# Global minimization of rational functions using semidefinite programming

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# Rational function minimization

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where $S$ is the *semi-algebraic set* given by

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**Applications:** Global and combinatorial optimization, statistics, geometry, economics ...

# Possible approaches

- If the infimum is attained one can solve the first order optimality condition equations. Excellent review: B. Sturmfels, *Solving Systems of Polynomial Equations*, AMS, 2002. If the inf is not attained ...

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- Global optimization codes — can converge to local minima.
- Today's talk: approaches involving semidefinite programming (SDP).

# Different cases

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- $S = \mathbb{R}^n$ and $n = 2$ (Unconstrained minimization: bivariate case);
- $S = \mathbb{R}^n$ and general $n$ (Unconstrained minimization: general case);
- $S$ is compact, connected and general $n$ (Constrained case);

# Unconstrained case

Consider the unconstrained problem.

$$\begin{aligned} p^* &:= \inf_{x \in \mathbb{R}^n} \frac{p(x)}{q(x)} \\ &= \sup \left\{ \rho \; : \; \frac{p(x)}{q(x)} - \rho \geq 0 \quad \forall x \in \mathbb{R}^n \right\} \end{aligned}$$

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We can replace the nonnegativity condition by a simpler one ...

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This leads us to the theory of *nonnegative polynomials*.

# Preliminaries

Let $p$ be a polynomial defined on $\mathbb{R}^n$.

- $p$ is a *sum of squares* (SOS) if there exist polynomials $p_i$ such that $p = \sum_i p_i^2$.

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- A form $p$ is *positive semidefinite* (PSD) if $p(x) \geq 0$ for all $x \in \mathbb{R}^n$.
- A form $p$ is *positive definite* (PD) if $p(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

# Homogenization

Each polynomial has an associated *form*, obtained by introducing a new homogenizing variable $t$ as follows:

$$p(x) \Rightarrow p\left(\frac{x}{t}\right) t^{\deg(p)}$$

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$$x^2 - 2x + 1 \Rightarrow x^2 - 2xt + t^2.$$

Nonnegativity and SOS properties are preserved under this transformation.

# Hilbert's 17th problem

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- Review paper: B. Reznick. Some concrete aspects of Hilbert's 17th Problem. In *Real algebraic geometry and ordered structures*, 251–272. AMS, 2000.

# Nonnegativity vs SOS

Consider real $n$-variate polynomials with degree $d$.
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- $n = 2$ and $d \leq 4$ (bivariate polynomials of degree at most $4$) (result by Hilbert);

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- $d = 2$ (quadratic polynomials on $n$ variables);
- $n = 2$ and $d \leq 4$ (bivariate polynomials of degree at most $4$) (result by Hilbert);

In all other cases counterexamples exist (e.g. the Motzkin form; see Reznick's paper).

# The sum of squares cone

We fix a basis of monomials

$$\tilde{x} := (1, x_1, \ldots, x_n, x_1^2, \ldots, x_n^d) \quad \text{dim:} \binom{n+d}{d}.$$

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**Notation:** We denote the cone in $\mathbb{R}^{\binom{n+2d}{2d}}$ generated by squares of polynomials on $\mathbb{R}^n$ of degree at most $d$ by $\Sigma_{n,d}^2$ (*sum-of-squares (SOS) cone*).

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(We drop the subscripts when they are clear from the context.)

# The sum of squares cone (cdt.)

**Theorem:** The cone $\Sigma^2_{n,d}$ is convex, closed, pointed, solid, and is the image of a linear map of the cone of PSD matrices of size $\binom{n+d}{d} \times \binom{n+d}{d}$.

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Theorem 17.1 in Y. Nesterov. Squared functional systems and optimization problems. In J.B.G. Frenk et al. eds., *High performance optimization*, 405–440. KAP, 2000.

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**Implication:** Conic linear optimization over the cone $\Sigma^2_{n,d}$ can be done using *semidefinite programming* (SDP) (the so-called *Gram matrix method*);

# Example (Parrilo)

Is $P(x) := 2x_1^4 + 2x_1^3x_2 - x_1^2x_2^2 + 5x_2^4$ a sum of squares?

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$$P(x) \;=\; \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}^T \begin{bmatrix} 2 & -\lambda & 1 \\ -\lambda & 5 & 0 \\ 1 & 0 & -1+2\lambda \end{bmatrix} \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}.$$

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**SDP problem:** is there a $\lambda$ such that $M(\lambda) \succeq 0$ (positive semidefinite)?

# Example (ctd.)

for $\lambda = 3$, $M(\lambda)$ is positive semidefinite, and

$$M(3) = L^T L, \quad L = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix},$$

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and consequently

$$P(x) = \tilde{x} L^T L \tilde{x} = \|L\tilde{x}\|^2,$$

where $\tilde{x} = [x_1^2 \;\; x_2^2 \;\; x_1 x_2]^T$.

Thus $P$ can be written as a sum of squares.

# Unconstrained univariate case

If $q$ does not change sign on $\mathbb{R}$, then

$$\begin{aligned}
\inf_{x\in\mathbb{R}} \frac{p(x)}{q(x)} &= \sup_{t,x} \{t \;:\; p(x) - tq(x) \geq 0 \;\forall x \in \mathbb{R}\} \\
&= \sup_{t,x} \{t \;:\; p(x) - tq(x) \in \Sigma^2\} \\
&= \sup_{t,x} \{t \;:\; p(x) - tq(x) = \tilde{x}^T M \tilde{x}\}
\end{aligned}$$

for some $M \succeq 0$, where

$$\tilde{x}^T = [1 \; x \; x^2 \ldots x^{\frac{1}{2}\max\{\deg(p),\deg(q)\}}].$$

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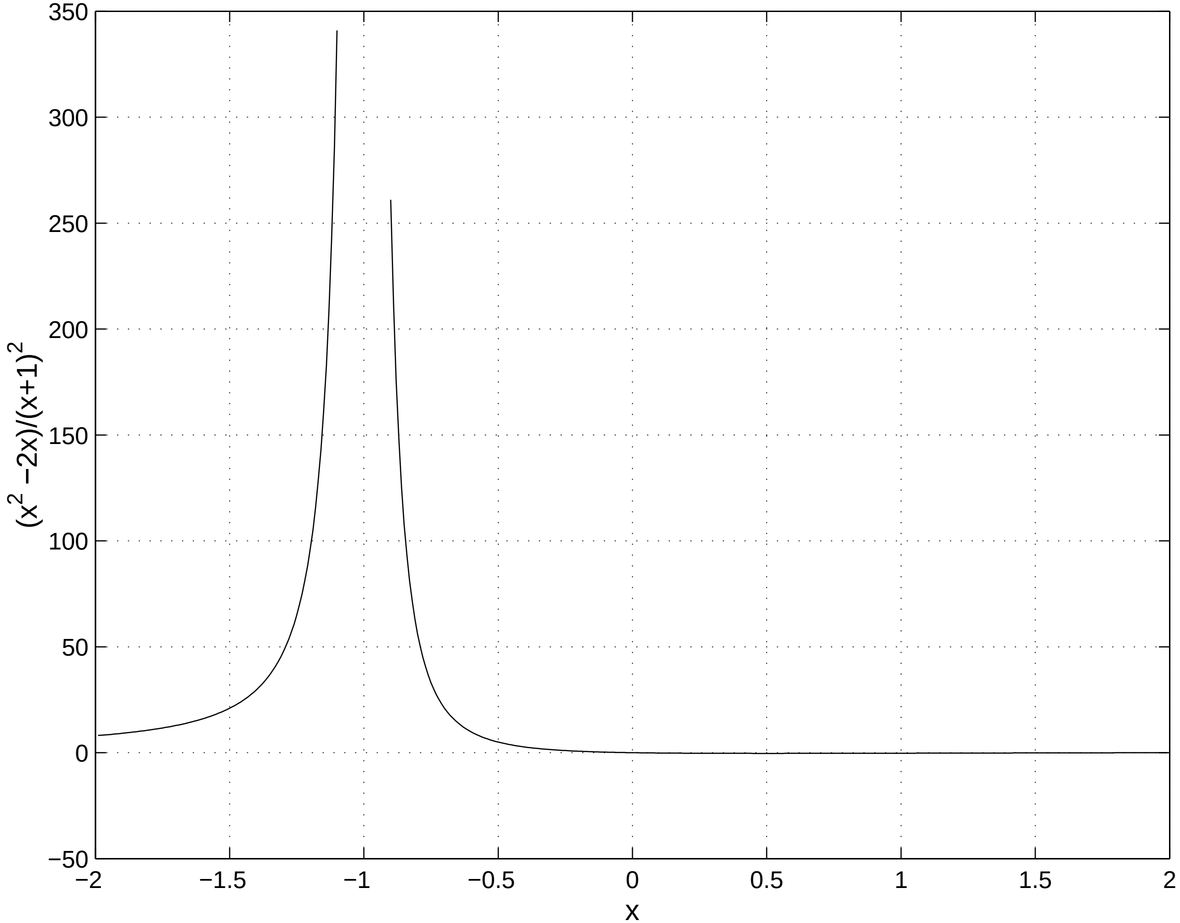
This is an SDP problem! (Result already obtained by Nesterov for $q(x) \equiv 1$.)

Y. Nesterov. Squared functional systems and optimization problems. In J.B.G. Frenk et al. eds., *High performance optimization*, 405–440. KAP, 2000.

# Example

$$\frac{p(x)}{q(x)} := \frac{x^2 - 2x}{(x+1)^2}.$$



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Equivalent problem: $\sup t$ such that

$$(1-t)x^2 - 2(1+t)x - t = \begin{bmatrix} 1 \\ x \end{bmatrix}^T \begin{bmatrix} M_{00} & M_{01} \\ M_{10} & M_{11} \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix}, \tag{2}$$

for some $M \succeq 0$.

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From (2):

$$M_{00} = -t, \quad M_{01} = M_{10} = -(1+t), \quad M_{11} = 1 - t.$$

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We therefore get

$$\min_{x \in \mathbb{R}} \frac{p(x)}{q(x)} = \max_{t,M} t$$

such that

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Note that the optimal value is $p^* = -1/3$.

# Unconstrained bivariate case

If $q$ does not change sign on $\mathbb{R}^2$, then

$$\inf_{x\in\mathbb{R}^2} \frac{p(x)}{q(x)} = \sup_{t,x} \left\{ t \ : \ p(x) - tq(x) \geq 0 \ \forall x \in \mathbb{R}^2 \right\}$$

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**Remark:** This problem also has an *exact SDP reformulation*, using results by De Klerk and Pasechnik, and by Nesterov.

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E. de Klerk, D.V. Pasechnik (2002). Products of positive forms, linear matrix inequalities, and Hilbert 17-th problem for ternary forms. *European J. of Operational Research*, to appear.

# Constrained case

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(We will return to the unconstrained problem presently.)

# Constrained case

**Theorem (Jibetean)** Assume that $S$ is full dimensional and connected. If $p^* > -\infty$ then $q$ does not change sign on $S$. If $q$ does not change sign on $S$, then

$$\frac{p(x)}{q(x)} \geq \alpha \; \forall x \in S \iff p(x) - \alpha q(x) \geq 0 \; \forall x \in S.$$

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Consequence

$$\inf_{x \in S} \frac{p(x)}{q(x)} = \sup \{\rho \; : \; p(x) - \rho q(x) > 0 \;\; \forall x \in S\}.$$

# Constrained univariate case

Univariate constrained problem: Assume $q$ does not change sign on $\mathbb{R}^2$ (else $p^* = -\infty$). Then

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where $S$ is a *line segment* or *an interval.*

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where $S$ is a *line segment* or *an interval.*

**Remark:** This problem has an *exact* SDP reformulation using the theorem by Jibetean and results by Nesterov.

# Constrained multivariate case

**Technical assumption:** $S$ is compact and there exists a

$$\bar{p} \in \Sigma^2 + p_1\Sigma^2 + \ldots + p_k\Sigma^2$$

such that $\{x \;:\; \bar{p}(x) \geq 0\}$ is *compact*.

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**Theorem (Putinar):** For a given polynomial $p_0$ one has $p_0(x) > 0$ for all $x \in S$ iff

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M. Putinar. Positive polynomials on compact semi-algebraic sets. *Ind. Univ. Math. J.* 42:969–984, 1993.

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By Putinar's and Jibetean's theorems we have

$$\begin{aligned}
p^* &= \sup \{\rho \ : \ p(x) - \rho q(x) > 0 \ \ \forall x \in S\} \\
&= \sup \left\{\rho \ : \ (p - \rho q) \in \Sigma^2 + p_1 \Sigma^2 + \ldots + p_k \Sigma^2 \right\} \\
&\geq \sup \left\{\rho \ : \ (p - \rho q) \in \Sigma^2_{n,t} + p_1 \Sigma^2_{n,t} + \ldots + p_k \Sigma^2_{n,t} \right\} \\
&:= \rho_t \ \ \text{(for any integer } t \geq 1).
\end{aligned}$$

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We have that $\rho_i \leq \rho_{i+1} \leq p^*$ and

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Computation of $\rho_t$: SDP problem with matrices of size $\binom{n+t}{t} \times \binom{n+t}{t}$ and at most $\max\{\deg(p), \deg(q)\}$ constraints — "polynomial" complexity for $t = O(1)$.

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These results by already obtained by Lasserre for $q(x) \equiv 1$ (polynomial objective function).

J.B. Lasserre. Global optimization with polynomials and the problem of moments. *SIOPT*, 11:296–817, 2001.

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No a priori choice for $R$ available in general.

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GloptiPoly and SOStools extremely useful to prove *global optimality* in small problems.

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- Need for large-scale (parallel?) SDP solvers to solve the large SDP relaxations.