Mathematics For Industry: A Personal Perspective

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1 Introduction

"I am an industrial mathematician."

When asked to identify my profession or academic field of study, this is the most concise answer I can provide. However, this seemingly straightforward statement is commonly greeted by a blank stare or an uncomfortable silence, regardless of whether I am speaking to a fellow mathematician or a non-mathematician. I usually follow up with the clarification: "I am an applied mathematician who derives much of my inspiration from the study of industrial problems that I encounter through collaborations with companies." This dispels some confusion, but unfortunately still leaves a great deal open to interpretation owing to the vagueness of the words *mathematics*, *industry* and *company*, each of which covers an extremely broad range of scientific or socio-economic activity.

To those academics who actually work in the field of *industrial mathematics* (and whose "perspective" referred to in the title is the focus of this article) this ambiguity is familiar and untroubling. However, for anyone less acquainted with the work of industrial mathematicians, some clarification is desirable especially for anyone who might be considering entering the field. This essay therefore aims to shed light upon the nature of research being done at the interface between mathematics and industry, paying particular attention to the following questions:

What is industrial mathematics? for which a proper answer depends sensitively on whether the mathematician in question is employed in industry or in academia, since these two sectors have demands, time scales and motives that are largely incommensurate. Because I am a university professor, this article naturally emphasizes the viewpoint of the academic mathematician.

Where is industrial mathematics? I presume that most readers will take no offense with the statement "mathematics is everywhere" which reflects the fact that math pervades the everyday operations of many organizations. Be that as it may, it is certainly not the case that every organization making use of mathematics is also a source of novel, challenging (and ultimately publishable) mathematical problems for an academic mathematician. The quest for interesting problems is really the *raison d'être* for the industrial mathematician and so I will discuss how a wide range of non-academic organizations spanning many industry sectors can give rise to stimulating mathematics … oftentimes in unexpected places.

How does one do industrial mathematics? which has two facets: the first concerns the actual process of mathematical problem-solving (which I won't attempt to address here); and the second relates to the more fundamental question of how one goes about initiating an industrial collaboration in the first place. The latter is commonly the most difficult stumbling block for anyone attempting to enter the field, owing at least in part to the disparities in culture and basic motivations that exist between academia and industry.

Why (or more precisely, what value is there in doing) industrial mathematics? There are obvious and well-documented economic and strategic benefits to industry in deploying advanced mathematical solutions to difficult problems [8, 12, 14, 16, 22]. However, perhaps more relevant to many readers of this article is the complementary question of what benefits academic mathematicians can derive from industry collaborations, and how such projects can enrich their careers in terms of research, teaching, training and mentorship, and personal professional development.

I will attempt to answer these questions by means of several case studies drawn from my own experience in tackling mathematical problems from industry. As a result, this account is necessarily a personal perspective that is influenced to a large extent by my own research interests bridging partial differential equations (mainly of parabolic type), scientific computing (mainly finite volume methods) and fluid dynamics. I should confess at the outset that the contents of my own *mathematical toolbox* place significant limitations on the class of problems and industrial applications that I am capable of making meaningful contributions to. I also make no claim to be the first person who has wrestled with the questions stated above. Indeed, there are many excellent accounts of the pressing need for mathematics in industry and the important role that mathematicians can play in solving real-world problems [2, 5, 17, 18, 25]. In particular, a paper by Bohun [3] appeared recently (while this article was under review) that addresses related issues regarding the role and importance of the field of industrial mathematics, and conveys the underlying message that the field is more of an art than a well-defined scientific discipline.

2 What is industrial mathematics?

Before attempting to answer this first question, it is most helpful to introduce working definitions for *industry* and *mathematics* in order to clarify how these terms are being used in the context of this essay:

Industry: Most everyday uses of the term *industry* follow the dictionary definition that emphasizes either "the process of making products by using machinery and factories" or "a group of businesses that provide a particular product or service" [15]. Here I assume a much broader and more inclusive definition that encompasses all *end-users* of mathematics, which includes not only public and private sector companies but also federal/provincial/-municipal government agencies, hospitals, foundations, not-for-profits, etc. By extension, I will use the word *company* to refer to any such non-academic organization. This inclusive definition is consistent with that advocated in the OECD Global Science Forum's *Report on Mathematics in Industry* as "any activity of economic or social value, including the service industry, regardless of whether it is in the public or private sector" [19].

Mathematics: I will take a similarly broad view of *mathematics* to refer to any branch of pure or applied mathematics, statistics or computational science[1] that can be employed to solve a problem of interest to industry. Moreover, I will refer primarily (although not exclusively) to advanced mathematics that is in the purview of academic mathematicians, and which relates to the the underlying mathematical structure of a problem as well as the derivation of exact or approximate solutions that are demonstrably (provably?) correct. This is in contrast with routine applications of well-known mathematical techniques, which is more commonly the approach used by engineers and other applied scientists.

2.1 Mathematics for industry

With the above definitions in hand, the term *industrial mathematics* (IM for short) refers to any mathematical treatment of problems that arise from industrial applications. Because this can encompass such a huge spectrum of mathematics, IM is not so much a field or sub-field of its own as it is a mathematical *modus operandi*.

In his 2013 CAIMS-Mprime Industrial Mathematics Prize lecture, Brian Wetton stated succinctly that "industrial mathematics is mathematics that industry is willing to pay for" [26]. I would like to propose an expanded definition of industrial mathematics that includes three classes of activity:

Mathematics IN industry (or MII) which refers to mathematics that is done by non-academic mathematicians who work as employees of a company. This is the perspective taken in the MII report commissioned by SIAM [22] and also in articles such as [2, 5].

Mathematics FOR industry (or MFI) which is performed by academic mathematicians within universities as part of a collaboration with a company. There is no necessity that the company is actually funding the research, but rather that the development of the mathematics is driven by needs of the partner organization and that there is a two-way collaboration between academics and industrialists. The terminology MFI is consistent with Tayler [25].

Mathematics INSPIRED BY industry (or MIBI) by which I refer to the mathematical analysis of problems that arise in an industrial setting, but where all of the mathematical work is performed in an academic setting that is largely isolated from the demands, pressures and constraints of industry and industry-university collaborations. For example, work that begins as MII or MFI can still be a rich source of more industrially-motivated problems

[1] Since numerical algorithms are simply the expression in computer code of a mathematical idea or procedure.

that are not of direct interest to the indunstry partner but nonetheless give rise to academic publications or other activities that fall into the realm of MIBI.

The boundaries between these three classes are fuzzy and there is significant overlap. Nevertheless, this division provides a useful high-level separation of industrial mathematics that may be represented symbolically as

$$\text{IM} = \text{MII} \cup \text{MFI} \cup \text{MIBI},$$

by making use of the acronyms above and slightly abusing set notation. This terminology is my own, and the reader should be aware that it conflicts with common usage in certain segments of the mathematical community. For example, in the UK and many parts of Europe, the term *maths-in-industry* [18] is often used to refer to the last two classes of activity (MFI $\cup$ MIBI), which is actually the complement of MII according to the SIAM definition! Furthermore, a *Journal of Mathematics in Industry* and a *European Consortium for Mathematics in Industry* (ECMI) have scope that effectively covers all of IM.

Using the above definitions, it is now possible to clearly identify the focus of the remainder of this article as the subset of IM that arises in the context of direct collaborations between academic mathematicians and industry and hence sits squarely within the second class of activity called *mathematics for industry* (MFI).

2.2 The industrial mathematician

Some insight into the desirable characteristics in an industrial mathematician is afforded by the following quotations:

> "Almost by definition, industrial research is interdisciplinary" (Ockendon, [18]).
>
> "[Real life mathematics] requires barbarians: people willing to fight, to conquer, to build, to understand, with no predetermined idea about which tool should be used" (Beauzamy, [2]).

The first author indicates that, unlike in university departments, problems cannot be neatly categorized into academic disciplines; rather, real industrial problems tend to be highly interdisciplinary in nature so that their solution requires expertise that crosses disciplinary boundaries. Therefore, industrial mathematicians must be willing and able to synthesize knowledge from other fields and formulate this knowledge mathematically.

The second quotation refers to the need for a diverse mathematical toolkit, since industrial problems are typically not clean textbook examples in which a single mathematical technique or numerical algorithm is sufficient to obtain a solution. As a result, the industrial mathematician must also be well-versed in more than one area of mathematics and should be flexible in their approach to problem-solving, displaying a willingness and ability to use whichever technique is most appropriate. For example, it is common for academic applied mathematicians to build a career on becoming an expert in a certain method (or class of methods) and then hunting for problems that they can apply their method(s) to. This approach of "choosing the problem to suit the method" is in stark contrast to the industrial mathematician who is typically forced to "choose the method best suited to solving the problem."

These qualities of breadth and agility, both within mathematics and outside of the discipline, are essential for a successful industrial mathematician. As the case studies in the next section will demonstrate, one of the primary rewards for investing the time and effort required to develop these qualities is to provide access to a wide variety of challenging and frequently novel mathematical problems.

3 Case Studies

The remaining three questions posed in the Introduction concern the "where, how and why" of industrial mathematics, and will be addressed through examples. In particular, I have chosen five case studies from among my own past and current industrial projects to illustrate the breadth of problems encountered in industrial settings as well as the stimulating mathematical questions that can arise. These examples exhibit tremendous diversity in a number of aspects:

- *Mathematical techniques:* ranging over simple algebra and trigonometry, partial differential equations, finite volume methods, integral transforms and Green's functions, Bayesian inversion, asymptotic analysis, and homogenization theory.

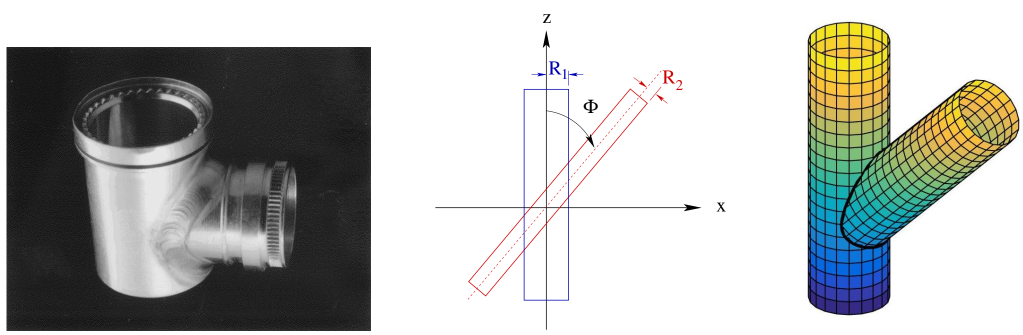
- *Scientific disciplines:* including engineering, image processing, electrochemistry, fluid mechanics, atmospheric science, and plant physiology.
- *Industrial partners:* representing industry sectors of manufacturing, defense, automotive, alternate energy, mining, and agriculture. The companies also cover the entire size spectrum, ranging from small engineering consulting firms, to medium-sized R&D-intensive companies, multi-national corporations, and non-profit network of industry associations.

For each case study, I will only provide a brief overview of the mathematical problem and solution, instead emphasizing aspects of each problem that address the questions posed in the Introduction. Wherever possible, I will provide references in which the interested reader can find more mathematical details. I will also describe how the collaboration was initiated and how the project was funded, which in most cases was through some form of joint industry-government R&D funding scheme.

3.1 Robotic pipe welding: A consulting contract

The first case study arose from a consulting project with a Vancouver engineering firm and was initiated through a personal introduction to one of the partner engineers at a neighbourhood social event[2]. I was drawn into conversation with this engineer who had recently entered into a contract with a pipe welding machinery manufacturer to write the control software for a robotic welder. He was struggling with how to prescribe the motion of the welding torch for a general class of joints involving pipes of different radii and intersection angles.

The problem geometry is illustrated in Figure 1, which includes a photograph of an actual pipe joint with a 90° intersection angle, as well as a diagram showing the general case where two pipes with different radii (R_1 and R_2) are joined at an arbitrary joint angle (Φ). I realized immediately that this was a straightforward exercise in



(a) (b) (c)

Figure 1: (a) Photograph of a pipe weld with joint angle $\Phi = 90°$. (b) Two cylinders with radii R_1 and R_2 intersecting at an angle Φ. (c) The pipe weld solution is shown as a black line for values of $R_1 = 1$, $R_2 = \frac{9}{10}$, $\Phi = 45°$.

algebra and trigonometry that could be solved analytically and hence easily implemented in software. By writing a parametric description of the pipes in cylindrical coordinates (θ_i, z_i) as follows

$$\underline{\text{Pipe 1:}} \quad \begin{aligned} x &= R_1 \cos\theta_1 \\ y &= R_1 \sin\theta_1 \\ z &= z_1 \end{aligned} \qquad\qquad \underline{\text{Pipe 2:}} \quad \begin{aligned} x &= R_2 \cos\theta_2 \cos\Phi - z_2 \sin\Phi \\ y &= R_2 \sin\theta_2 \\ z &= R_2 \cos\theta_2 \sin\Phi + z_2 \cos\Phi \end{aligned}$$

and equating x, y and z components, the two intersection curves can be obtained in parametric form as

$$x = \pm\sqrt{R_1^2 - R_2^2 + R_2^2 \cos^2\theta_2}, \qquad y = R_2 \sin\theta_2, \qquad z = \frac{R_2 \cos\theta_2 \mp \cos\Phi \sqrt{R_1^2 - R_2^2 + R_2^2 \cos^2\theta_2}}{\sin\Phi}.$$

The solution derivation was a straightforward application of the Maple software package[3] and formed the basis of an article appearing in the *MapleTech* journal [23]. Some interesting implementation-related issues arose sur-

[2]This is an excellent example of how fruitful industrial connections can arise in the most unexpected situations!

[3]An expanded version of the Maple code was later posted on MapleSoft's Application Center at `http://www.maplesoft.com/applications/view.aspx?SID=3773`.

rounding automatic selection of the correct solution branch as well as preventing the welding torch assembly from interfering with the pipes at small joint angles.

This is the first and so far only consulting contract that I have been involved with. While the financial reward was attractive at the time (especially for a PhD student living on a student-sized salary) this reward has since been far surpassed by other more academic spin-offs. First of all, I have used this problem to great effect in my undergraduate teaching and outreach to high school students as a prime example of the potential value to industry of expertise in mathematics. Furthermore, I have subsequently been approached on a regular basis by other welding companies or engineering firms to request assistance in implementing the analytical pipe weld solution, when they encountered my web page or the MapleSoft Application site.

3.2 Land mine trip-wire detection: A study group problem

The next case study is an image processing problem that was brought to the second Industrial Problem-Solving Workshop (commonly known as a study group outside of Canada) held in Calgary in 1998 by the Pacific Institute for the Mathematical Sciences (PIMS). The problem posed by the industrial partner ITRES was to automatically identify land-mine trip-wires within an image that is cluttered by vegetation, terrain, man-made structures, etc. A sample image file is provided in Figure 2(a).

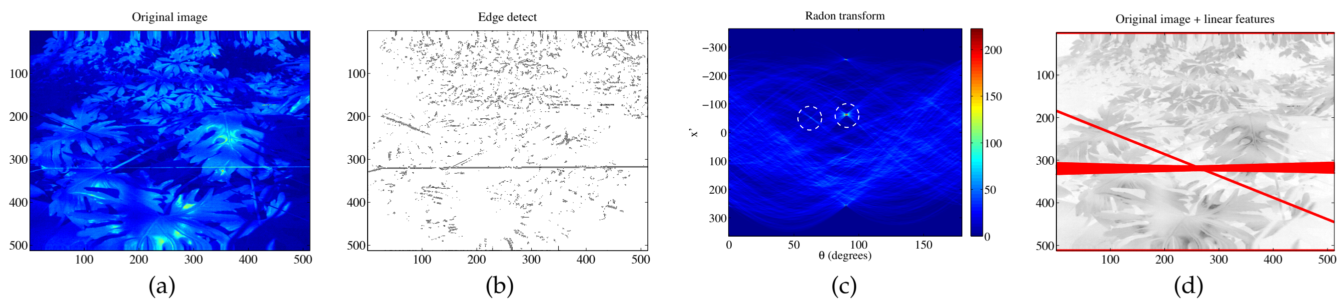


Figure 2: (a) Original image containing two linear features: a bright horizontal line and a less obvious dark oblique line running downward from left to right. (b) Preprocessed image after Laplacian and edge detection filters. (c) Radon transform image $U(r, \theta)$ with white circles around peaks that exceed the threshold T. (d) Linear features (corresponding to the two peaks) are overlaid on the original image.

Because trip-wires appear as linear features in an image, the team of academics tackling this problem were naturally led to consider the *Radon transform* that is a type of integral transform in which the integrals are taken over straight lines. If the intensity of a 2D image $\mathcal{I}$ is represented as a function $u(x, y)$ for $(x, y) \in \mathcal{I}$, then the Radon transform may be written as

$$U(\rho, \theta) = \int_{\mathcal{I}} u(x, y)\, \delta(x \cos\theta - y \sin\theta - \rho)\, dx\, dy,$$

where (ρ, θ) are polar coordinates and $\delta(\cdot)$ is the Dirac delta function. Because the equation $x \cos\theta - y \sin\theta - \rho = 0$ represents a straight line in the original image, the above integral has the effect that a linear feature with high intensity u generates a single point (ρ, θ) at which the transformed intensity U is correspondingly large. Hence, the largest values of the Radon transform correspond to the primary linear features in the original image.

When applying the Radon transform by itself to the test images provided, it was not possible to reliably detect trip-wires. Instead, it was necessary to introduce a pre-processing step in which two filters accentuate regions where the intensity changes rapidly. The resulting algorithm is summarized as Algorithm 1. The images obtained after steps 2 and 3 of the algorithm are shown in Figure 2(b,c) and the final linear features identified by the algorithm are overlaid on top of the original image in Figure 2(d).

These results were presented at the conclusion of the Workshop [4] after which we had no further communication with the industrial partner. This is perhaps not unexpected for a problem having such obvious military applications. However, a publication with co-authors from ITRES did appear a few years later in a conference proceedings [1], suggesting that our results were eventually exploited by the company. This is another problem that has been highlighted in many outreach activities as an example of the potential value of mathematics in industry[4].

[4] Two examples of such outreach activities are at http://plus.maths.org/content/saving-lives-mathematics-tomography

Algorithm 1 Detecting linear features in an image using the Radon transform.

1: Apply a *Laplacian filter* to each point in the raw image u to accentuate regions of high curvature.
2: Perform *edge detection* on the filtered image.
3: Calculate the *Radon transform* U of the edge-detected image.
4: Find all points (ρ, θ) that *exceed a threshold*, $U(\rho, \theta) > T$.
5: Apply the *inverse Radon transform* of each point found in step 4 to determine the linear features of interest.

3.3 Mathematical modelling of PEM fuel cells: A multi-university collaboration

The next case study derives from a long collaboration with Ballard Power Systems, an R&D-intensive company based in Burnaby, British Columbia that is a world leader in the development of polymer electrolyte membrane (PEM) fuel cell technology. This activity was part of a large multi-disciplinary, multi-university project funded jointly by Ballard and Mitacs (a Canadian research network dedicated to funding research projects at the interface between mathematics and industry). I joined this project as a postdoctoral fellow in 1998 and continued as a faculty investigator until roughly 2005.

The primary aim of the project was to develop mathematical models of fuel cell processes and components that could support the Company's internal R&D efforts. Space doesn't permit a detailed description of the wide range of problems investigated and so I will only provide a high-level overview of a few main modelling efforts:

- A 2D model of heat and mass transport in a *unit cell* that captures flow of a multi-component, multi-phase (gas-liquid) mixture through a porous fuel cell electrode.
- Reduced models that focus solely on electrical coupling between unit cells and can be used to validate the results of cyclic voltammetry and other similar diagnostic tests.
- Reduced models of an entire fuel cell that consist of algebraic equations, ODEs and simplified PDEs. The goal here was to construct models that capture the essential physics but are simple enough to permit development of efficient algorithms for simulating fuel cell stacks (consisting of hundreds of unit cells connected in series).
- A detailed model of the membrane that lies at the heart of the PEM fuel cell, and whose complicated physics is still not very well understood.
- Nano-scale models for the catalyst layer, which is a complex multi-scale composite material consisting of a porous electrically-conducting electrode, impregnated by the ionomer membrane and platinum catalyst.

These and other projects led to a host of academic publications (a few of which I was personally involved with) that are detailed in a comprehensive review article on the mathematics of PEM fuel cells co-authored by the project's principal investigators [20].

Before closing, I want to draw attention to the issue of *intellectual property* (or IP) which is an important aspect of many industrial collaborations. Ballard's business depends critically on their PEM fuel cell-related inventions and patents, and so they are naturally extremely sensitive to IP. It was initially a struggle to craft an IP agreement that was acceptable to the Company and the universities involved, and the ultimate solution was both elegant and surprisingly simple: the mathematics and the algorithms belonged to us mathematicians, whereas the data and any simulations run on the data were owned by the Company. This left us relatively free to publish mathematical results in the open literature with only a short delay for the Company to review drafts of papers. I have come to believe that mathematics has a significant advantage over other disciplines (such as engineering) in the sense that *mathematical IP* is often considered by companies to be of less obvious concern ... although perhaps it should be!

3.4 Atmospheric pollutant dispersion: Graduate student internships

I will next describe a collaboration with Teck, which is a multi-national mining corporation headquartered in Vancouver. This project was initiated in 2005 by a "cold call" to SFU from an Environmental Superintendent at Teck's lead-zinc smelter in Trail, BC[5]. The Company had taken a series of ground-level measurements of particulate material accumulated in *dust-fall jars* around the site, and wanted to know how they could exploit this data to

and `http://www.whydomath.org/node/tomography/landmine-wires.html`.

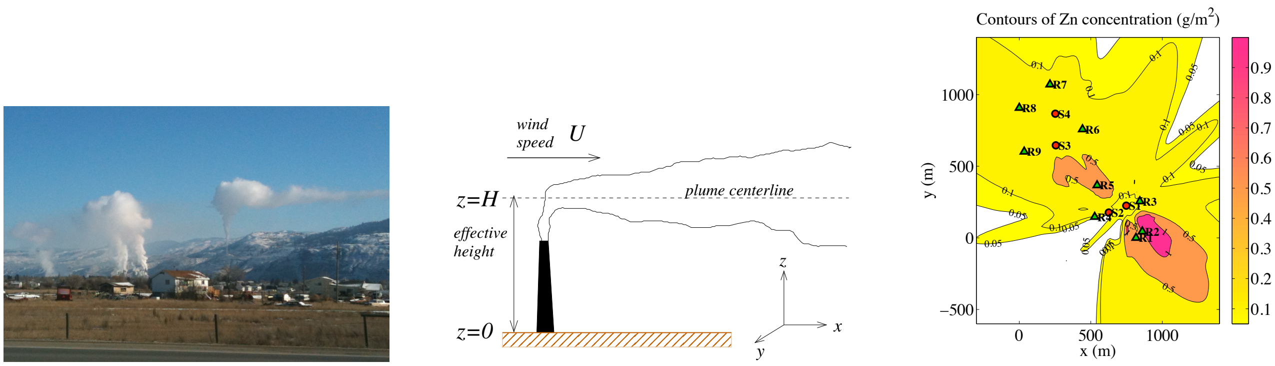
[5]In my experience, such cold calls (in either direction) are very rarely successful, which makes the manner in which this collaboration was initiated particularly notable.

estimate the corresponding airborne contaminant emissions from a certain set of sources that they were unable to measure directly. The Company's goal was to provide additional backing for (and perhaps even improvements to) the engineering estimates of emissions that Teck is legally obliged to report annually to Environment Canada.

Inspired by the fledgling Mitacs Graduate Internship Program (now known as *Mitacs Accelerate*) I convinced Teck to co-fund a Master's student intern (E. Lushi). Within a few short months, she came up with a surprisingly simple approach based on the *Gaussian plume solution* for contaminant concentration

$$C(x,y,z) = \frac{Q}{2\pi U \sigma_y \sigma_z} \exp\left(-\frac{y^2}{2\sigma_y^2}\right) \left[\exp\left(-\frac{(z-H)^2}{2\sigma_z^2}\right) + \exp\left(-\frac{(z+H)^2}{2\sigma_z^2}\right)\right],$$

which is incidentally a very nice illustration of the application of Green's functions or Laplace transform methods to solving the advection-diffusion equation. Referring to the problem geometry depicted in Figure 3(a), the symbols appearing in this equation are the emission rate Q, source height H, constant wind velocity U (in the x-direction), and dispersion coefficients $\sigma_{y,z}$. A sample ground-level zinc concentration map is shown in Figure 3(b). Taking advantage of the key fact that concentration (and hence also deposition) is linear in Q, multiple sources may be linearly superimposed and integrated in time to yield cumulative depositions that afford a direct comparison with dust-fall measurements. The original question of determining emission rates based on deposition measurements is an inverse problem that can therefore be reformulated as a simple application of linear least squares.



(a) (b) (c)

Figure 3: (a) Plumes generated by a pulp mill in Kamloops, BC. (b) Schematic of a contaminant plume emitted from a source at location $(0, 0, H)$ in a steady wind with velocity $(U, 0, 0)$. (c) Contours of ground-level zinc concentration over the smelter site, with four sources (Sn, red circles) and nine dust-fall measurements (Rn, green triangles).

This work has continued through four follow-up internships with Teck, giving rise to a number of publications [11, 13, 24] and a Master's thesis [10] in which a finite volume discretization of the advection-diffusion equation was used to validate the results from the inverse problem. Furthermore, a PhD student (B. Hosseini) is currently studying more advanced Bayesian techniques for source inversion problems and their application to both atmospheric and groundwater contamination on the Teck smelter site.

3.5 Maple sap exudation: A sweet project in search of a partner

This last case study has an interesting background story that is worth relating. My first exposure to the science behind maple sap was stumbling across an article in the tree physiology literature that described a long-standing controversy over the bio-physical mechanisms driving sap exudation during the spring thaw in trees like maple[6]. Having spent my childhood in prime sugarbush territory in southern Ontario, I was immediately intrigued by this quintessentially Canadian problem begging for a mathematical solution. However, without any existing connections to tree physiologists or the maple syrup industry, I took the problem no further (although as an inveterate problem-hunter I naturally filed it away for future reference). As luck would have it, a few years later I encountered a call for proposals to the Research Fund of the North American Maple Syrup Council – a non-profit network

[6] *Exudation* refers to the unusual ability of certain deciduous trees such as maple, birch and walnut to generate stem pressure in a leafless state (i.e., in winter or early spring when transpiration is inactive). When a sugar maple tree is *tapped* during the harvest season, the exudation pressure is sufficient to cause the sweet sap to seep out in large enough quantities that it can be harvested commercially.

of local industry associations representing mostly small maple syrup producers. After submitting a proposal for a modest research grant in 2011, and leveraging funds from a Mitacs postdoctoral fellowship program, I was thrilled (indeed, even a bit surprised) when both proposals were successful. The mathematical study of maple sap was on!

The first phase of the project involved developing a microscale model for the *Milburn-O'Malley process*, which is a freeze-thaw mechanism that operates at the scale of individual wood cells and is currently the most favoured hypothesis to account for maple's ability to generate positive stem pressure. This process involves an interplay between liquid/frozen sap and trapped gas within two types of wood cell called fibers and vessels (refer to Figure 4):

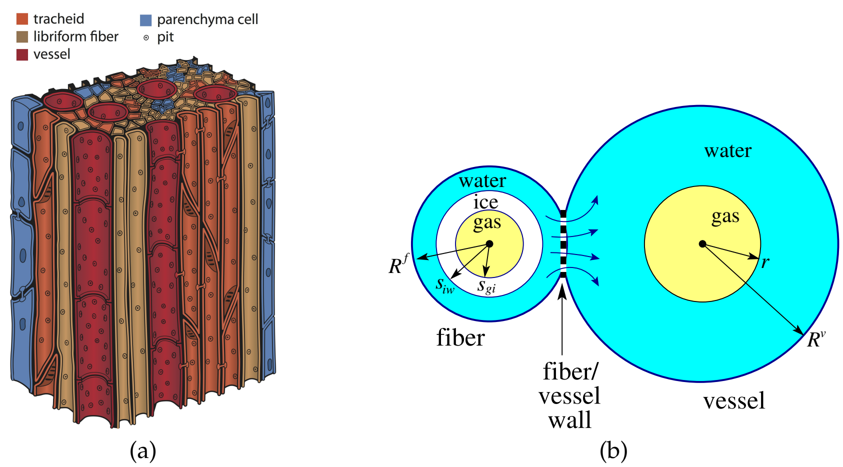


Figure 4: (a) 3D view of hardwood structure, showing vessels surrounded by (libriform) fibers and tracheids. (b) 2D cross-section through a fiber–vessel pair during the thawing process, showing the liquid, gas and ice regions, along with the phase interfaces.

- In late autumn as temperatures drop below freezing, ice crystals form on the inner wall of the mostly gas-filled fiber cells, which compresses the gas trapped inside.
- During the sap harvest season when temperatures rise above freezing, the ice layer melts and the pressure of the trapped gas drives the melted sap through the fiber walls into the vessels, hence causing the elevated pressures observed during sap exudation.
- Pressures are sufficiently high that significant quantities of gas dissolve within the sap.
- Flow between the fiber and vessel passes through a selectively permeable membrane that sustains an osmotic pressure difference.

Along with postdoctoral fellow M. Ceseri, I developed a mathematical model of the thawing half of the process that consists of a coupled system of nonlinear differential-algebraic equations incorporating heat diffusion, porous media flow, Stefan conditions for phase interface motion, osmosis, gas dissolution, and mass conservation. By employing a quasi-steady approximation for temperature (along with other simplifying assumptions) we obtained a much simpler system of three ODEs governing the location of the various gas-liquid interfaces [6] that captures the expected exudation behaviour. Work is currently underway (with a second postdoc, I. Graf) to develop a corresponding model for the freezing half of the process.

This project is a superb example of how an industrial collaboration can be a rich source of stimulating mathematical problems. For example, the phase interface motion in our fiber-vessel model exhibits a non-trivial separation of time scales that can be captured very accurately using an asymptotic analysis [7]. Furthermore, the multi-scale wood cell structure lends itself naturally to methods of periodic homogenization [9]. These and other mathematical analyses support our two ultimate goals of (1) addressing outstanding fundamental questions related to the biophysical mechanisms governing sap exudation, and (2) applying these insights to problems more relevant to the maple syrup industry such as developing improved sap harvesting strategies or explaining the impact of climate change on sap yields. There is certainly no shortage of mathematical problems related to maple sap that will keep the author and his students/postdocs busy for years to come.

4 Conclusions

I hope that the preceding examples have managed to convey some of my own enthusiasm for industrial problems, as well as convincing you of the opportunities available to mathematicians within the field of research I have termed *mathematics for industry*. Industrial collaborations can be a rich source of interesting and challenging mathematical research problems that lead to many spin-offs of an academic and pedagogical nature.

It is worth highlighting a few recurring themes in this article that lead naturally to some words of advice for academics who might be interested in undertaking work at the interface between mathematics and industry:

- Interesting and novel mathematical problems can arise in unexpected places, and the people best suited to recognize and exploit these opportunities are mathematicians who have an interdisiplinary training and a broad knowledge of mathematics.
- Serendipity plays an important role in identifying fruitful industrial collaborations. One must be vigilant for opportunities and also willing and able to communicate with (potential) partners in clear and non-technical language.
- There is a huge demand (and an even larger need, that is frequently unrecognized) for advanced mathematics in industry. It is often difficult for companies to engage with academics to gain entry to this expertise, and so anything that we as academic mathematicians can do to lower these barriers to access can be of huge mutual benefit. A few suggestions are to create a web page that explains your work in lay terms, to introduce yourself to non-academic participants at conferences, to attend trade shows or other events that attract industrialists, and to give public or outreach lectures.
- Be flexible, agile and willing to explore new areas. The time investment required to move into a new sub-field of mathematics or application area introduces some degree of risk and delay for an academic mathematician, but the potential rewards are substantial.
- Industrial collaborations can provide excellent opportunities for students and postdocs in terms of novel research projects and professional development. Problems arising from these collaborations are often suitable for incorporating into undergraduate classes or outreach activities as "modules" that poignantly illustrate the value of mathematics to industry.

Finally, let me close with a few words of inspiration regarding industrial mathematics from John Ockendon, who expresses his "confidence in the intellectual viability of an activity that always has and surely always will spring mathematical surprises at a rate that could never be matched by most academic mathematicians pursuing their trade in the traditional way" [18].

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