Porous Medium Model of Sap Flow in Trees

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Abstract

In an effort to better understand the dynamics of sap flow, with a particular interest in winter sap flow in maple trees, a PDE model is created and tested. The model follows the work done by Chuang et al., and treats the wood as a porous medium. Darcy's Law and conservation laws are used to link sap flux to transpiration. A comparison of results to those of Chuang et al. is made to determine the validity of the model.

Introduction

Developing a complete model for water transport in tree boles remains an attractive topic in the field of physiological ecology.

A commonly used model for the water transport is the RC-model, which likens the system to an electrical circuit. This model is often employed because while relatively simple, employing only ODEs, it is still able to produce results similar to experimental data. In particular, the RC model is able to mimic the lag between water uptake through the roots, flux through the tree, and transpiration in the crown.

A detailed RC-model is formulated and explored by Loustau et al. [1] who claim the model provides a basis for interpreting vertical variations in sap flux. They also provide a method to obtain model parameters R and C through experimentation.

The common use of RC-models is in spite of a number of criticisms made against the model. Major problems with the model noted by Chuang et al. include unaccounted for loses of water when artificially shifting and altering flux curves to match transpiration curves, and the possibility of negative saturations with no real physical meaning. Aumann and Ford [2] also show that the definition for the capacitance leads to an unrealistic notion that a during a change in pressure lasting minutes, water saturation would

change as much as if that same change in pressure lasted days.

Arguably the greatest problem with the RC-model is fundamental – there simply is no physical reason to choose this model. Trees are not electrical circuits, and the analogy is never fully explained. RC-models with varying amounts of complexity can produce results that have been shown to be reasonably accurate, but the fact that the model begins without a physical basis limits the ability to improve it, and makes the model a difficult choice to justify.

An alternative model based on actual physiology is a PM (porous media) model. Such a model considers the wood to be similar to soils or other porous media. They combine Darcy's Law for fluid flow in a porous medium, conservation laws, and include an addition term to account for sinks through the leaves.

Chuang et al. [3] have derived a PM model which they use as a link between sap flux and transpiration. The model does not have the artificial loss of water they critiqued the RC models for, nor does it allow negative saturation, or have the unrealistic capacitance.

This report primarily attempts to reproduce the findings of Chuang et al. by applying a PM model similar to the one used by them.

Nomenclature

A	Cross sectional area (m^2)
C	Hydraulic capacitance (m^{-1})
D	Local tree diameter (m)
E	Transpiration rate $(kg/m^2 s)$
j	Volume flux (m^3/s)
K	Hydraulic conductivity (m/s)
l	Leaf area per stem length (m)
p	Pressure $(kg/m^2 s^2)$
Q	Sink term (s^{-1})
s	Saturation (m^3/m^3)
t	Time (s)
v	Darcy velocity (m/s)
z	Position (m)
Δz	Distance between cells (m)
λ	Transpiration scaling term
ψ	Pressure head
τ	Period of fluctuation $(8.64 \times 10^4\ s)$

Constants and Parameters

g	Gravitational constant $(9.8\ m/s^2)$
H	Tree height $(6.7\ m)$
K_{max}	Max. conductivity $(5.36 \times 10^{-7}\ m/s)$
n	Fitting parameter (400)
N	Number of cell divisions (100)
p_o	Retention Potential coefficient $(6.8 \times 10^6\ kg/m^2 s^2)$
s_{max}	Max. Saturation (0.5735)
β	Fitting parameter (3.5)
ε	Fitting parameter (10^{-5})
ρ	Water density $(1 \times 10^3\ kg/m^3)$
ψ_o	Min. Pressure head (2.93×10^5)

Mathematical Model

The model used to represent the tree involved several approximations, some of which are reasonable, many of which are very useful approximations but are questionably accurate. Ideally these could be refined in future study. The first such approximation is made in describing the tree as a one dimensional object. Cross sectional area is considered, but each cross section is considered uniform, and sap flow is described as being vertical but not radial.

The equations governing the flow of sap are then created by combining conservation laws with Darcy's Law for porous media flow. First, we consider a thin disc shaped cross section of the tree. The change in the volume of liquid in this region,

$$\int_a^b (sA)dz\,,$$

must equal the net flow of fluid into the region,

$$vA|_{z=a} -\ vA|_{z=b}\,.$$

Here s is the saturation (the volume fraction V_{fluid}/V_{Total}), A (m^2) is the cross sectional area, and $v(m/s)$ is the velocity of the fluid. By the Fundamental Theorem of Calculus, this is equivalent to

$$-\int_a^b \frac{\partial}{\partial z}(vA)dz\,.$$

Allowing for a source term, we then have

$$\frac{\partial}{\partial t}\int_a^b (sA)dz = -\int_a^b \frac{\partial}{\partial z}(vA)dz + \int_a^b QAdz\,.$$

The choice of a and b is arbitrary, so we can reduce this to

$$\frac{\partial}{\partial t}(sA) + \frac{\partial}{\partial z}(vA) =\ QA$$

or further reduce to

$$\frac{\partial s}{\partial t} + \frac{1}{A}\frac{\partial}{\partial z}(vA) =\ Q\,. \qquad (1)$$

The next part of the model comes from Darcy's Law for porous media, which describes the volume flux j (m^3/s)

$$j = -\frac{KA}{\rho g}\frac{\partial p}{\partial z}\,.$$

Here g (m/s^2) is the gravitational constant, p (Pa) is the pressure, and K (m/s) is the Hydraulic Conductivity.

Because the flow is occurring in the vertical direction where gravitational forces will have an impact, we adapt this to

$$j = -\frac{KA}{\rho g}\frac{\partial}{\partial z}(p + \rho g z)$$

Next, we define the pressure head ψ (m) by

$$\psi = \frac{p}{\rho g}\ .$$

In our case, ρ is the density of water and we can take it to be a constant. We then have that

$$j = -\frac{KA}{\rho g}\frac{\partial}{\partial z}(\psi \rho g + \rho g z) = -KA\frac{\partial}{\partial z}(\psi + z)\ . \quad (2)$$

If we note that $= j/A$, we get an equation for v:

$$v = -K\frac{\partial}{\partial z}(\psi + z) = -K\left(1 + \frac{\partial \psi}{\partial z}\right). \quad (3)$$

We can then substitute (2) back into (1)

$$\frac{\partial s}{\partial t} - \frac{1}{A}\frac{\partial}{\partial z}(KA) - \frac{1}{A}\frac{\partial}{\partial z}\left(KA\frac{\partial \psi}{\partial z}\right) = Q\ .$$

By assuming ψ is a function of saturation, and vice versa, we can define another variable, the hydraulic capacitance, as $C = \frac{\partial s}{\partial \psi}$ (m^{-1}). We then have

$$\frac{\partial s}{\partial t} - \frac{1}{A}\frac{\partial}{\partial z}(KA) - \frac{1}{A}\frac{\partial}{\partial z}\left(\frac{KA}{C}\frac{\partial s}{\partial z}\right) = Q\ . \quad (4)$$

This is a PDE for s only; we need only to define the transport coefficients, K and C, the cross sectional area A, and the source term Q for the model to be well defined.

Transport Coefficients, Area, and Sink Term

The hydraulic conductivity, K(s), and the hydraulic capacitance, C(s), are both dependant on saturation and are generally determined through experimentation. A primary goal of this research is to compare the results of this model to the results found by Chuang et al. [3], so for that reason we use here the expressions for these parameters described in their paper. For

hydraulic conductivity they proposed the Weibull function

$$K(s) = K_{max}\,exp\left[-\left(\frac{-\rho g \psi}{p_o}\right)^{\beta}\right],$$

where

$$\psi(s) = \psi_o\left[1 - \left(\frac{S_{max}}{s}\right)^{1/n}\right].$$

Here, $K_{max} = 5.36 \times 10^{-7}\ m/s$, $\quad p_o = 6.8 \times 10^6$, $\beta = 3.5$, and $\psi_o = 2.93 \times 10^5$. s_{max} represents the maximum possible saturation, and is equal to 0.5735.

For the hydraulic capacitance, Chuang et al. proposed the equation

$$C(s) = \frac{n\,s_{max}}{\psi_o}\left(\frac{s}{s_{max}}\right)^{(n+1)/n}$$

with $n = 400$ as a fitting parameter.

Chuang et al. also provide tree diameter, D, as a function of height,

$$D = 0.129\,exp(-1.42z/H)$$

where H is the height of the tree, in this case 6.7m. This is easily converted to an equation for A,

$$A(z) = \pi\left(\frac{0.129}{2}exp(-1.42z/H)\right)^2.$$

Lastly, the sink term, $Q(z,t)$, is given by

$$Q(z,t) = -\frac{l\,E}{\rho\,A}\ . \quad (5)$$

Here $l(z)$ (m^2/m) is the leaf area density and represents the quantity of leaf area per unit length of trunk at a given height, and is given by

$$l(z) = \frac{91.9}{H}\,sech^2\left(\frac{6\,z}{H} - 2.4\right).$$

This equation fails to consider localized details, such as the exact locations and sizes of the branches which we assume are the means by which water is extracted from the tree, but we

assume that these variations will have little effect on the behaviour of the tree on the larger scale.

$E(t, z)$ $(kg/m^2 s)$ is the transpiration rate, and gives the rate at which water leaves the tree per unit area. Chuang et al. calculated this as a result of their paper; however we instead use a sine curve as an approximation. E seems to vary with the quantity of daylight, so that it is high in midday and nearly zero at night. The sine wave provides a convenient approximation for this, although it should be noted it fails to capture some of the characteristics of the actual curve, particularly during the night where the actual transpiration rate remains close to zero for several hours. The equation for the transpiration rate is

$$E(t, z) = \varepsilon \, \lambda \, sin^2(\pi \, t/\tau), \qquad (6)$$

with fitting parameter $\varepsilon = 10^{-5}$ and period $\tau = 8.64 \times 10^4$, the number of seconds in a day.

$\lambda(z)$ is a scaling term introduced to give a height dependence to E, as necessary to match experimental data. This dependence likely exists due to leaves lower in the tree being shaded by those above them, resulting in reduced effective light and a lower transpiration rate. The equation for λ which was found to provide results most similar to experimental data taken from [1] is

$$\lambda(z) = 2.3 \, atan(10 \, z + 1.67).$$

Boundary Conditions

In addition to (3), we enforced boundary conditions at the roots ($z = 0$) and at the crown ($z = H$).

We assume that at all times the roots are able to uptake however much water is required by tree, and therefore that a reasonable boundary condition at the bottom of the tree would be maximum saturation in the wood,

$$s = s_{max} \, |_{z=0}. \qquad (7)$$

At the top of the tree, we impose a limitation on the flux. It would make little physical sense to suppose that sap flows out of the top of the tree (and not through the leaves, a possible sink which is already included in the source term), so we enforce a zero flux condition. Setting the flux equal to zero, we find our upper boundary condition

$$j = -KA\frac{\partial}{\partial z}(\psi + z) = -KA - \frac{KA}{C}\frac{\partial s}{\partial z} = 0 \quad (8)$$

$$\frac{1}{C}\frac{\partial s}{\partial z}\bigg|_{z=H} = -1.$$

Turning attention from Chuang et al., we consider another possible upper boundary condition. In the case of a maple tree, the area of interest might be only the lower trunk. In this case there would be little or no branches within the area of interest, and the sink term (5) would be equal to zero. Instead of a distributed sink term, the sink could be incorporated into the upper boundary condition by setting the flux to some positive value, j_o

$$j = KA + \frac{KA}{C}\frac{\partial s}{\partial z}\bigg|_{z=H} = j_o(t).$$

Nadezhdina Et al. [4] provide data for sap flux which we can use to find an approximation of j_o. Their data shows $6 \; cm^3/cm^2 hr = 2.16 \times 10^{-7} \; m^3/m^2 s$ is a typical peak value, so we use for j_o

$$j_o = 2.16 \times 10^{-7} \, sin^2(\pi \, t/\tau). \qquad (9)$$

Discretization

To solve for the saturation s throughout the tree, (4) was first discretized in space and then the resultant system of ODEs was solved over time.

We divided the tree, which covers the vertical length [0,H], into $N = 100$ equally spaced cells of length $\Delta z = H/N$. We consider only the values for saturation at the centers of these cells, which are taken as approximate representative averages of the saturation throughout the subdivision.

We thus are considering the points $z_1, \ldots, z_N$, where $z_i = \left(i - \frac{1}{2}\right)\Delta z$, on which are defined the N variables $s_1, \ldots, s_N$, where $s_i(t) = s(t, z_i)$.

The discrete formula for (4) is,

$$
\frac{ds_i}{dt} - \frac{1}{A_i \Delta z}\left(K_{i+\frac{1}{2}}A_{i+\frac{1}{2}} - K_{i-\frac{1}{2}}A_{i-\frac{1}{2}}\right)
$$

$$
- \frac{1}{A_i(\Delta z)^2}\left[\frac{K_{i+\frac{1}{2}}A_{i+\frac{1}{2}}}{C_{i+\frac{1}{2}}}(s_{i+1} - s_i)\right.
$$

$$
\left. - \frac{K_{i-\frac{1}{2}}A_{i-\frac{1}{2}}}{C_{i-\frac{1}{2}}}(s_i - s_{i-1})\right] = Q_i \qquad (9)
$$

Here the spatial derivatives have been replaced by central difference approximations,

$$
\frac{df(z)}{dz} = \frac{1}{\Delta z}\left(f(z + h) - f(z - h)\right).
$$

The subscripts $i \pm \frac{1}{2}$ on A, K and C variables mean that they are to be defined on the cell boundaries. We give these values as

$$
A_{i\pm\frac{1}{2}} = A\left(s_{i\pm\frac{1}{2}}\right) = A\left(\frac{s_{i\pm1} + s_i}{2}\right)
$$

And similarly for K and C.

The boundary conditions describe conditions at the cell edges at the limits of the defined area. To derive a discrete formulation of these we need to consider two additional imaginary cells, each also of length Δz, at the lower and upper boundaries of the tree. We call the saturations at the midpoints of these cells s_0 and s_{N+1} respectively.

We find the discrete form of the lower boundary condition, (7), by noting that

$$
s_{1/2} = (s_0 + s_1)/2 = s_{max}
$$

Simplifies easily to

$$
s_0 = 2\, s_{max} - s_1 \ . \qquad (10)
$$

The upper boundary condition, (8), becomes

$$
\frac{1}{\Delta z\, C_{N+\frac{1}{2}}}(s_{N+1} - s_N) = -1
$$

Or

$$
s_{N+1} = s_N - \Delta z\, C_N \ . \qquad (11)
$$

Note that in (11) we have defined C using the interior value s_N rather than the boundary value, reducing the solution at this point to first order accuracy. The benefit of this is in keeping the equation linear for s_{N+1}, and the loss in accuracy at this single point should have negligible effect on the results for the interior points.

In summation, the system of ODEs given by (9) applied to each s_i along with (10) and (11) give N+2 equations for N+2 variables. We wrote the equations out in a Matlab code and solved them using a standard ODE solver, ode15s, providing ourselves with values for saturation at each height for each time step over the simulations duration.

We used a time step of one hour, and used a total duration of six days, although in each case results of the first day were identical to those for the final day – the system quickly arrived at the resultant behaviours independent of the initial conditions used.

Results

Computing the results of the system of ODEs, we found values of saturation at each height for each time step. The available experimental data was given as flux measurements, so in order to make comparisons we then used (7) to compute flux measurements at each height and time.

Some sets of the flux measurements were multiplied by $\rho/(A(z))$ in order to change the units from m^3/s to kg/(m^2 s) in order to match units with available data for comparison.

The first goal was to find a steady state, reached with a constant zero transpiration. This was done primarily as a starting point from which additional complexity could be added, but it does highlight what would happen during extended periods of zero transpiration, such as during winter, under this model. The motion here was driven only by diffusion and gravity. The result (not shown) was a linear decrease in saturation with height, and a near zero flux throughout the tree.

Following this, we set the transpiration to a small positive constant value (10^{-6}), introducing a constant driving pressure to the system. Next the transpiration rate was set equal to the sine function in (6), with $\lambda=1$ so that it varied with time but not height. This gave the system daily fluctuations as would be expected in a realistic model.
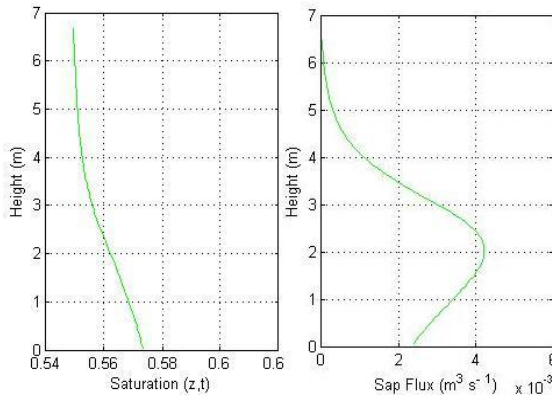
Fig. 1: Saturation and flux data plotted against height, data calculated with constant transpiration value.
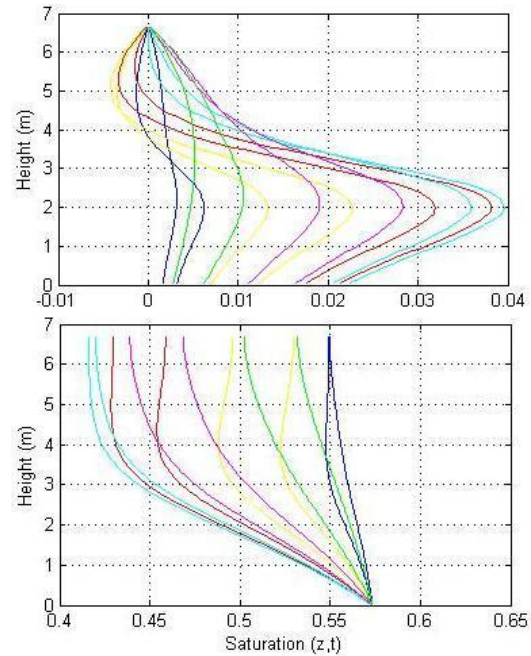
Fig. 2: Saturation and flux data plotted against height, transpiration value included time dependency as in (6) but no height dependency.

Finally, we added height dependence by taking values of peak sap flux as reported in [1] at different heights in the tree and fitting an arctangent to the data, a choice made after extensive testing of different possible fits. The arctangent best captured the steep incline necessary to capture the behaviour of the sap flux.
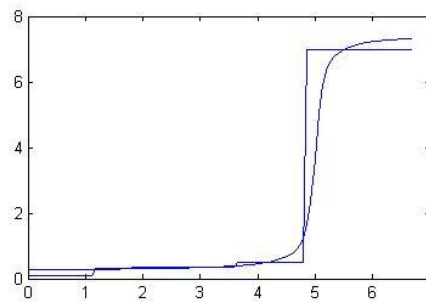

Fig. 3: Fit of arctangent used to provide vertical scaling to transpiration.

The result of this dependence was a shift from fluctuations of a fairly constant shape to a system which is bimodal during the midday and almost parabolic at night.
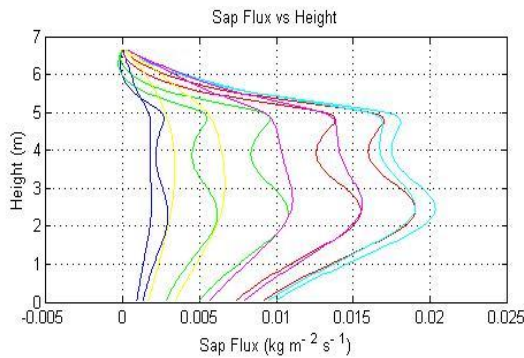

Fig. 4: Flux data plotted against height, calculated with a transpiration dependant on both time and height.

These results are an excellent qualitative match to the measured data presented by Chuang et al. [3]. It should be noted that the time of day in figure 5 does not match the same times as in figure 7. This is a problem with the inaccuracy in our approximation for *E*. Chuang et al. calculated a transpiration rate that remained at nearly zero for several hours during the night. (6) instead has E only briefly reach zero during the night, as a result the flux does not reach values quite as low as seen by Chuang et al. However, the transpiration rate does drop sufficiently for us to observe the relaxation phenomenon resulting in the single node.
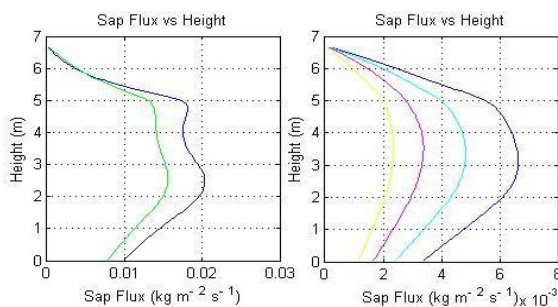

Fig. 5: Sap flux plotted against height. The left plot has a curve for 12 to 4pm, and the right 10pm to 1am.
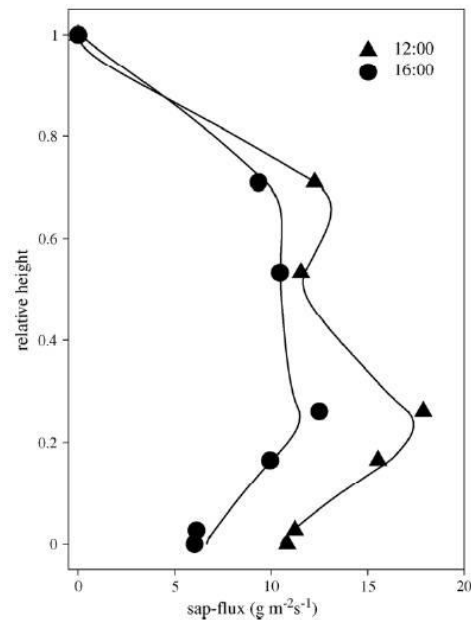

Fig. 6: Taken from [3], experimentally measured midday flux data vs relative height.
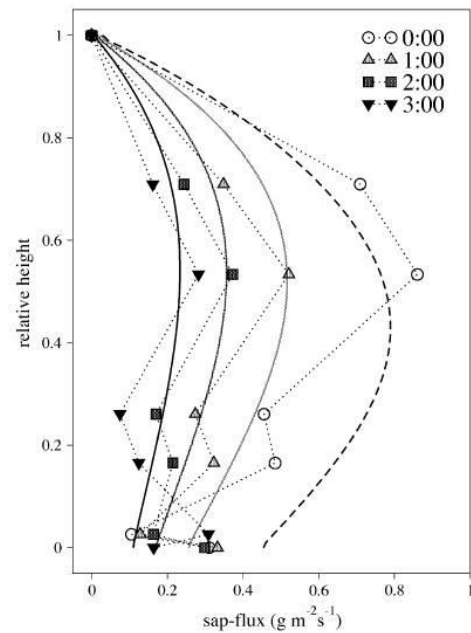

Fig. 7: Taken from [3], experimentally measured night time flux data vs relative height.

Physically, each mode shown during midday can be explained. The lower mode, present before height dependence, is simply a result of l; this peak is at the location where l is greatest and therefore Q is high. The higher node is a result of the transpiration E being much greater at the top of the tree, a result of multiplying E by λ. By adding a small constant to λ and therefore reducing the shrinking effects on the lower regions of the tree, or subtracting a constant to increase the effect, it is possible to make either node dominant. Fitting parameters were chosen for λ such that the final flux matched experimental data.
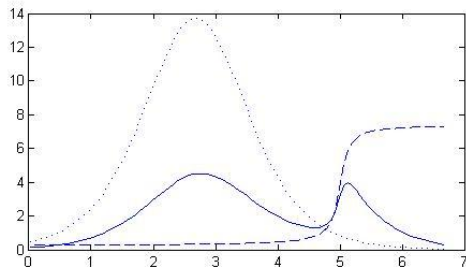

Fig. 8: This plot shows the leaf area per stem length (dotted line), the transpiration scaling coefficient (dotted line), and their product (solid line). The bimodal characteristic is clearly visible here.
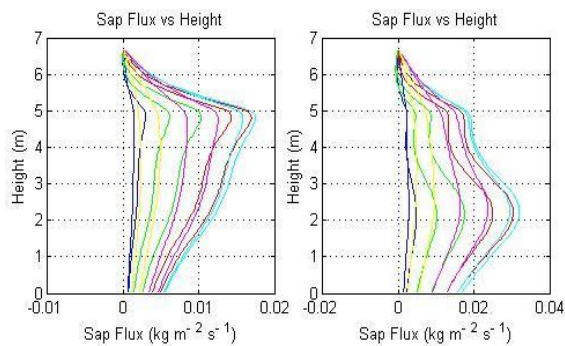

Fig. 9: Demonstration of emphasizing either mode by adding or subtracting a small constant to λ. Here the left plot was made by subtracting 0.2 from λ, and the right by adding 0.3.

Next, we altered the model by removing the distributed sink term and changing the zero flux upper boundary condition to incorporate the transpiration. The formulation for A was also changed; when considering a section of trunk only it no longer makes sense to have the peak be such a small fraction of the area of the base – instead the area was taken to be a constant value equal to $A(z = 0)$. Again, we started with a constant transpiration. In this case, flux throughout the trunk was very nearly constant, as would be necessary for the conservation of water volume.

We then reintroduced the time dependence of the transpiration by setting the upper boundary flux to (8). Initially the size of the flux was kept very small, then was slowly increased, allowing us to observe the transition.
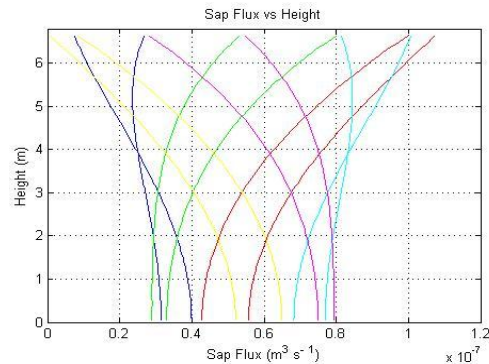

Fig. 10: Sap flux plotted against height. Here the sink term has been incorporated into the upper boundary.

Here we can see that as the system is forced from the top of the trunk area, the sap flux trough the rest of the tree simply lags behind; the flux at any height is approximately a sinusoidal function of time. This is a simpler result than the patterns observed with a distributed sink term, and therefore this would likely be the better implementation of the sink term in cases where we are interested only in the trunk section of a tree.

References

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