidate k-paths when the position of e is fixed. Since e can take any of the k positions in a candidate k-path, there are at most $k \cdot 2^{k-1}$ candidate k-paths containing e. □ □

As an immediate consequence of Corollary 4.2, we have an upper bound of 2^{n-2} on the number of Hamiltonian cycles in a Halin graph. To see this, let r be the number of Hamiltonian cycles in H. From each such cycle, we can generate n distinct Hamiltonian paths (candidate $(n-1)$-paths) by ejecting an edge. Repeating this for all Hamiltonian cycles in H, we get rn candidate $(n-1)$-paths and all these paths are distinct. Hence rn is a lower bound on the number of candidate $(n-1)$-paths in H. By corollary 4.2, $(n-1)2^{n-2}$ is an upper bound on the number of candidate $(n-1)$-paths in H. Thus, $rn \leq (n-1)2^{n-2}$ and hence $r \leq \frac{n-1}{n} 2^{n-2} \leq 2^{n-2}$.

Noting that there are at most $2(n-1)$ edges in H and that the number of quadratic costs which are relevant is the number of candidate k-paths after applying 4.14, it follow from Corollary 4.2, that the number of quadratic costs which are relevant is bounded above by $k \cdot 2^k \cdot (n-1) = O(n)$ for any fixed k and $O(n^t)$ if $k \leq t \log n$.

Note that any face of H must contain an outer edge. Moreover, the following corollary will prove useful.

Corollary 4.3. *If H is embedded in the plane such that it is planar and C defines the outer face, for any outer edge e which is contained in the outer face and face F_e, every tour which does not contain e must contain all other edges of F_e.*

4.1. **TSP(3) on Halin graphs.** As indicated earlier, TSP(1) is the same as TSP, which is solvable in linear time on Halin graphs [9]. TSP(2) can also be solved in linear time by appropriate modifications of the algorithm of [26] as indicated in [33]. However, for $k \geq 3$, such modifications seem not a viable option. We now develop a linear time algorithm to solve TSP(3).

Let us start with an alternative formulation of TSP(3). For any subgraph G of H, let $P_3(G)$ be the collection of all distinct candidate 3-paths in G. For each candidate 3-path $(e, f, g) \in P_3(H)$, define

$$q(e,f,g) = q(e,g) + \frac{q(e,f)+q(f,g)}{2} + \frac{c(e)+c(f)+c(g)}{3}. \tag{4.1}$$

Now consider the simplified problem:

$$\begin{aligned} STSP(3): \quad & \text{Minimize} \quad \sum_{(e,f,g)\in P_3(\tau)} q(e,f,g) \\ & \text{Subject to} \quad \tau \in \mathcal{F}. \end{aligned}$$

Theorem 4.4. *Any optimal solution to the STSP(3) is also optimal solution to TSP(3).*

Proof. For any $\tau \in \mathcal{F}$,

$$\begin{aligned} \sum_{(e,f,g)\in P_3(\tau)} q(e,f,g) &= \sum_{(e,f,g)\in P_3(\tau)} \left(q(e,g) + \frac{q(e,f)+q(f,g)}{2} + \frac{c(e)+c(f)+c(g)}{3} \right) \\ &= \sum_{(e,f)\in\delta(3,\tau)} q(e,f) + \sum_{e\in\delta(\tau)} c(e). \end{aligned}$$

Thus, the objective function values of STSP(3) and TSP(3) are identical for identical solutions. Since the family of feasible solutions of both these problems are the same, the result follows. □ □

In view of Theorem 4.4, we restrict our attention to STSP(3).

For TSP(1), Cornuejols et al. [9] identified costs of new edges generated by a fan contraction operation by solving a linear system of equations. This approach cannot be extended for any k-neighbour TSP for $k \geq 3$ as it leads to an over-determined system of equations which may be infeasible. Instead, we extend the penalty approach used in Phillips et al. [26]. The idea here is to introduce a node-weighted version of the problem STSP(3) where we use a penalty function for the nodes of C, the value of which depends on the edges chosen to enter and exit the node, along with some other 'candidate' edges. We iteratively contract the fans in H, storing the appropriate values of suitable subpaths as we traverse the fans in a recursive way. Once we reach a wheel, we can compute an optimal tour for the resulting problem. Backtracking by recovering appropriate subpaths from contracted fans in sequence, an optimal solution can be identified.

To formalize the general idea discussed above, let us first discuss the case where H is a Halin graph which is not a wheel. In this case, H will have at least two fans.

Let F be an arbitrary fan in H with w as the centre. Label the outer nodes of F in the order they appear in C, say, $u_1, u_2, \ldots, u_r$ $(r \geq 2)$. Let $\{j, k, l\}$ be the 3-edge cutset $\varphi(F)$ which disconnects F from H such that j is adjacent to u_1, k is adjacent to w and l is adjacent to u_r. Let $j = (u_1, u_0)$, $k = (w, x)$ and $l = (u_r, u_{r+1})$. There are exactly two edges not connected to F which are cofacial with k and incident on x. The first edge which follows k in the clockwise orientation of edges incident on x is denoted α_5, and the other edge incident on x and cofacial with k is denoted by α_6. (See Figure 9.) There are exactly two edges not connected to F and incident on u_0. These edges are labelled α_1, α_2. Likewise, there are exactly two edges not connected to F and incident on u_{r+1}. These edges are labelled α_3, α_4. (See Figure 9.) Without loss of generality α_1, α_3 are in C and α_2, α_4 are in T. It is possible that α_2 could be the same as α_5 and also possible that α_4 could be the same as α_6.

To complete a fan contraction operation, we consider the 3 types of traversals of F. We define a penalty function stored at nodes (pseudonodes) of C which contains attributes of a minimum traversal of F of each type. For any left- or right-traversal of F, there is a single path through F using all cycle edges. Any tour which includes j and k must pass through one edge of the pair incident on u_0 lying outside F together with edges $y_1, \ldots, y_{r-1}, t_r, k, \alpha_6$. Similarly, any tour which includes k and l must pass through one edge of the pair incident on u_{r+1} lying outside F together with edges $y_{r-1}, \ldots, y_1, t_1, k, \alpha_5$. Any tour which includes j and l must also pass through one edge in each of the pairs of edges incident on u_0 (or u_{r+1}) lying outside F. That is, every tour τ containing j and l must contain a path containing one collection of edges from the set $\{(\alpha_1, j, l, \alpha_3), (\alpha_1, j, l, \alpha_4), (\alpha_2, j, l, \alpha_3), (\alpha_2, j, l, \alpha_4)\}$. We refer to a centre-traversal of F which bypasses $y_1 \in F \cap C$ as a left path, one which bypasses $y_s \in F \cap C$ for some $s \in [2, r-2]$ as a middle path, and one which bypasses $y_{r-1} \in F \cap C$ as a right path.

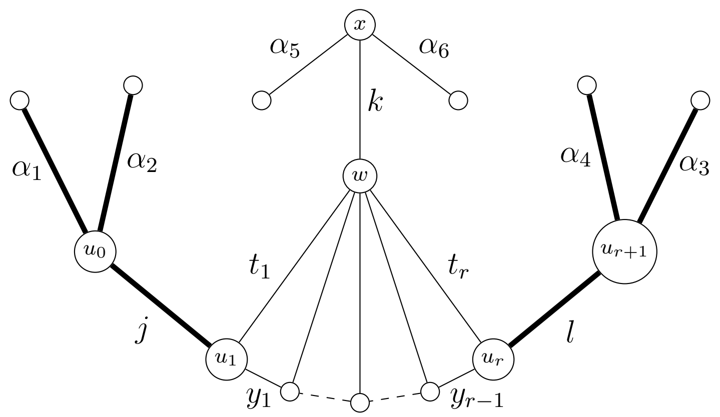

FIGURE 9. A fan F with centre w. Every τ containing j and k contains edges $y_1, \ldots, y_{r-1}, t_r$. Every τ containing k and l contains edges $y_1, \ldots, y_{r-1}, t_1$. Every τ containing j and l must contain one of the subpaths from the set $\{\alpha_1 - j - l - \alpha_3, \alpha_1 - j - l - \alpha_4, \alpha_2 - j - l - \alpha_3, \alpha_2 - j - l - \alpha_4\}$.

Let $\mathcal{S}$ be the set of nodes (pseudonodes) in C at some iteration in the contraction process. In TSP(3), a number of quadratic costs are 'absorbed' during each fan contraction operation, depending both on the edges in F and within a distance 2 from F, so care must be taken to retain the proper information. That is, in order to develop an extension of the penalty approach used in Phillips et al. [26] for TSP(3), we extend the penalty function stored at the nodes (pseudonodes) in C that depends on an additional parameter ρ, which specifies the structure of edges around each pseudonode. Note that due to the recursive property of pseudonodes where a fan contraction operation may 'absorb' pseudonodes, the parameter ρ specifies a shape of the structure rather than explicitly stating the edges surrounding a pseudonode. Further, we define a function β which stores the penalty values associated with ρ.

Let A^1 be the collection of ordered pairs $\{M = (0,0), L = (1,0), R = (0,1), B = (1,1)\}$ and A^2 be the collection of ordered pairs $\{(1,3), (1,4), (1,6), (2,3), (2,4), (2,6), (5,3), (5,4)\}$. At each node $i \in \mathcal{S}$ we define $\rho_i = (\rho_i^1, \rho_i^2)$ where $\rho_i^1 \in A^1$ and $\rho_i^2 \in A^2$. ρ_i indicates which penalty value (to be defined shortly) stored at pseudonode i is to contribute to the objective function value. The first component of ρ_i is a binary vector of length 2 which specifies the inner structure of i (edges y_1, t_1, y_{r-1} and t_r prior to any fan contractions such that the first component is 0 if y_1 is selected, and 1 if t_1 is instead, and the second component is 0 if y_{r-1} is selected, and 1 if t_r is instead), and the second component of ρ_i, the outer structure (α_1 to α_6 prior to any contraction of adjacent pseudonodes). Let ρ be the vector containing ρ_i for every $i \in \mathcal{S}$. Let $\rho_{H/F}$ be the restriction of ρ to the vertices in H/F and augmented by $\rho_{v_F} \in \{(a,b)\}$. Let $\beta_i(\rho_i)$ be the penalty that is incurred if ρ_i occurs at i.

For $i \in C$ define:

$$\mathcal{P}_i(\tau, \rho_i) = \begin{cases} \beta_i(\rho_i) & \text{if } \rho_i \text{ is defined,} \\ 0 & \text{otherwise.} \end{cases}$$

Since the penalties stored at $i \in \mathcal{S}$ depend on edges which are not incident with i, and the dependent edges may be 'absorbed' into adjacent pseudonodes so that not every ρ is feasible for a given τ. That is, the inner and outer structures of adjacent pseudonodes in C must agree. Formally, we say that ρ is feasible for τ if the following conditions are satisfied for every pseudonode i

(1) $j,k \in \tau \iff \rho_i^{22} = 6$ and
(2) $k,l \in \tau \iff \rho_i^{21} = 5$,

and for every pair of consecutive pseudonodes $i, i+1 \in C$

(1) $\rho_i^{12} = 1 \iff \rho_{i+1}^{21} = 2$,
(2) $\rho_i^{12} = 0 \iff \rho_{i+1}^{21} = 1$,
(3) $\rho_i^{22} = 3 \iff \rho_{i+1}^{11} = 0$ and
(4) $\rho_i^{22} = 4 \iff \rho_{i+1}^{11} = 1$.

Let $\mathcal{F}'$ be the set of all feasible (τ, ρ) pairs. For an example of a feasible (τ, ρ) pair, see Figure 10.

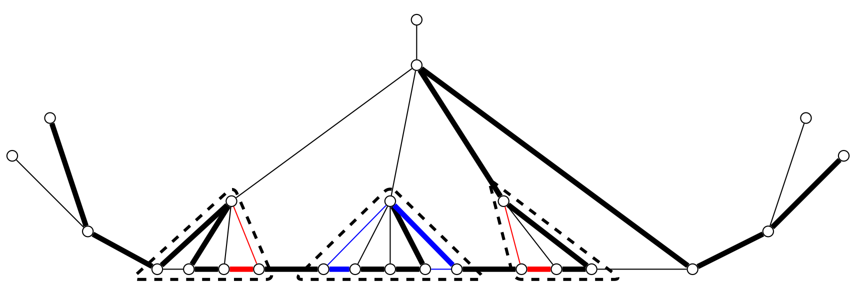

FIGURE 10. A subgraph of H which becomes a fan after contracting F_1, F_2 and F_3. The bold edges depict a centre-traversal of F contained in Hamilton cycle τ. A (τ, ρ) pair containing $\rho_{v_{F_1}} = ((1,0),(2,3))$, $\rho_{v_{F_2}} = ((0,1),(1,3))$ and $\rho_{v_{F_3}} = ((0,0),(2,6))$ is feasible. The edges which correspond to the inner structure of $\rho_{v_{F_2}}$ are coloured in blue and the edges which correspond to the outer structure of $\rho_{v_{F_2}}$ are coloured in red.

The problem now contains a cost for every triplet of consecutive edges in tour τ and additionally, a penalty at each outer node i. Consider the modified 3-Neighbour TSP on a Halin graph defined as follows:

$$MTSP(3): \quad \text{Minimize} \quad z(\tau,\rho) = \sum_{(e,f,g) \in P_3(\tau)} q(e,f,g) + \sum_{i \in C} \mathcal{P}_i(\tau, \rho_i)$$
$$\text{Subject to} \quad (\tau, \rho) \in \mathcal{F}'.$$

The necessary costs to construct MTSP(3) can be obtained as required by applying 4.1 or by first embedding H in the plane, and evaluating the candidate k-paths from Corollary 4.2. Note that $\mathcal{P}_v(\tau, \rho_v)$ can be computed in $O(1)$ time by storing penalty 24-tuples containing the β_v-values described in table 1, at each cycle node v. Also note that there may be $O(2^n)$ feasible ρ vectors for a given τ, however, we show that the

	Penalty	Description
1	$\beta_{v_F}((0,*),(1,6))$	traversal of F with inner structure $(0,*)$ and outer structure $(1,6)$
2	$\beta_{v_F}((1,*),(1,6))$	traversal of F with inner structure $(1,*)$ and outer structure $(1,6)$
3	$\beta_{v_F}((0,*),(2,6))$	traversal of F with inner structure $(0,*)$ and outer structure $(2,6)$
4	$\beta_{v_F}((1,*),(2,6))$	traversal of F with inner structure $(1,*)$ and outer structure $(2,6)$
5	$\beta_{v_F}((*,0),(5,3))$	traversal of F with inner structure $(0,*)$ and outer structure $(5,3)$
6	$\beta_{v_F}((*,1),(5,3))$	traversal of F with inner structure $(1,*)$ and outer structure $(5,3)$
7	$\beta_{v_F}((*,0),(5,4))$	traversal of F with inner structure $(0,*)$ and outer structure $(5,4)$
8	$\beta_{v_F}((*,1),(5,4))$	traversal of F with inner structure $(1,*)$ and outer structure $(5,4)$
9	$\beta_{v_F}((0,0),(1,3))$	traversal of F with inner structure $(0,0)$ and outer structure $(1,3)$
10	$\beta_{v_F}((0,1),(1,3))$	traversal of F with inner structure $(0,1)$ and outer structure $(1,3)$
11	$\beta_{v_F}((1,0),(1,3))$	traversal of F with inner structure $(1,0)$ and outer structure $(1,3)$
12	$\beta_{v_F}((1,1),(1,3))$	traversal of F with inner structure $(1,1)$ and outer structure $(1,3)$
13	$\beta_{v_F}((0,0),(1,4))$	traversal of F with inner structure $(0,0)$ and outer structure $(1,4)$
14	$\beta_{v_F}((0,1),(1,4))$	traversal of F with inner structure $(0,1)$ and outer structure $(1,4)$
15	$\beta_{v_F}((1,0),(1,4))$	traversal of F with inner structure $(1,0)$ and outer structure $(1,4)$
16	$\beta_{v_F}((1,1),(1,4))$	traversal of F with inner structure $(1,1)$ and outer structure $(1,4)$
17	$\beta_{v_F}((0,0),(2,3))$	traversal of F with inner structure $(0,0)$ and outer structure $(2,3)$
18	$\beta_{v_F}((0,1),(2,3))$	traversal of F with inner structure $(0,1)$ and outer structure $(2,3)$
19	$\beta_{v_F}((1,0),(2,3))$	traversal of F with inner structure $(1,0)$ and outer structure $(2,3)$
20	$\beta_{v_F}((1,1),(2,3))$	traversal of F with inner structure $(1,1)$ and outer structure $(2,3)$
21	$\beta_{v_F}((0,0),(2,4))$	traversal of F with inner structure $(0,0)$ and outer structure $(2,4)$
22	$\beta_{v_F}((0,1),(2,4))$	traversal of F with inner structure $(0,1)$ and outer structure $(2,4)$
23	$\beta_{v_F}((1,0),(2,4))$	traversal of F with inner structure $(1,0)$ and outer structure $(2,4)$
24	$\beta_{v_F}((1,1),(2,4))$	traversal of F with inner structure $(1,1)$ and outer structure $(2,4)$

TABLE 1. Description of penalty 24-tuple stored at pseudonodes in C.

optimal (τ,ρ) pair can be found in $O(n)$-time. It is also important to note that the set of pseudonodes is retained for reasons which will become apparent.

In the initial graph, and for all ρ_i, $i\in C$, set $\beta_i(\rho_i)=0$. For fan F in H, the penalties must be updated to store the costs of traversing F when F is contracted to pseudonode v_F. Let K represent the traversal of F which contains only edges in C. That is, $K=j-y_1-\cdots-y_{r-1}-l$. Then $q(K)=\sum_{e-f-g\in K}q(e,f,g)$ represents the cost incurred by selecting the edges in K. Let $\tau(F)$ and $\rho(F)$ be the restrictions of τ and ρ to F, respectively.

Assign the minimum cost of the right-traversal (which contains α_s, $s\in\{1,2\}$ and α_6), with inner structure of the first pseudonode $a\in\{L=(1,0),M=(0,0)\}$ to $\beta_{v_F}(a,(s,6))$. That is, assign to $\beta_{v_F}(a,(s,6))$ the sum of the costs along the traversal, $q(\alpha_s-j-y_1-\cdots-y_{r-1}-t_r-k-\alpha_6)$, together with the minimum feasible set of penalties on the outer nodes contained in F, $u_1,u_2\ldots,u_r$. Note that for the case that $u_1\notin\mathcal{S}$, it is not possible to have an inner structure L or B, and $\beta_{v_F}(L,(s,6))=\beta_{v_F}(B,(s,6))=\infty$. Otherwise

$$\begin{aligned}\beta_{v_F}(a,(s,6)) &= q(\alpha_s-j-y_1-\cdots-y_{r-1}-t_r-k-\alpha_6)+\min_{\substack{(\tau(F),\rho(F))\in\mathcal{F}'(F):\\ \rho^1_{u_1}=a\text{ or }B,\rho^{22}_{u_r}=6}}\left\{\sum_{i=1}^{r}\beta_{u_i}(\rho_{u_i})\right\}\\ &= q(\alpha_s,j,y_1)+q(K)-q(y_{r-2},y_{r-1},l)+q(y_{r-2},y_{r-1},t_r)+q(y_{r-1},t_r,k)+q(t_r,k,\alpha_6)\\ &\quad+\min_{\substack{(\tau(F),\rho(F))\in\mathcal{F}'(F):\\ \rho^1_{u_1}=a\text{ or }B,\rho^{22}_{u_r}=6}}\left\{\sum_{i=1}^{r}\beta_{u_i}(\rho_{u_i})\right\}.\end{aligned}\tag{4.2}$$

We will explain how the minimum in 4.2 can be calculated efficiently later in this paper.

Similarly, assign the minimum cost of the left-traversal (which contains α_t, $t \in \{3,4\}$ and α_5) with inner structure $a \in \{M=(0,0), R=(0,1)\}$ to $\beta_{v_F}(a,(5,t))$. In the case that $u_r \notin \mathcal{S}$, it is not possible to have an inner structure R or B, and $\beta_{v_F}(R,(5,t)) = \beta_{v_F}(B,(5,t)) = \infty$. Otherwise

$$\begin{aligned} \beta_{v_F}(a,(5,t)) &= q(\alpha_5,k,t_1)+q(k,t_1,y_1)+q(t_1,y_1,y_2)+q(K)-q(j,y_1,y_2)+q(y_{r-1},l,\alpha_t) \\ &+ \min_{\substack{(\tau(F),\rho(F))\in\mathcal{F}'(F):\\ \rho^1_{u_r}=a \text{ or } B,\rho^{21}_{u_1}=5}} \left\{\sum_{i=1}^{r}\beta_{u_i}(\rho_{u_i})\right\}. \end{aligned} \tag{4.3}$$

Let $K(y_i), i \in \{1,\ldots,r-1\}$, be the centre-traversal of F which does not contain y_i. Then $q(K(y_i))$ represents the cost incurred by the edges in $K(y_i)$. That is,

$$q(K(y_1)) = q(K)+q(j,t_1,t_2)+q(t_1,t_2,y_2)+q(t_2,y_2,y_3)-q(j,y_1,y_2)-q(y_1,y_2,y_3),$$

$$\begin{aligned} q(K(y_p)) &= q(K)+q(y_{p-2},y_{p-1},t_p)+q(y_{p-1},t_p,t_{p+1})+q(t_p,t_{p+1},y_{p+1})+q(t_{p+1},y_{p+1},y_{p+2}) \\ &- q(y_{p-2},y_{p-1},y_p)-q(y_{p-1},y_p,y_{p+1})-q(y_p,y_{p+1},y_{p+2}), \end{aligned}$$

for $p \in \{2,\ldots,r-2\}$, and

$$\begin{aligned} q(K(y_{r-1})) &= q(K)+q(y_{r-3},y_{r-2},t_{r-1})+q(y_{r-2},t_{r-1},t_r)+q(t_{r-1},t_r,l) \\ &- q(y_{r-3},y_{r-2},y_{r-1})-q(y_{r-2},y_{r-1},l). \end{aligned}$$

Assign the minimum cost of the centre-traversal which contains α_s, $s \in \{1,2\}$, and α_t, $t \in \{3,4\}$, which has inner structure $a \in \{L=(1,0), M=(0,0), R=(0,1), B=(1,1)\}$ to $\beta_{v_F}(a,(s,t))$. In the case that $u_1 \notin \mathcal{S}$ and $a = L$, there is a single path traversing F with inner structure L, namely, $j-t_1-t_2-y_2-\cdots-y_{r-1}-l$, so

$$\beta_{v_F}(L,(s,t)) = q(\alpha_s,j,t_1)+q(K(y_1))+q(y_r,l,\alpha_t)+\min_{\substack{(\tau(F),\rho(F))\in\mathcal{F}'(F):\\ \rho^{12}_{u_r}=0,\rho^{22}_{u_2}=6}} \left\{\sum_{i=2}^{r}\beta_{u_i}(\rho_{u_i})\right\}, \tag{4.4}$$

and when $u_1 \in \mathcal{S}$, we assign the cost of the minimum centre-traversal with inner structure L. Note that this path detours some y_g, $g \in \{2,\ldots,r-1\}$.

$$\beta_{v_F}(L,(s,t)) = \min_{g\in\{2,\ldots,r-1\}} \left\{ q(\alpha_s,j,t_1)+q(K(y_g))+q(y_r,l,\alpha_t)+\min_{\substack{(\tau(F),\rho(F))\in\mathcal{F}'(F):\\ \rho^{11}_{u_1}=1,\rho^{12}_{r}=0,\\ \rho^{21}_{u_g}=5,\rho^{22}_{u_{g+1}}=6}} \left\{\sum_{i=1}^{r}\beta_{u_i}(\rho_{u_i})\right\} \right\}. \tag{4.5}$$

Similarly, when $a = R$ and $u_r \notin \mathcal{S}$, there is a single path traversing F with inner structure R, namely, $j-y_1-\cdots-y_{r-2}-r_{r-1}-t_r-l$ so

$$\beta_{v_F}(R,(s,t)) = q(\alpha_s,j,y_1)+q(K(y_1))+q(t_r,l,\alpha_t)+\min_{\substack{(\tau(F),\rho(F))\in\mathcal{F}'(F):\\ \rho^{11}_{u_1}=0,\rho^{22}_{u_{r-1}}=6}} \left\{\sum_{i=1}^{r-1}\beta_{u_i}(\rho_{u_i})\right\}, \tag{4.6}$$

and when $u_r \notin \mathcal{S}$, we assign the cost of the minimum centre-traversal with structure R. Note that this path detours some y_g, $g \in \{1, \dots, r-2\}$.

(4.7)

$$\beta_{v_F}(R,(s,t)) = \min_{g\in\{1,\dots,r-2\}} \left\{ q(\alpha_s, j, y_1) + q(K(y_g)) + q(y_r, l, \alpha_t) + \min_{\substack{(\tau(F),\rho(F))\in\mathcal{F}'(F):\\ \rho^{11}_{u_1}=0,\rho^{12}_{r}=1,\\ \rho^{21}_{u_g}=5,\rho^{22}_{u_{g+1}}=6}} \left\{ \sum_{i=1}^{r} \beta_{u_i}(\rho_{u_i}) \right\} \right\}.$$

Now consider $a = M$. If $u_1 \in \mathcal{S}, u_r \in \mathcal{S}$ then $\beta_{v_F}(M,(s,t))$ is assigned the cost of the minimum path detouring y_g for $g \in \{1, \dots, r-1\}$. If u_1 (u_r) is not a pseudonode then the centre-traversal must contain y_1 (y_{r-1}) and 1 ($r-1$) is removed from g from the following equation.

(4.8)

$$\beta_{v_F}(M,(s,t)) = \min_{g\in\{1,\dots,r-1\}} \left\{ q(\alpha_s, j, y_1) + q(K(y_g)) + q(y_r, l, \alpha_t) + \min_{\substack{(\tau(F),\rho(F))\in\mathcal{F}'(F):\\ \rho^{11}_{u_1}=0,\rho^{12}_{u_r}=0,\\ \rho^{21}_{u_g}=5,\rho^{22}_{u_{g+1}}=6}} \left\{ \sum_{i=1}^{r-1} \beta_{u_i}(\rho_{u_i}) \right\} \right\}.$$

Now consider $a = B$. This is the same as the case where $a = M$ except for the inner structure.

(4.9)

$$\beta_{v_F}(B,(s,t)) = \min_{g\in\{1,\dots,r-1\}} \left\{ q(\alpha_s, j, y_1) + q(K(y_g)) + q(y_r, l, \alpha_t) + \min_{\substack{(\tau(F),\rho(F))\in\mathcal{F}'(F):\\ \rho^{11}_{u_1}=1,\rho^{12}_{u_r}=1,\\ \rho^{21}_{u_g}=5,\rho^{22}_{u_{g+1}}=6}} \left\{ \sum_{i=1}^{r-1} \beta_{u_i}(\rho_{u_i}) \right\} \right\}.$$

All β_{v_F} which has not been assigned are not associated with a feasible ρ_{v_F} and are assigned a value ∞.

Theorem 4.5. *Suppose $H = T \cup C$ is a Halin graph which is not a wheel and F is a fan in H. If (τ^*, ρ^*) is an optimal tour pair in H then there exists a feasible $\rho^*_{H/F}$ in H/F such that $(\tau^*/F, \rho^*_{H/F})$ is optimal in H/F and $z(\tau^*, \rho^*) = z(\tau^*/F, \rho^*_{H/F})$.*

Proof. Let $\mathcal{S}$ be the set of pseudonodes in H. Let v_F be the pseudonode which results from the contraction of F, and label the vertices and edges of F as in Figure 9. Given (τ^*, ρ^*), we construct $(\tau^*/F, \rho^*_{H/F})$ as follows. Let $\rho^*_{H/F,i} = \rho^*_i \ \forall i \notin F$ and $\rho^*_{H/F,v_F}$ corresponding to the structure of (τ^*, ρ^*) around F in H. That is, τ contains an α_i, α_j path through F in H, so let

$$\rho^*_{H/F,v_F} = ((a,b),(c,d))$$

where

$$a = \begin{cases} 0 & \text{if } (u_1 \notin \mathcal{S} \text{ and } y_1 \in \tau^*) \text{ or } (u_1 \in \mathcal{S} \text{ and } \rho^{*11}_{u_1} = 0) \\ 1 & \text{if } (u_1 \notin \mathcal{S} \text{ and } y_1 \notin \tau^*) \text{ or } (u_1 \in \mathcal{S} \text{ and } \rho^{*11}_{u_1} = 1), \end{cases}$$

$$b = \begin{cases} 0 & \text{if } (u_r \notin \mathcal{S} \text{ and } y_{r-1} \in \tau^*) \text{ or } (u_r \in \mathcal{S} \text{ and } \rho^{*12}_{u_1} = 0) \\ 1 & \text{if } (u_r \notin \mathcal{S} \text{ and } y_{r-1} \notin \tau^*) \text{ or } (u_r \in \mathcal{S} \text{ and } \rho^{*12}_{u_1} = 1), \end{cases}$$

$$c = \begin{cases} 1 & \text{if } (u_1 \notin \mathcal{S} \text{ and } \alpha_1 \in \tau^*) \text{ or } (u_1 \in \mathcal{S} \text{ and } \rho^{*21}_{u_0} = 1) \\ 2 & \text{if } (u_1 \notin \mathcal{S} \text{ and } \alpha_2 \in \tau^*) \text{ or } (u_1 \in \mathcal{S} \text{ and } \rho^{*21}_{u_0} = 2) \\ 5 & \text{if } (u_1 \notin \mathcal{S} \text{ and } \alpha_5 \in \tau^*) \text{ or } (u_1 \in \mathcal{S} \text{ and } \rho^{*21}_{u_0} = 5), \end{cases}$$

and

$$d = \begin{cases} 3 & \text{if } (u_r \notin \mathcal{S} \text{ and } t_r \notin \tau^*) \text{ or } (u_r \in \mathcal{S} \text{ and } \rho^{*22}_{u_r} = 3) \\ 4 & \text{if } (u_r \notin \mathcal{S} \text{ and } t_r \in \tau^*) \text{ or } (u_r \in \mathcal{S} \text{ and } \rho^{*22}_{u_r} = 4) \\ 6 & \text{if } (u_r \notin \mathcal{S} \text{ and } t_r \in \tau^*) \text{ or } (u_r \in \mathcal{S} \text{ and } \rho^{*22}_{u_r} = 6). \end{cases}$$

Then $(\tau^*/F, \rho^*_{H/F,v_F})$ is feasible in H/F.

Using equations 4.2-4.9 and noting that all new triples which contain v_F have $q(e, f, g) = 0$, we get

$$\begin{aligned} z(\tau^*, \rho^*) &= \sum_{(e,f,g)\in P_3(\tau^*)} q(e, f, g) + \sum_{i\in C} \mathcal{P}_i(\tau^*, \rho^*_i) \\ &= \sum_{\substack{(e,f,g)\in \\ P_3(\tau^*-F)}} q(e, f, g) + \sum_{\substack{(e,f,g)\in \\ P_3(\tau^*)\setminus P_3(\tau^*-F)}} q(e, f, g) + \sum_{i\in C\setminus F} \beta_i(\rho^*_i) + \sum_{i\in F} \beta_i(\rho^*_i) \\ &= \sum_{\substack{(e,f,g)\in \\ P_3(\tau^*/F-v_F)}} q(e, f, g) + \sum_{i\in C\setminus F} \beta_i(\rho^*_i) + \beta_{v_F}(\rho_{v_F}) \\ &= z(\tau^*/F, \rho^*_{H/F}). \end{aligned}$$

It remains to show that $(\tau^*/F, \rho^*_{H/F})$ is optimal in H/F. Towards a contradiction, suppose there exists a tour pair $(\tau'/F, \rho'_{H/F}) \neq (\tau^*/F, \rho^*_{H/F})$ such that $z(\tau'/F, \rho'_{H/F}) < z(\tau^*/F, \rho^*_{H/F})$. $\tau'/F \neq \tau^*/F$ otherwise this contradicts the calculations of the minimums in equations 4.2-4.9. Using equations 4.2-4.9, we can expand F, extending $(\tau'/F, \rho'_{H/F})$ to (τ'', ρ'') in H with $z(\tau'/F, \rho'_{H/F}) = z(\tau'', \rho'')$. Then $z(\tau'', \rho'') = z(\tau'/F, \rho'_{H/F}) < z(\tau^*/F, \rho^*_{H/F}) = z(\tau^*, \rho^*)$, contradicting the optimality of (τ^*, ρ^*). Hence $(\tau^*/F, \rho^*_{H/F})$ is optimal in H/F. □ □

We now show that β_{v_F} can be updated in $O(|F|)$-time, by introducing a structure which allows to optimally chain together the β-values for consecutive nodes in $F \cap C$. Refer to the subgraph induced by the nodes $\{w\} \cup \{u_a, u_{a+1}, \ldots, u_b\}$ as *pseudo-fan* $PF_{a,b}$ for $(1 \leq a \leq b \leq r)$. Define the minimum penalty associated with pseudo-fan $PF_{a,b}$ to be $PF_{a,b}(c, d)$ for $a, b \in \{1, \ldots, r\}$ with inner structure $c \in \{0, 1\}^2$ and outer structure $d = (d_1, d_2) \in \{(1, 3), (1, 4), (2, 3), (2, 4)\}$. Note that the inner structure of pseudo-fan $PF_{a,b}$ refers to the edges y_1, t_1 and y_{r-1}, t_r within fans F_a and F_b (which are contracted to pseudonodes u_a and u_b), respectively. The outer structure refers to the edges α_1, α_2 and α_3, α_4 with respect to fans F_a and F_b, if u_a and u_b are considered as pseudonodes.

To compute $PF_{a,b}(c, d)$ we use the following recursion which chains together the minimum β-values through consecutive nodes in C while maintaining feasibility of ρ.

For $c_1, c_2 \in \{0, 1\}$, $d_1 \in \{1, 2\}$ and $d_2 \in \{3, 4\}$, and $1 \leq i \leq r - 1$

$$PF_{1,1}((c_1, c_2), (d_1, d_2)) = \beta_{u_1}((c_1, c_2), (d_1, d_2)) \tag{4.10}$$

$$PF_{r,r}((c_1, c_2), (d_1, d_2)) = \beta_{u_r}((c_1, c_2), (d_1, d_2)) \tag{4.11}$$

$$PF_{1,i+1}((c_1, c_2), (d_1, d_2)) = \min_{s\in\{0,1\}, t\in\{3,4\}} \{PF_{1,i}((c_1, s), (d_1, t)) + \beta_{u_{i+1}}((t - 3, c_2), (s + 1, d_2))\} \tag{4.12}$$

and

$$PF_{i-1,r}((c_1,c_2),(d_1,d_2)) = \min_{s\in\{0,1\},t\in\{1,2\}} \{\beta_{u_{i-1}}((c_1,t-1),(d_1,s+3)) + PF_{i,r}((s,c_2),(t,d_2))\} \tag{4.13}$$

To prove that the recursions defined by 4.10-4.13 are correct, first consider the optimal $PF_{1,i+1}((c_1,c_2),(d_1,d_2))$. The minimum feasible assignment of penalties for the nodes within pseudo-fan $PF_{1,i+1}$ is simply the minimum among optimal assignment of penalties for the nodes within pseudo-fan $PF_{1,i}$ and the penalty at pseudonode u_{i+1}, such that these penalties are feasible. That is, the outer structure of $PF_{1,i}$ must match the inner structure of u_{i+1} and vice-versa. By definition, this is precisely $\min_{s\in\{0,1\},t\in\{3,4\}} PF_{1,i}((c_1,s),(d_1,t)) + \beta_{u_{i+1}}((t-3,c_2),(s+1,d_2))$. An analogous argument holds for $PF_{i-1,r}((c_1,c_2),(d_1,d_2))$.

It is now possible to compute the minimum traversals of F used in equations 4.2-4.9. For example, middle paths through F have a cost which minimizes the sum of the penalties on $PF_{1,i-1} + \beta_i$ and $\beta_{i+1} + PF_{i+2,r}$, such that both pairs are feasible. Note that the minimum $\rho(F)$ has been found while performing the recursion defined by equations4.10-4.13. In the worst case, when all outer nodes in F belong to $\mathcal{S}$, we must compute $PF_{1,1},\ldots,PF_{1,r-1}$ and $PF_{r,r},\ldots,PF_{2,r}$, which can be done in $O(|F|)$-time. By pre-computing these, one can evaluate the minimum traversals of F used in equations 4.2-4.9, in $O(|F|)$-time and hence one can update β_{v_F} in $O(|F|)$-time.

We iteratively perform the fan contraction operation, updating costs and penalties until we are left with a wheel. The optimal tour in H skirts the cycle C and detours exactly once through centre w, skipping exactly one edge of C. Orient the cycle C in the clockwise direction. τ contains all edges in C except for the skipped edge, say $c_i = (u_i,u_{i+1})$, together with the two edges which detour around c_i. Define function $\phi(c_i)$ for each edge $c_i = (u_i,u_{i+1}) \in E(C)$. Let t_i and t_{i+1} be the tree edges adjacent to u_i and u_{i+1}, respectively.

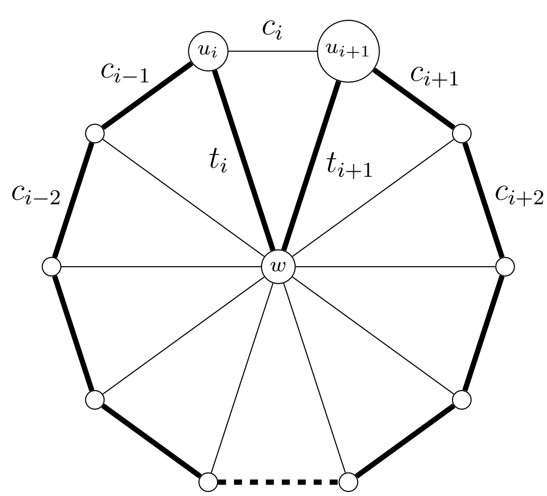

FIGURE 11. A tour τ in a wheel, which skips edge c_i.

We will define

$$\begin{aligned}\phi(c_i) \quad = \quad & q(c_{i-2},c_{i-1},t_i) + q(c_{i-1},t_i,t_{i+1}) + q(t_i,t_{i+1},c_{i+1})\\ & +q(t_{i+1},c_{i+1},c_{i+2}) - q(c_{i-2},c_{i-1},c_i) - q(c_{i-1},c_i,c_{i+1})\\ & -q(c_i,c_{i+1},c_{i+2}).\end{aligned}$$

Then, the optimal tour pair has

$$z(\tau^*, \rho^*) = q(C) + \min_{i:c_i \in C} \left\{ \phi(c_i) + \min_{\text{feasible } \rho} \sum_{u_j \in C} \beta_{u_j}(\rho_{u_j}) \right\}.$$

Suppose that we fix an edge c_i in τ. Then H can be considered to be a fan as shown in Figure 12.

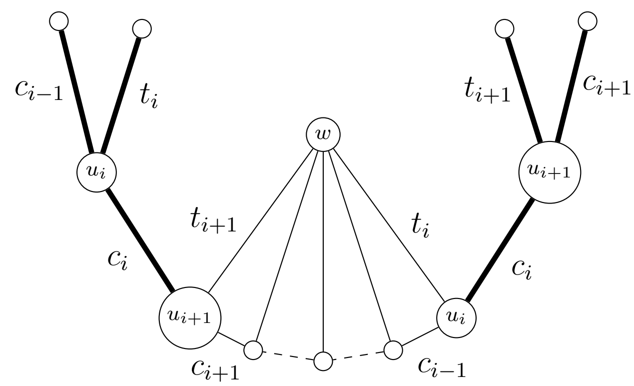

FIGURE 12. Wheel H with centre w considered as a fan. Note that edge c_i is fixed and the fan 'wraps' around the wheel, reusing edges $c_{i-1}, c_i, c_{i+1}, t_{i-1}, t_i$ and t_{i+1}.

Fix edge c_r in τ and consider H to be a fan F_{c_r}. Then the minimum tour in H can be determined by calculating the minimum of the minimum centre-traversal of F_{c_r} and the tour which bypasses c_r (using, say, fan F_{c_1}). This can be computed in $O(r)$-time using the pseudofan technique described above.

The preceeding discussion yields the following algorithm.

Algorithm 1 HalinTSP(3)(H, q, β)

Input: Halin graph H, quadratic cost function q, and penalty function β
  if H is a wheel **then**
    Use the wheel procedure to find an optimal tour τ in H
  else
    Let F be a fan in H
    Contract F to a single node v_F, using the Case 1 procedure. That is, assign penalties β to v_F, assign cost 0 to all triples in H/F and assign costs q to all remaining triples which are 3-neighbours in H.
    HalinTSP(3)(H/F, q, β)
  end if
  Expand all pseudonodes in reverse order and update τ
  return τ

Each time that a fan is contracted, the number of tree nodes is reduced by 1, and hence the fan contraction operation is performed one less than the number of non-leaf nodes in H. The fan contraction operation can be performed in $O(|F|)$-time, and each time it is performed, the number of nodes in H is reduced by $|F|-1$. Since the wheel procedure takes $O(n)$-time, the total time for the algorithm is $O(n)$.

4.2. **TSP(k).** We now show that the previous ideas can be extended to solve TSP(k). For any subgraph G of H, let $P_k(G)$ be the collection of all distinct candidate k-paths in G. For each candidate k-path

$(e_1, e_2, \dots e_k) \in P_k(H)$, define

$$q(e_1, e_2, \dots, e_k) = q(e_1, e_k) + \frac{q(e_1, e_{k-1}) + q(e_2, e_k)}{2} + \frac{q(e_1, e_{k-2}) + q(e_2, e_{k-1}) + q(e_3, e_k)}{3} + \dots + \frac{c(e_1) + c(e_2) + \dots + c(e_k)}{k}. \tag{4.14}$$

Now consider the simplified problem:

$$STSP(k): \quad \text{Minimize} \quad \sum_{(e_1, e_2, \dots, e_k) \in P_k(\tau)} q(e_1, e_2, \dots, e_k)$$
$$\text{Subject to} \quad \tau \in \mathcal{F}.$$

Theorem 4.6. *Any optimal solution to the STSP(k) is also optimal for TSP(k).*

Proof. Using equation 4.14, the proof of this follows along the same way as that of Theorem 4.4 and hence is omitted. □ □

As a result of Corollary 4.2, the preceding algorithm can be extended to solve TSP(k) in by extending the penalty functions at outer nodes to accommodate subpaths of length $2^{\lceil (k+1)/2 \rceil}$.

A complete description of these varies as the information that needs to be stored is more involved and is hence omitted. Some details however will be available in [32]. The complexity increases by a factor of $(2^{\lceil (k+1)/2 \rceil})$, which is constant for fixed k and plynomially bounded when $k = t \log n$.

5. TSP(k) on fully reducible graph classes

We say that a class $\mathcal{C}$ of 3-connected graphs is *fully reducible* if it satisfies the following:

(1) If $G \in \mathcal{C}$ has a 3-edge cutset which partitions G into components S and $\bar{S}$, then both G/S and $G/\bar{S}$ are in $\mathcal{C}$ and we call G a *reducible* graph in $\mathcal{C}$; and
(2) TSP can be solved in polynomial time for the graphs in $\mathcal{C}$ that do not have non-trivial 3-edge cutsets. We call such graphs *irreducible.*

For instance, Halin graphs can be understood as graphs built up from irreducible fans connected to the remainder of the graph via 3-edge cutsets. Cornuejols et al. [10] show that the ability to solve TSP in polynomial time on irreducible graphs in $\mathcal{C}$ allows to solve TSP in polynomial time on all of $\mathcal{C}$ using facts about the TSP polyhedron.

We remark that the algorithm of Section 4 can be used to show a somewhat similar statement for TSP(k). Here we consider a graph class $\mathcal{C}$ that is *fully k-reducible* in the sense that either it can be subdivided into irreducible graphs via 3-edge cutsets, or it is irreducible and it is possible to solve the k-neighbour Hamilton path problem in polynomial time.

This requires solving the following problem:

$$MTSP(k): \quad \text{Minimize} \quad \sum_{(e_1, \dots, e_k) \in P_k(\tau)} q(e_1, \dots, e_k) + \sum_{i \in V} \mathcal{P}_i(\tau)$$
$$\text{Subject to} \quad \tau \in \mathcal{F}$$

where $\mathcal{P}_i(\tau)$ is a penalty function for the pseudonode which depends on how tour τ traverses i, analogous to the construction for the 3-neighbour TSP of section 4.

We recursively perform the contraction operation on the irreducible subgraphs of G, storing the necessary tour information in the penalty at the resulting pseudonode. A similar result to Corollary 4.2 may be derived to show that for any fixed k, this requires a polynomial number of penalties. The least cost traversals of S can be computed in polynomial time using a generalization of the pseudo-fan strategy above.

Suppose the contraction operation is performed on a subgraph of size r in time $O(P(r))$, where $P(r)$ is a polynomial in r. Each time this operation is performed, the number of nodes in the graph is reduced by r. This operation is performed at most n times and it follows that the entire algorithm can be performed in polynomial time.

6. Conclusions

In this paper, we have shown that QTSP is NP-hard even when the costs are restricted to taking 0-1 values on Halin graphs. We have presented a polynomial time algorithm to solve a restriction of QTSP, denoted TSP(k) on any fully k-reducible graph for any fixed k. To illustrate this, we have given an algorithm which solves TSP(3) on a Halin graph in $O(n)$ time.

The k-neighbour bottleneck TSP on a Halin graph can be solved by solving $O(log(n))$ problems of the type TSP(k). However, it is possible to solve the problem faster. Details will be reported elsewhere.

We would also like to thank Ante Custic for his useful comments.

7. References

References

[1] A. Aggarwal, D. Coppersmith, S. Khanna, R. Motwani, and B. Schieber. The angular-metric traveling salesman problem. *SIAM Journal on Computing*, 29:697–711, 2000.

[2] D. Applegate, R. Bixby, V. Chvatal, and W. Cook. *The traveling salesman problem: a computational study*. Princeton University Press, 2011.

[3] E. Arkin, Y. Chiang, J. Mitchell, S. Skiena, and T. Yang. On the maximum scatter traveling salesperson problem. *SIAM Journal on Computing*, 29:515–544, 1999.

[4] E. Balas, R. Carr, M. Fischetti, and N. Simonetti. New facets of the STS polytope generated from known facets of the ATS polytope. *Discrete Optimization*, 3:3–19, 2006.

[5] E. Balas, M. Fischetti, and W. Pulleyblank. The precedence-constrained asymmetric traveling salesman polytope. *Mathematical Programming*, 68:241–265, 1995.

[6] H. Bodlaender. Dynamic programming on graphs with bounded treewidth. In *Automata, languages and programming (Tampere, 1988)*,*Lecture Notes in Comput. Sci.*, 317:105–118. Springer, Berlin, 1988.

[7] W. Chin and S. Ntafos. Optimum watchman routes. In *Proceedings of the Second Annual Symposium on Computational Geometry*, 24–33. ACM, 1986.

[8] W. Cook. *In pursuit of the traveling salesman: mathematics at the limits of computation*. Princeton University Press, 2012.

[9] G. Cornuéjols, D. Naddef, and W. Pulleyblank. Halin graphs and the travelling salesman problem. *Mathematical Programming*, 26:287–294, 1983.

[10] G. Cornuéjols, D. Naddef, and W. Pulleyblank. The traveling salesman problem in graphs with 3-edge cutsets. *Journal of ACM*, 32:383–410, April 1985.

[11] B. Courcelle. The monadic second-order logic of graphs. I. Recognizable sets of finite graphs. *Inform. and Comput.*, 85:12–75, 1990.

[12] F. Della Croce, V. Paschos, and R. Calvo. Approximating the 2-peripatetic salesman problem. In *Proc. Workshop on Modelling and Algorithms for Planning and Scheduling Problems, MAPSP*, 114–116, 2005.

[13] O. Ergun and J. Orlin. A dynamic programming methodology in very large scale neighborhood search applied to the traveling salesman problem. *Discrete Optimization*, 3:78–85, 2006. The Traveling Salesman Problem.

[14] A. Fischer. An analysis of the asymmetric quadratic traveling salesman polytope. *SIAM Journal on Discrete Mathematics*, 28:240–276, 2014.

[15] A. Fischer and F. Fischer. An extended approach for lifting clique tree inequalities. *Journal of Combinatorial Optimization*, 30:489–519, 2015.

[16] A. Fischer, F. Fischer, G. Jäger, J. Keilwagen, P. Molitor, and I. Grosse. Exact algorithms and heuristics for the quadratic traveling salesman problem with an application in bioinformatics. *Discrete Applied Mathematics*, 166:97–114, 2014.

[17] A. Fischer and C. Helmberg. The symmetric quadratic traveling salesman problem. *Mathematical Programming*, 142:205–254, 2013.

[18] G. Gutin and A. Punnen, editors. *The traveling salesman problem and its variations*, volume 12 of *Combinatorial Optimization*. Springer New York, 2002.

[19] G. Jäger and P. Molitor. Algorithms and experimental study for the traveling salesman problem of second order. In Boting Yang, Ding-Zhu Du, and CaoAn Wang, editors, *Combinatorial Optimization and Applications*, *Lecture Notes in Computer Science*, 5165:211–224. Springer Berlin, 2008.

[20] J. Krarup and I. Spadille. The peripatetic salesman and some related unsolved problems. *Combinatorial programming: methods and applications: proceedings of the NATO Advanced Study Institute held at the Palais des Congrès, Versailles, France, 2-13 September 1974*, 173, 1975.

[21] J. LaRusic and A. Punnen. The asymmetric bottleneck traveling salesman problem: algorithms, complexity and empirical analysis. *Computers & Operations Research*, 43:20–35, 2014.

[22] J. LaRusic, A. Punnen, and E. Aubanel. Experimental analysis of heuristics for the bottleneck traveling salesman problem. *Journal of heuristics*, 18:473–503, 2012.

[23] E. Lawler, J. Lenstra, A. Kan, and D. Shmoys. *The Traveling Salesman Problem: a guided tour of combinatorial optimization*, volume 3. Wiley New York, 1985.

[24] J. Le Ny, E. Feron, and E. Frazzoli. On the Dubins Traveling Salesman Problem. *Automatic Control, IEEE Transactions on*, 57:265–270, 2012.

[25] D. Lou. An algorithm to find the optimal matching in Halin graphs. *IAENG International Journal of Computer Science*, 34:220–226, 2007.

[26] J. Phillips, A. Punnen, and S. Kabadi. A linear algorithm for the bottleneck traveling salesman problem on a Halin graph. *Information Processing Letters*, 67:105–110, 1998.

[27] G. Reinelt. *The traveling salesman: computational solutions for TSP applications*. Springer-Verlag, 1994.

[28] B. Rostami, F. Malucelli, P. Belotti and S. Gualandi. Quadratic TSP: A lower bounding procedure and a column generation approach *Computer Science and Information Systems (FedCSIS), 2013 Federated Conference on*, 377-384, September 2013.

[29] B. Rostami, F. Malucelli, P. Belotti, and S. Gualandi. Lower bounding procedure for the asymmetric quadratic traveling salesman problem. *European Journal of Operational Research*, 253:584–592, 2016.

[30] R. Staněk, Problems on tours and trees in combinatorial optimization, PhD Thesis, Graz University of Technology, 2016.

[31] B. Woods. Generalized traveling salesman problems on Halin graphs. Master's thesis, Simon Fraser University, Canada, 2010.

[32] B. Woods The quadratic travelling salesman problem: complexity, approximations, and exponential neighbourhoods. PhD. thesis *In preparation*, Simon Fraser University, Canada.

[33] B. Woods and A. Punnen. The quadratic traveling salesman problem on Halin graphs. *Working paper.*, Simon Fraser University, Canada, 2016.