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# A rigorous bound on the vertical transport of heat in Rayleigh-Bénard convection at infinite Prandtl number with mixed thermal boundary conditions

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A rigorous upper bound on the Nusselt number is derived for infinite Prandtl number Rayleigh-Bénard convection for a fluid constrained between no-slip, mixed thermal vertical boundaries. The result suggests that the thermal boundary condition does not affect the qualitative nature of the heat transport. The bound is obtained with the use of a nonlinear, stably stratified background temperature profile in the bulk, notwithstanding the lack of boundary control of the temperature due to the Robin boundary conditions. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4896223]

## I. INTRODUCTION

Density variations underpin one of the fundamental driving forces for fluid dynamics in nature and engineering. From the earth's oceanic and atmospheric circulation to a pot of boiling water, the effects of convective phenomena pervade our lives, yet there remain open questions regarding the underlying physics (see Refs. 1 and 2 for a review). A mathematically tractable framework for convection driven by a prescribed temperature gradient was originally formulated by Rayleigh.[3] His simple model, with variations, has historically stimulated investigations in nonlinear dynamics[4] and the early development of amplitude[5] and modulation[6] equations, and has become a paradigm for studies in pattern formation in complex systems.[7] However, the nature of the dynamics in the limit of strong thermal driving is still a matter of ongoing experimental investigation and dispute regarding the possible transition to an asymptotic "ultimate" regime (see, for instance, Refs. 8 and 9 for recent examples). We will consider the effect of varying thermal boundary conditions on the heat transport for the simplified model of infinite Prandtl number convection, when the forcing applied to the system is asymptotically strong.

The quantity of most interest in the study of Rayleigh-Bénard convection is the enhancement of heat transport due to convection (relative to that in a stationary purely conductive state). This is quantified by the non-dimensional Nusselt number *Nu*, the ratio of the total to the conductive heat transport, and much effort has been put into finding the functional dependence of *Nu* on the non-dimensional measure of the force described by the Rayleigh number *Ra*, the material properties of the fluid described by the Prandtl number *Pr* (ratio of kinematic viscosity to thermal diffusivity), and the geometry of the system. Early theories developed to ascertain the effect of varying these parameters on the heat transport for sufficiently strong forcing yielded conflicting results.[10,11] A more comprehensive theory has subsequently been developed based on experimental data to predict the behavior of the heat transport and bulk flow Reynolds number.[12–15]

Starting with the insightful work by Howard[16] and Busse,[17] and continuing with the development of the background method by Doering and Constantin,[18] rigorous bounds on the Nusselt number

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have been shown for a variety of contexts and boundary conditions; see, for instance, Refs. 19 and 20 concerning variations in thermal boundaries, and Refs. 21–25 regarding the effect of stress-free versus no-slip vertical boundaries, as well as Refs. 26–32 for a discussion of the limiting and simplified case of infinite Prandtl number as described below. These results indicate that different velocity boundary conditions result in different scaling laws for the transport of heat, and that the transition from finite (and possibly small) *Pr* to infinite *Pr* also has a marked impact on scaling. On the other hand, both analytical bounds[19,20,33] and numerical simulations[34,35] suggest that there appears to be little noticeable difference in the *Nu*-*Ra* scaling when variations in the thermal boundary conditions are considered. In this paper we will extend the results of Refs. 19 and 20 to the case of infinite Prandtl number to indicate that, at least at the level of rigorous bounds, the scaling of the heat transport is the same (up to logarithmic corrections) irrespective of the thermal boundary condition at infinite *Pr*.

Convection at large Prandtl numbers is often well-approximated by assuming the Prandtl number tends to infinity; this limit is often proposed for some silicone oils, the earth's mantle and many gases under high pressure (see Refs. 36–38). This approximation, while difficult to imitate in the laboratory, greatly simplifies the analysis, slaving the velocity to the temperature (buoyancy) field. In particular, while the qualitative bound $Nu \leq C Ra^{1/2}$ reached in Ref. 16 and established rigorously in Ref. 18 (similar to the "ultimate regime" predicted by Kraichnan's original calculation[11]) for no-slip, fixed temperature convection in three dimensions has not yet been improved on for arbitrary *Pr* (although the prefactor *C* has since been computed and decreased significantly), work on the infinite *Pr* problem has resulted in a bound of $Nu \leq C Ra^{1/3}$ modulo logarithmic corrections (see Refs. 29 and 39 for the most recent results in this case) that is eerily reminiscent of Malkus' original prediction (see Ref. 10). We show in the following that these results for infinite *Pr* are generic for no-slip boundaries and reasonably chosen thermal boundary conditions on the top and bottom plates.

Section II introduces the equations of motion, discussing the boundary conditions considered in this paper as well as the generic formulation of the background method for this instance. Section III introduces the specific background profile used here, and discusses the potential impacts of this choice. Section IV details the necessary calculations to demonstrate the eventual bound, and Sec. V implements these estimates to bound the heat transport (Nusselt number) in terms of the forcing for all the relevant boundary conditions considered here. The main result is an overall asymptotic bound of the form $Nu \lesssim C\, Ra^{1/3}(\ln Ra)^{1/2}$, with $C = 0.29149$ for fixed temperature boundaries and $C = 0.30962$ for general imperfectly conducting boundaries (nonzero Biot number): see Eqs. (82) and (90). Section VI describes some conclusions and remaining open questions in this area. Some detailed calculations are relegated to the appendices so as not to detract from the results contained in the bulk of the paper.

## II. FORMULATION OF THE BOUNDING PROBLEM

### A. Governing equations, boundary conditions, and relevant statistical quantities

We begin with the governing equations for convection of an incompressible Boussinesq fluid at infinite Prandtl number (see Ref. 40 for a justification of this limit), nondimensionalized as in Ref. 20:

$$\nabla p = \nabla^2 \mathbf{u} + R\, T\, \mathbf{e_z}, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{2}$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \nabla^2 T. \tag{3}$$

The control parameter $R$ is defined in terms of the temperature scale $\Theta$ as

$$R = \frac{\alpha g h^3}{\nu_f \kappa_f}\, \Theta\,, \tag{4}$$

where in the case of fixed temperature boundary conditions $\Theta$ is just the imposed temperature drop across the fluid, and $R$ coincides with the usual Rayleigh number $Ra$. For general thermal boundary conditions, however, the (averaged) temperature drop across is *a priori* unknown, and $\Theta$ is defined in terms of the given periodic boundary conditions (BCs).[20] In this case, we need to estimate the nondimensional time- and horizontally-averaged temperature difference

$$\Delta T = \langle \overline{T}|_{z=0} - \overline{T}|_{z=1} \rangle, \tag{5}$$

where $\overline{T}$ denotes the horizontal average and $\langle \cdot \rangle$ the long-time average (notation as in Ref. 20). The Rayleigh number $Ra$ is then related to the control parameter $R$ via[19]

$$Ra = R\, \Delta T. \tag{6}$$

We supplement (1)–(3) with horizontally BCs in all variables, and no-slip velocity BCs in the vertical direction, $\mathbf{u}|_{z=0} = \mathbf{u}|_{z=1} = \mathbf{0}$. General mixed (Robin) thermal boundary conditions on the plates are given (in dimensionless form) in terms of the Biot number $\eta$ as

$$T - \eta\, T_z = 1 + \eta \ \text{ on } \ z = 0, \quad T + \eta\, T_z = -\eta \ \text{ on } \ z = 1, \tag{7}$$

where in the special case $\eta = 0$ we recover the well-known *fixed temperature* (Dirichlet) thermal BCs

$$T = 1 \ \text{ on } \ z = 0, \qquad T = 0 \ \text{ on } \ z = 1, \tag{8}$$

while the limit $\eta \to \infty$ gives the *fixed flux* (Neumann) conditions

$$T_z = -1 \ \text{ on } \ z = 0 \text{ and } z = 1. \tag{9}$$

These variable boundary conditions on the temperature can be considered as a simplification of a fluid contained between two plates of finite conductivity (see Ref. 20).

We define the dimensionless, time- and horizontally-averaged boundary heat flux

$$\beta = \left\langle -\overline{T}_z \right\rangle\Big|_{z=0} = \left\langle -\overline{T}_z \right\rangle\Big|_{z=1}, \tag{10}$$

which is fixed *a priori* at $\beta = 1$ (only) for fixed flux BCs $\eta = \infty$. Substituting the definitions (5) and (10) into the fixed Biot number BCs (7), we immediately obtain a relationship between $\Delta T$ and $\beta$ for $0 < \eta < \infty$:

$$\Delta T + 2\eta\, \beta = 1 + 2\eta. \tag{11}$$

Now the Nusselt number $Nu = 1 + \langle \int_0^1 \overline{wT}\, dz \rangle / \Delta T$ may readily be computed to be[20]

$$Nu = \frac{\beta}{\Delta T}, \tag{12}$$

so that via (6) we also have $Nu\, Ra = R\, \beta$. Note that, for $0 < \eta < \infty$, to obtain an upper bound on $Nu$ it is sufficient to bound *either* $\beta$ from above *or* $\Delta T$ from below, since $\Delta T$ and $\beta$ are related by (11).

As noted previously, the reduction of the momentum equation to (1) leads to significant insights into the heat transport at infinite Prandtl number. Key to these developments is noting that the momentum equation can be rewritten as

$$\nabla^4 w = -R(T_{xx} + T_{yy}) \equiv -R \nabla_H^2 T, \tag{13}$$

where $w$ is the vertical component of the velocity field. The no-slip velocity BCs on $\mathbf{u}$ indicate (via incompressibility) that the vertical velocity has boundary conditions

$$w = w_z = 0 \quad \text{at} \quad z = 0 \text{ and } 1. \tag{14}$$

Thus the horizontal components of velocity have only an indirect effect on the flow's evolution and resultant heat transport.

### B. Background field

Using the "background" approach,[18] we decompose the temperature field via

$$T(\mathbf{x}, t) = \tau(z) + \theta(\mathbf{x}, t), \tag{15}$$

where the background temperature profile $\tau(z)$ is chosen to satisfy the inhomogeneous thermal BCs (7), $\tau(0) - \eta\tau'(0) = 1 + \eta$, and $\tau(1) + \eta\tau'(1) = -\eta$, so that the "fluctuating field" $\theta(\mathbf{x}, t)$ satisfies the corresponding homogeneous BCs, which for general fixed Biot number in our geometry are

$$\theta - \eta\,\theta_z = 0 \ \text{ at } \ z = 0, \qquad \theta + \eta\,\theta_z = 0 \ \text{ at } \ z = 1. \tag{16}$$

Requiring for simplicity that $\tau'(0) = \tau'(1)$, and introducing the notation

$$\Delta\tau \equiv \tau(0) - \tau(1), \qquad \gamma \equiv -\tau'(0) = -\tau'(1), \tag{17}$$

the BCs on $\tau(z)$ imply the relation (compare (11))

$$\Delta\tau + 2\eta\,\gamma = 1 + 2\eta. \tag{18}$$

Using the decomposition (15) we can find the evolution of the temperature fluctuations from (3), and the velocity slaving relation (13):

$$\frac{\partial\theta}{\partial t} + \mathbf{v}\cdot\nabla\theta = \nabla^2\theta + \tau'' - w\tau' \tag{19}$$

and

$$\nabla^4 w = -R\nabla_H^2\theta. \tag{20}$$

The time evolution of the norm of the temperature fluctuation is derived by multiplying (19) by $\theta$ and integrating (by parts), to give

$$\frac{1}{2}\frac{d}{dt}\,\|\theta\|^2 = -\,\|\nabla\theta\|^2 + A\,\overline{\theta\theta_z}\big|_{z=0}^{1} - \int_f \theta_z\tau' + A\,\overline{\theta}\tau'\big|_{z=0}^{1} - \int_f w\theta\tau', \tag{21}$$

where $A$ is the dimensionless cross-sectional area, and the $\mathrm{L}^2$ norm of a function $h$ is defined via $\|h\|^2 = \int_f h^2$, where $\int_f h = A\int_0^1 \overline{h}(z)\,dz$ denotes a volume integral over the fluid layer. We also note from (15) that the norms of the gradients of $T$ and $\theta$ are related by

$$\|\nabla T\|^2 = \|\nabla\theta\|^2 + 2\int_f \theta_z\tau' + \int_f \tau'^2. \tag{22}$$

### C. Bounding principle

These identities are combined to form a bounding principle as follows. First add 2 × (21) to (22) to eliminate the linear (in the fluctuations) $\int\theta_z\tau'$ term; then taking the time average and integrating by parts, we see that

$$\left\langle \overline{TT_z}\big|_{z=0}^{1}\right\rangle = \frac{1}{A}\left\langle\|\nabla T\|^2\right\rangle = \frac{1}{A}\int_f \tau'^2 + 2\left\langle\overline{\theta\theta_z}\big|_{z=0}^{1} + \overline{\theta}\tau'\big|_{z=0}^{1}\right\rangle - \frac{1}{A}\mathcal{Q}_\tau[w,\theta], \tag{23}$$

where the first equality is derived from (3) and we have defined the quadratic form $\mathcal{Q}_\tau$ (for given $\tau(z)$, a quadratic functional of the fields $w$ and $\theta$ related through (20)) by

$$\mathcal{Q}_\tau[w,\theta] = \left\langle\int_f\left[|\nabla\theta|^2 + 2\tau' w\theta\right]\right\rangle. \tag{24}$$

Using the decomposition (15) on the left-hand side of (23) with the definitions (5), (10), and (17), and rearranging, we find

$$\beta\Delta\tau - \gamma\Delta T = \int_0^1 \tau'^2\,dz - \gamma\Delta\tau + \frac{1}{A}\widetilde{\mathcal{Q}}_\tau[w,\theta], \tag{25}$$

where the modified quadratic form $\widetilde{\mathcal{Q}}_\tau$ is given by

$$\widetilde{\mathcal{Q}}_\tau[w,\theta] = \mathcal{Q}_\tau[w,\theta] - A\left\langle \overline{\theta\theta_z}\Big|_{z=0}^{1}\right\rangle. \tag{26}$$

In the general case of mixed thermal BCs with finite Biot number ($0 \le \eta < \infty$), we can now use (11) and (18) to substitute for $\Delta T$ and $\Delta\tau$; after some simplification, this gives the following identity in terms of $\beta$ and $\gamma$:

$$\beta(1+2\eta) = \int_0^1 \tau'^2\, dz + 2\eta\gamma^2 - \frac{1}{A}\widetilde{\mathcal{Q}}_\tau[w,\theta]. \tag{27}$$

From this equation, we can now see how to obtain an upper bound on the Nusselt number (for $\eta < \infty$): If for all "allowed" scalar fields $\theta(\mathbf{x})$ and $w(\mathbf{x})$, and for a given background $\tau(z)$, we have $\widetilde{\mathcal{Q}}_\tau[w,\theta] \ge 0$, then (27) implies an upper bound $\mathcal{B}_\eta$ on the boundary heat flux $\beta$,

$$\beta \le \frac{1}{1+2\eta}\left(\int_0^1 \tau'^2\, dz + 2\eta\gamma^2\right) \equiv \mathcal{B}_\eta[\tau], \tag{28}$$

and hence (via (11) and (12)) on the Nusselt number *Nu*. Here the class of *allowed* fields is those $w(\mathbf{x})$ and $\theta(\mathbf{x})$ satisfying the constraints of the problem, that is the homogeneous (no-slip) velocity BCs (14), the thermal BCs (16), and the infinite Prandtl number relation (20) slaving $w$ to $\theta$; note that all solutions of (19) and (20) are allowed in this sense.

In the fixed flux limit $\eta \to \infty$, the above bounding formulation becomes inapplicable, but we instead consider the lower bound $\mathcal{D}_\eta$ on the averaged temperature drop $\Delta T$ which may be derived (using (11) and (18)) for any $\eta > 0$:

$$\Delta T \ge 2\Delta\tau - \frac{2\eta}{1+2\eta}\left(\int_0^1 \tau'^2\, dz + \frac{\Delta\tau^2}{2\eta}\right) \equiv \mathcal{D}_\eta[\tau] \tag{29}$$

(where $\mathcal{D}_\eta[\tau] + 2\eta\mathcal{B}_\eta[\tau] = 1 + 2\eta$). The Nusselt number is then bounded above by

$$Nu \le \mathcal{N}_\eta[\tau] = \mathcal{B}_\eta[\tau]/\mathcal{D}_\eta[\tau]. \tag{30}$$

For reference, let us consider the two limiting cases:

- For fixed temperature (Biot number $\eta = 0$), the BCs (8) yield $\Delta T = 1$, so $Nu = \beta$ and $Ra = R$; and the requirement that the background $\tau(z)$ satisfy the thermal BCs enforces $\Delta\tau = 1$. We thus have $\mathcal{D}_0[\tau] = 1$, while

$$Nu = \beta \le \int_0^1 \tau'^2\, dz \equiv \mathcal{B}_0[\tau] = \mathcal{N}_0[\tau].$$

- For fixed flux ($\eta = \infty$) the Neumann thermal BCs (9) imply $\beta = 1$, so $Nu = 1/\Delta T$; and the condition on $\tau(z)$ is $\gamma = 1$. In this case we have $\mathcal{B}_\infty[\tau] = 1$ and

$$Nu^{-1} = \Delta T \ge 2\Delta\tau - \int_0^1 \tau'^2\, dz \equiv \mathcal{D}_\infty[\tau] = (\mathcal{N}_\infty[\tau])^{-1}.$$

### D. Admissible backgrounds: The spectral constraint

The positivity requirement on the quadratic form $\widetilde{\mathcal{Q}}_\tau$ may be interpreted as a condition on the function $\tau$; so we say that a background field $\tau(z)$ satisfying the BCs (7) is *admissible* for a given $R$ if $\widetilde{\mathcal{Q}}_\tau[w,\theta] \ge 0$ for all allowed scalar fields $w$ and $\theta$. Since the condition $\widetilde{\mathcal{Q}}_\tau \ge 0$ may be rewritten as a condition that the lowest eigenvalue of the elliptic operator associated with the quadratic form $\widetilde{\mathcal{Q}}_\tau$ is non-negative, the admissibility criterion is also referred to as a *spectral constraint* on $\tau$.[18] The bounding problem is thus to show (for given $R$) that admissible fields exist, and then to find the lowest bound $\mathcal{N}_\eta[\tau]$ over all admissible $\tau$.

Now in general, the quadratic form $\widetilde{\mathcal{Q}}_\tau$ defined in (26) contains a boundary term $A\langle \overline{\theta\theta_z}\big|_{z=0}^{1}\rangle$ (although we note that this term vanishes in both the fixed temperature and fixed flux limits, in which cases $\widetilde{\mathcal{Q}}_\tau = \mathcal{Q}_\tau$), and it is unclear how to exploit or bound this term in an explicit estimate. Fortunately for $0 < \eta < \infty$ this term is in fact stabilizing, since by (16) we find

$$\overline{\theta\theta_z}\big|_{z=0}^{1} = -\eta\left(\overline{\theta_z^2}|_{z=0} + \overline{\theta_z^2}|_{z=1}\right) \leq 0, \tag{31}$$

which substituted into (26) implies that $\widetilde{\mathcal{Q}}_\tau[w,\theta] \geq \mathcal{Q}_\tau[w,\theta]$ for all $\theta$ satisfying the thermal BCs. The boundary conditions (16) are difficult to incorporate into the variational statements that follow, so we impose a stronger requirement on $\tau$: We say that a background field $\tau(z)$ is *strongly admissible* (for a given control parameter $R$) if it satisfies the thermal BCs (7) and if $\mathcal{Q}_\tau[w,\theta] \geq 0$ for all sufficiently smooth scalar fields $w$ and $\theta$ satisfying $w = w_z = 0$ at $z = 0, 1$ and the constraint (20); note that we do not specify BCs on $\theta$ at this stage in the development.

Since $\widetilde{\mathcal{Q}}_\tau \geq \mathcal{Q}_\tau$, strong admissibility implies admissibility, and thus to bound the heat transport it suffices to work with the quadratic form $\mathcal{Q}_\tau[w,\theta]$ defined in (24). The $\eta$-dependence encapsulated in the thermal BCs is then incorporated solely through the construction of $\tau(z)$. We note though that unlike in the fixed temperature case considered in Refs. 28 and 29, we have no boundary control of $\theta$ to aid in the estimates.

Horizontal periodicity enables us to rewrite the admissibility criterion (spectral constraint) $\mathcal{Q}_\tau[w,\theta] \geq 0$ in horizontal Fourier space. We express the temperature fluctuation field $\theta(\mathbf{x})$ in terms of its Fourier expansion,

$$\theta(x,y,z) = \sum_{\mathbf{k}} \mathrm{e}^{\mathrm{i}(k_x x + k_y y)}\hat{\theta}_{\mathbf{k}}(z), \tag{32}$$

and similarly for the vertical velocity $w(\mathbf{x})$; here $\mathbf{k} = (k_x, k_y) = (2\pi n_x/L_x, 2\pi n_y/L_y)$ is the horizontal wave vector, where $L_x$ and $L_y$ are the periods in the $x$- and $y$-directions, respectively. The infinite Prandtl number constraint (20) then reduces to a relationship between individual Fourier components $\hat{w}_{\mathbf{k}}$ and $\hat{\theta}_{\mathbf{k}}$,

$$\left(\mathrm{D}^2 - k^2\right)^2 \hat{w}_{\mathbf{k}} = k^2 R\, \hat{\theta}_{\mathbf{k}} \tag{33}$$

(where $\mathrm{D} = d/dz$ and $k^2 = |\mathbf{k}|^2$). Moreover, in Fourier space the quadratic form becomes

$$\mathcal{Q}_\tau[w,\theta] = \int_f \left[|\nabla\theta|^2 + 2\tau'\,w\theta\right] = A\sum_{\mathbf{k}}\int_0^1 \left[\left(k^2|\hat{\theta}_{\mathbf{k}}|^2 + |\mathrm{D}\hat{\theta}_{\mathbf{k}}|^2\right) + 2\tau'\mathrm{Re}[\hat{w}_{\mathbf{k}}\hat{\theta}_{\mathbf{k}}^*]\right] dz \equiv A\sum_{\mathbf{k}} \mathcal{Q}_{\mathbf{k}}, \tag{34}$$

decomposing into a sum of terms $\mathcal{Q}_{\mathbf{k};\tau}[\hat{w}_{\mathbf{k}}, \hat{\theta}_{\mathbf{k}}]$, which permits us to enforce the admissibility condition separately for each Fourier mode $\mathbf{k}$.

The condition on the background field $\tau(z)$ to yield a Nusselt number bound is thus: For a given $R$, a background $\tau(z)$ is strongly admissible if and only if

$$\mathcal{Q}_{\mathbf{k}}[\hat{w}_{\mathbf{k}}, \hat{\theta}_{\mathbf{k}}] = \int_0^1 \left[\left(k^2|\hat{\theta}_{\mathbf{k}}|^2 + |\mathrm{D}\hat{\theta}_{\mathbf{k}}|^2\right) + 2\tau'\mathrm{Re}[\hat{w}_{\mathbf{k}}\hat{\theta}_{\mathbf{k}}^*]\right] dz \geq 0 \tag{35}$$

for each $\mathbf{k}$, where $\hat{w}_{\mathbf{k}}$ and $\hat{\theta}_{\mathbf{k}}$ are coupled through (33), and where by (14) we require velocity BCs $\hat{w}_{\mathbf{k}} = \mathrm{D}\hat{w}_{\mathbf{k}} = 0$ at $z = 0, 1$, while no boundary conditions are imposed on the $\hat{\theta}_{\mathbf{k}}$ *a priori*.

## III. BACKGROUND PROFILE

We now consider a specific family of background fields $\tau(z) = \tau_{\delta;\,\eta}(z)$ that explicitly enforce a boundary layer of size $\delta$ near the top and bottom boundaries. Following the intuition provided in Refs. 28 and 29, we use a stably stratified interior profile, so that (due to the coupling between $w$ and $\theta$ via (20)) there is a positive contribution from $\int \tau' w\theta$ in the bulk which balances the indefinite boundary layer terms, permitting a looser restriction on the size of the boundary layer.

Define the background profile $\tau(z)$ for $0 < \delta < \frac{1}{2}$ as

$$\tau(z) = \begin{cases} \tau(0) - \gamma z, & 0 \le z \le \delta, \\ \frac{1}{2} + \mu\, G(z), & \delta < z < 1 - \delta, \\ \tau(1) - \gamma(z-1), & 1 - \delta \le z \le 1; \end{cases} \tag{36}$$

here $\mu > 0$ is a constant that we leave free for the moment. Requiring $G(z)$ to have odd symmetry about $z = 1/2$ enforces the condition that $\tau\left(\frac{1}{2}\right) = \frac{1}{2}$ which is consistent with the restriction (due to the Robin boundary conditions) that for $\eta < \infty$, $(\tau(0) + \tau(1))/2 = \frac{1}{2}$. Enforcing continuity of the profile leads to $\tau(0) = \frac{1}{2} + \gamma\delta + \mu G(\delta)$ and $\tau(1) = \frac{1}{2} - \gamma\delta - \mu G(\delta)$, and requiring that $G' > 0$ (to ensure that the contribution from the bulk is indeed stably stratifying) we note that $G(\delta) < 0$, and for convenience define the constant $\lambda = -\mu G(\delta) > 0$. Using this definition we see that

$$\Delta\tau = \tau(0) - \tau(1) = 2\left(\gamma\delta + \mu G(\delta)\right) \equiv 2\left(\gamma\delta - \lambda\right). \tag{37}$$

The thermal boundary conditions (18) also imply that $\gamma$ and $\lambda$ are related by

$$\gamma = \frac{1 + 2\eta + 2\lambda}{2(\delta + \eta)}. \tag{38}$$

As $\tau(z)$ rarely appears in the variational formulation, but its derivative does, we write $\tau'(z)$ out explicitly here for future reference:

$$\tau'(z) = \begin{cases} -\gamma, & 0 \le z < \delta, \\ \mu\, g(z), & \delta < z < 1 - \delta, \\ -\gamma, & 1 - \delta < z \le 1 \end{cases} \tag{39}$$

where $g(z) = G'(z)$.

### A. Power law profile

Akin to Refs. 28 and 29 we choose $G$ such that $\tau'(z)$ exhibits a power law behavior in the interior. The lack of control of $\theta$ near the boundaries does not allow the choice $g(z) \sim z^{-1}$ as originally proposed in Ref. 28, but we can allow

$$g(z) = \frac{1}{z^{\alpha}} + \frac{1}{(1-z)^{\alpha}}, \tag{40}$$

where the exponent $\alpha < 1$ will be fixed later. As one might expect from the results of Refs. 28 and 29, the resultant bound is optimized for $1 - \alpha \ll 1$. The antiderivative $G(z)$ appearing in $\tau$ (36) is thus

$$G(z) = \frac{1}{1-\alpha}\left(z^{1-\alpha} - (1-z)^{1-\alpha}\right) \tag{41}$$

for $\alpha \neq 1$ (the value $\alpha = 1$, giving $G(z) = \ln[z/(1-z)]$, corresponds to the background used in Refs. 28 and 29). The price of not considering the explicit logarithmic background profile ($\alpha = 1$) is a more significant logarithmic correction to the $\frac{1}{3}$ power bound than that found for fixed temperature boundary conditions;[28] however, $\alpha < 1$ is required when there is no uniform bound on $\theta$ (as dictated by the lack of boundary control of $\theta$). Hence the final result in this paper is not as tight as those found in Refs. 28 and 29, but is applicable to more generic thermal boundary conditions.

To compute the bounds on $\beta$ and $\Delta T$ (and hence on $Nu$) for the given background $\tau'$ given by (39), we begin by evaluating

$$\int_0^1 \tau'^2\, dz = \int_0^{\delta} \gamma^2\, dz + \mu^2 \int_{\delta}^{1-\delta} (g(z))^2\, dz + \int_{1-\delta}^{1} \gamma^2\, dz = 2\delta\gamma^2 + T_B, \tag{42}$$

where $2\delta\gamma^2$ is the contribution to $\int_0^1 \tau'^2\, dz$ from the (linear) boundary layer, and we let $T_B$ be the contribution due to the (nonlinear) interior, or bulk. For the power law profile (40) this bulk term is (for $\alpha \neq \frac{1}{2}$)

$$T_B = \mu^2 \int_\delta^{1-\delta} (g(z))^2 \, dz = 2\frac{\lambda^2}{G(\delta)^2} \int_\delta^{1/2} \left[\frac{1}{z^\alpha} + \frac{1}{(1-z)^\alpha}\right]^2 dz$$
$$= \frac{2(1-\alpha)^2\lambda^2}{\left[(1-\delta)^{1-\alpha} - \delta^{1-\alpha}\right]^2} \left[\frac{1}{2\alpha - 1}\left(\delta^{1-2\alpha} - (1-\delta)^{1-2\alpha}\right) + 2\int_\delta^{1/2} \frac{dz}{z^\alpha(1-z)^\alpha}\right]; \tag{43}$$

the cross term is inconvenient to compute explicitly, but can be bounded using

$$\int_\delta^{1/2} \frac{dz}{z^\alpha(1-z)^\alpha} \leq \int_\delta^{1/2} \frac{dz}{z^\alpha(1/2)^\alpha} = \frac{2^\alpha}{1-\alpha}\left(\frac{1}{2^{1-\alpha}} - \delta^{1-\alpha}\right). \tag{44}$$

From (28) and (42), and using (38) to substitute for $\gamma$, we thus find that (for admissible $\tau(z)$) the averaged boundary heat flux $\beta$ is bounded by

$$\beta \leq \mathcal{B}_\eta[\tau] = \frac{1}{1+2\eta}\left(\int_0^1 \tau'^2 \, dz + 2\eta\gamma^2\right) = \frac{1}{1+2\eta}\left(2(\delta+\eta)\gamma^2 + T_B\right)$$
$$= \frac{1}{1+2\eta}\left(\frac{(1+2\eta+2\lambda)^2}{2(\delta+\eta)} + T_B\right). \tag{45}$$

Similarly, via (29) and (42), substituting for $\Delta\tau$ and $\gamma$ using (37) and (38) and simplifying, the averaged temperature drop $\Delta T$ is bounded below by

$$\Delta T \geq \mathcal{D}_\eta[\tau] = 2\Delta\tau - \frac{2\eta}{1+2\eta}\int_0^1 \tau'^2 \, dz - \frac{\Delta\tau^2}{1+2\eta} = \left(2 - \frac{\Delta\tau}{1+2\eta}\right)\Delta\tau - \frac{2\eta}{1+2\eta}\left(2\delta\gamma^2 + T_B\right)$$
$$= \frac{\delta(1+2\eta)}{\delta+\eta} - \frac{4\eta\lambda(1+2\eta+\lambda)}{(\delta+\eta)(1+2\eta)} - \frac{2\eta}{1+2\eta}T_B. \tag{46}$$

Now, we need to determine the constraints on $\delta$ imposed by the admissibility (spectral constraint) of the background $\tau(z)$; together with the above formulas, we may then deduce an appropriate form for $\lambda$ and the scaling of the bounds.

## IV. POSITIVITY OF THE QUADRATIC FORM

As discussed in Secs. II C–II D, the bounds on $\beta$ and $\Delta T$, and hence on the Nusselt number $Nu$, hold provided the background field $\tau(z)$ is admissible, which is a non-negativity condition on the quadratic form. Substituting $\tau'$ defined piecewise in (39) into the formula for $\mathcal{Q}_\mathbf{k}[\hat{w}_\mathbf{k}, \hat{\theta}_\mathbf{k}]$, and adding and subtracting $\int 2\mu g(z)\mathrm{Re}[\hat{w}_\mathbf{k}\hat{\theta}_\mathbf{k}^*]$ terms at the boundaries — and also dropping the hats and subscripts on $\hat{w}_\mathbf{k}(z)$ and $\hat{\theta}_\mathbf{k}(z)$ to simplify notation — this condition in its Fourier space form (35) becomes

$$\int_0^1 \left(k^2|\theta|^2 + |\theta'|^2\right) dz + 2\mu\int_0^1 g(z)\mathrm{Re}[w\theta^*]\, dz$$
$$- 2\int_0^\delta [\gamma + \mu g(z)]\,\mathrm{Re}[w\theta^*]\, dz - 2\int_{1-\delta}^1 [\gamma + \mu g(z)]\,\mathrm{Re}[w\theta^*]\, dz \geq 0, \tag{47}$$

where $w = w' = 0$ at $z = 0$ and 1, and by (33)

$$\theta = \frac{1}{k^2 R}\left(\mathrm{D}^2 - k^2\right)^2 w. \tag{48}$$

Specializing to the particular power law form (40) of $g(z)$, and rearranging slightly as in Ref. 28, we now write out in detail the admissibility criterion (spectral constraint) that the background must satisfy: we require

$$0 \le \int_0^{1/2} \left(k^2|\theta|^2 + |\theta'|^2\right)\, dz + 2\mu \int_0^1 \frac{\mathrm{Re}[w\theta^*]}{z^\alpha}\, dz - 2\int_0^\delta \left[\gamma + \mu\left(\frac{1}{z^\alpha} + \frac{1}{(1-z)^\alpha}\right)\right] \mathrm{Re}[w\theta^*]\, dz$$
$$+ \int_{1/2}^1 \left(k^2|\theta|^2 + |\theta'|^2\right)\, dz + 2\mu \int_0^1 \frac{\mathrm{Re}[w\theta^*]}{(1-z)^\alpha}\, dz - 2\int_{1-\delta}^1 \left[\gamma + \mu\left(\frac{1}{z^\alpha} + \frac{1}{(1-z)^\alpha}\right)\right] \mathrm{Re}[w\theta^*]\, dz \quad (49)$$

for $w$ satisfying the appropriate BCs and $\theta$ related to $w$ via (48).

As in Ref. 28, we shall verify the above inequality by enforcing the non-negativity of each line on the right-hand side of (49) separately. We shall carry out the calculation for the first line, which involves controlling the indefinite term near the lower boundary $z = 0$; by symmetry the corresponding estimates for the second line are identical upon replacing $z \mapsto 1 - z$.

### A. Evaluation of the bulk term

Since $w$ and $\theta$ are coupled through (48), we rewrite the bulk term in the first line of (49) explicitly as definite quadratic terms in $w$ weighted by powers of $z$. To do so, we first substitute for $\theta$ from (48),

$$\int_0^1 \frac{\mathrm{Re}[w\theta^*]}{z^\alpha}\,dz = \int_0^1 \frac{\mathrm{Re}[\theta w^*]}{z^\alpha}\,dz = \frac{1}{2Rk^2}\int_0^1 \frac{(w'''' - 2k^2 w'' + k^4 w)w^* + \text{c.c.}}{z^\alpha}\,dz.$$

Integrating by parts multiple times we compute (using the formulae from Appendix A for no-slip BCs, specifically (A2) and (A6)):

$$\begin{aligned} Rk^2\int_0^1 \frac{\mathrm{Re}[w\theta^*]}{z^\alpha}\,dz &= \int_0^1 \left\{ \frac{|w''|^2}{z^\alpha} - 2\alpha(\alpha+1)\frac{|w'|^2}{z^{\alpha+2}} + \tfrac{1}{2}\alpha(\alpha+1)(\alpha+2)(\alpha+3)\frac{|w|^2}{z^{\alpha+4}} \right\} dz \\ &\quad + k^2\int_0^1 \left\{ 2\frac{|w'|^2}{z^\alpha} - \alpha(\alpha+1)\frac{|w|^2}{z^{\alpha+2}} \right\} dz + k^4\int_0^1 \frac{|w|^2}{z^\alpha}\,dz, \end{aligned} \tag{50}$$

where an analogous identity holds with $z \mapsto 1 - z$.

### B. Estimates on the boundary layer integral

In order to control the boundary integral $\int_0^\delta (\cdots)\,dz$ in (49), we require pointwise estimates on $\mathrm{Re}[w\theta^*]$ near the lower boundary. As there is no control of $\theta$ near $z = 0$, we proceed as in Refs. 19 and 20, and write

$$w(z)\theta^*(z) = \int_0^z \mathrm{D}(w\theta^*)\,d\zeta = \int_0^z \left\{ w(\theta^*)' + w'\theta^* \right\} d\zeta. \tag{51}$$

The no-slip BCs $w(0) = w'(0) = 0$ allow us to obtain uniform bounds on the vertical velocity $w(z)$ and its derivative $w'(z) = \mathrm{D}w(z)$; we do so using appropriately weighted integrals, as we wish to obtain a form that allows comparison with the bulk terms in (50).

For real numbers $\nu_1$ and $\nu_2$ such that $|w| = o(z^{\nu_1})$ as $z \to 0^+$ and $\nu_2 > -\frac{1}{2}$, and using the Fundamental Theorem of Calculus and the Cauchy-Schwarz Inequality for $0 < z \le 1$ we estimate

$$\begin{aligned} |w(z)| &= \left| z^{\nu_1}\frac{w(z)}{z^{\nu_1}} \right| = z^{\nu_1}\left| \int_0^z \zeta^{\nu_2}\frac{1}{\zeta^{\nu_2}}\mathrm{D}\left(\frac{w}{\zeta^{\nu_1}}\right) d\zeta \right| \\ &\le z^{\nu_1}\left( \int_0^z \zeta^{2\nu_2}\,d\zeta \right)^{1/2} \left( \int_0^z \frac{1}{\zeta^{2\nu_2}}\left| \mathrm{D}\left(\frac{w}{\zeta^{\nu_1}}\right) \right|^2 d\zeta \right)^{1/2} \\ &\le \left( \frac{1}{2\nu_2+1} \right)^{1/2} z^{\nu_1+\nu_2+1/2} \left( \int_0^1 \frac{1}{\zeta^{2\nu_2}}\left| \mathrm{D}\left(\frac{w}{\zeta^{\nu_1}}\right) \right|^2 d\zeta \right)^{1/2} \\ &= \left( \frac{1}{2\nu_2+1} \right)^{1/2} z^{\nu_1+\nu_2+1/2} \left( \int_0^1 \left\{ \frac{|w'|^2}{\zeta^{2(\nu_1+\nu_2)}} - \nu_1(\nu_1+2\nu_2+1)\frac{|w|^2}{\zeta^{2(\nu_1+\nu_2+1)}} \right\} d\zeta \right)^{1/2}, \end{aligned} \tag{52}$$

where we used (A7) in the last equality (converging with vanishing boundary term provided $|w| = o(z^{\nu_1+\nu_2+1/2})$ as $z \to 0^+$).

Similarly, replacing $w$ with $w'$ gives

$$|w'(z)| \leq \left(\frac{1}{2\nu_2+1}\right)^{1/2} z^{\nu_1+\nu_2+1/2} \left(\int_0^1 \left\{\frac{|w''|^2}{\zeta^{2(\nu_1+\nu_2)}} - \nu_1(\nu_1+2\nu_2+1)\frac{|w'|^2}{\zeta^{2(\nu_1+\nu_2+1)}}\right\} d\zeta\right)^{1/2} \tag{53}$$

subject to analogous convergence constraints. Since $|w'| = \mathcal{O}(z)$ and hence $|w| = \mathcal{O}(z^2)$ (this holds for the given boundary conditions provided $\theta \in L^2$, as shown in Ref. 27), the above estimates are valid provided $\nu_1 < 1$ and $\nu_1 + \nu_2 < \frac{1}{2}$. Note that we do not consider the limiting cases $\nu_1 = 1$ or $\nu_1 + \nu_2 = \frac{1}{2}$ in the current paper, although the discussion in the Appendix of Ref. 28 indicates that such bounds would hold in these cases as well.

Using the pointwise estimates (52) and (53) in (51), we can control $w\theta^*$ pointwise near the boundary: Provided $\nu_1 + \nu_2 > -1$, for $z \leq \frac{1}{2}$,

$$\begin{aligned}
\left|w(z)\theta^*(z)\right| &\leq \int_0^z \left\{|w||\theta'| + |w'||\theta|\right\} d\zeta \\
&\leq \frac{1}{(2\nu_2+1)^{1/2}} \left(\int_0^1 \left\{\frac{|w'|^2}{\zeta^{2(\nu_1+\nu_2)}} - \nu_1(\nu_1+2\nu_2+1)\frac{|w|^2}{\zeta^{2(\nu_1+\nu_2+1)}}\right\} d\zeta\right)^{1/2} \\
&\qquad\qquad \times \int_0^z \zeta^{\nu_1+\nu_2+1/2}|\theta'|\, d\zeta \\
&\quad + \frac{1}{(2\nu_2+1)^{1/2}} \left(\int_0^1 \left\{\frac{|w''|^2}{\zeta^{2(\nu_1+\nu_2)}} - \nu_1(\nu_1+2\nu_2+1)\frac{|w'|^2}{\zeta^{2(\nu_1+\nu_2+1)}}\right\} d\zeta\right)^{1/2} \\
&\qquad\qquad \times \int_0^z \zeta^{\nu_1+\nu_2+1/2}|\theta|\, d\zeta \\
&\leq \frac{1}{(2\nu_2+1)^{1/2}} \left(\int_0^1 \left\{\frac{|w'|^2}{\zeta^{2(\nu_1+\nu_2)}} - \nu_1(\nu_1+2\nu_2+1)\frac{|w|^2}{\zeta^{2(\nu_1+\nu_2+1)}}\right\} d\zeta\right)^{1/2} \\
&\qquad\qquad \times \left(\int_0^z \zeta^{2\nu_1+2\nu_2+1}\, d\zeta\right)^{1/2} \left(\int_0^z |\theta'|^2\, d\zeta\right)^{1/2} \\
&\quad + \frac{1}{(2\nu_2+1)^{1/2}} \left(\int_0^1 \left\{\frac{|w''|^2}{\zeta^{2(\nu_1+\nu_2)}} - \nu_1(\nu_1+2\nu_2+1)\frac{|w'|^2}{\zeta^{2(\nu_1+\nu_2+1)}}\right\} d\zeta\right)^{1/2} \\
&\qquad\qquad \times \left(\int_0^z \zeta^{2\nu_1+2\nu_2+1}\, d\zeta\right)^{1/2} \left(\int_0^z |\theta|^2\, d\zeta\right)^{1/2} \\
&= \frac{1}{[2(2\nu_2+1)(\nu_1+\nu_2+1)]^{1/2}}\, z^{\nu_1+\nu_2+1} \\
&\quad \times \left\{\left(\int_0^1 \left\{\frac{|w'|^2}{\zeta^{2(\nu_1+\nu_2)}} - \nu_1(\nu_1+2\nu_2+1)\frac{|w|^2}{\zeta^{2(\nu_1+\nu_2+1)}}\right\} d\zeta\right)^{1/2} \left(\int_0^{1/2} |\theta'|^2\, d\zeta\right)^{1/2}\right. \\
&\quad \left. + \left(\int_0^1 \left\{\frac{|w''|^2}{\zeta^{2(\nu_1+\nu_2)}} - \nu_1(\nu_1+2\nu_2+1)\frac{|w'|^2}{\zeta^{2(\nu_1+\nu_2+1)}}\right\} d\zeta\right)^{1/2} \left(\int_0^{1/2} |\theta|^2\, d\zeta\right)^{1/2}\right\}.
\end{aligned} \tag{54}$$

With this pointwise estimate we can now estimate the entire boundary term near $z = 0$ in (47). Recalling that $\gamma + \mu g(z) > 0$ (see (38)–(40)), for positive constants $a_1$ and $a_2$ we have the bound

$$
\begin{aligned}
&\left|2\int_0^\delta [\gamma+\mu g(z)]\,\mathrm{Re}[w(z)\theta^*(z)]\,dz\right| \le 2\int_0^\delta [\gamma+\mu g(z)]\left|w(z)\theta^*(z)\right|\,dz\\
&\quad\le \frac{2}{[2(2\nu_2+1)(\nu_1+\nu_2+1)]^{1/2}}\int_0^\delta [\gamma+\mu g(z)]\,z^{\nu_1+\nu_2+1}\,dz\\
&\quad\times\left\{\left(\int_0^1\left\{\frac{|w'|^2}{\zeta^{2(\nu_1+\nu_2)}}-\nu_1(\nu_1+2\nu_2+1)\frac{|w|^2}{\zeta^{2(\nu_1+\nu_2+1)}}\right\}d\zeta\right)^{1/2}\left(\int_0^{1/2}|\theta'|^2\,d\zeta\right)^{1/2}\right.\\
&\qquad\left.+\left(\int_0^1\left\{\frac{|w''|^2}{\zeta^{2(\nu_1+\nu_2)}}-\nu_1(\nu_1+2\nu_2+1)\frac{|w'|^2}{\zeta^{2(\nu_1+\nu_2+1)}}\right\}d\zeta\right)^{1/2}\left(\int_0^{1/2}|\theta|^2\,d\zeta\right)^{1/2}\right\}\\
&\quad\le \frac{2}{(2\nu_2+1)(\nu_1+\nu_2+1)}\left(\int_0^\delta [\gamma+\mu g(z)]\,z^{\nu_1+\nu_2+1}\,dz\right)^2\\
&\quad\times\left[\frac{a_1}{2}\int_0^1\left\{\frac{|w'|^2}{z^{2(\nu_1+\nu_2)}}-\nu_1(\nu_1+2\nu_2+1)\frac{|w|^2}{z^{2(\nu_1+\nu_2+1)}}\right\}dz\right.\\
&\qquad\left.+\frac{a_2}{2}\int_0^1\left\{\frac{|w''|^2}{z^{2(\nu_1+\nu_2)}}-\nu_1(\nu_1+2\nu_2+1)\frac{|w'|^2}{z^{2(\nu_1+\nu_2+1)}}\right\}dz\right]\\
&\quad+\frac{1}{2a_1}\int_0^{1/2}|\theta'|^2\,dz+\frac{1}{2a_2}\int_0^{1/2}|\theta|^2\,dz.
\end{aligned}
\tag{55}
$$

### C. Sufficient condition for admissibility

As discussed previously, the spectral constraint (positive definiteness of the quadratic form $\mathcal{Q}_{\mathbf{k}}$), and hence the admissibility of the background $\tau(z)$, follows if

$$
0 \le k^2\int_0^{1/2}|\theta|^2\,dz+\int_0^{1/2}|\theta'|^2\,dz+2\mu\int_0^1\frac{\mathrm{Re}[w\theta^*]}{z^\alpha}\,dz-2\int_0^\delta[\gamma+\mu g(z)]\,\mathrm{Re}[w\theta^*]\,dz, \tag{56}
$$

with an analogous inequality for the second line of (49), where $g(z)$ for the chosen background profile used here is given by (40). Substituting the identity for the bulk term (50) and using the boundary estimate (55), for positivity of the quadratic form it is sufficient that the following is satisfied (where as yet $a_1 > 0$, $a_2 > 0$, and—subject to some constraints—the exponents $\nu_1$ and $\nu_2$ are still unspecified):

$$
\begin{aligned}
0 &\le k^2\int_0^{1/2}|\theta|^2\,dz+\int_0^{1/2}|\theta'|^2\,dz\\
&\quad+\frac{2\mu}{Rk^2}\left[\int_0^1\left\{\frac{|w''|^2}{z^\alpha}-2\alpha(\alpha+1)\frac{|w'|^2}{z^{\alpha+2}}+\tfrac{1}{2}\alpha(\alpha+1)(\alpha+2)(\alpha+3)\frac{|w|^2}{z^{\alpha+4}}\right\}dz\right.\\
&\qquad\left.+k^2\int_0^1\left\{2\frac{|w'|^2}{z^\alpha}-\alpha(\alpha+1)\frac{|w|^2}{z^{\alpha+2}}\right\}dz+k^4\int_0^1\frac{|w|^2}{z^\alpha}\,dz\right]\\
&\quad-\frac{2}{(2\nu_2+1)(\nu_1+\nu_2+1)}\left(\int_0^\delta[\gamma+\mu g(z)]\,z^{\nu_1+\nu_2+1}\,dz\right)^2
\end{aligned}
$$

$$\times\left[\frac{a_1}{2}\int_0^1\left\{\frac{|w'|^2}{z^{2(\nu_1+\nu_2)}}-\nu_1(\nu_1+2\nu_2+1)\frac{|w|^2}{z^{2(\nu_1+\nu_2+1)}}\right\}dz\right.$$
$$\left.+\frac{a_2}{2}\int_0^1\left\{\frac{|w''|^2}{z^{2(\nu_1+\nu_2)}}-\nu_1(\nu_1+2\nu_2+1)\frac{|w'|^2}{z^{2(\nu_1+\nu_2+1)}}\right\}dz\right]$$
$$-\frac{1}{2a_1}\int_0^{1/2}|\theta'|^2\,dz-\frac{1}{2a_2}\int_0^{1/2}|\theta|^2\,dz. \tag{57}$$

The $\theta$-dependent terms are eliminated by choosing $a_1 = 1/2$, $a_2 = 1/2k^2$, while in order for a balance of the highest-derivative terms in $w$ to be possible, we clearly require $2(\nu_1 + \nu_2) = \alpha$. Substituting for $a_1$, $a_2$ and $\nu_1 = \frac{1}{2}\alpha - \nu_2$, and dividing by $2\mu/R$, (57) becomes

$$0 \le \frac{1}{k^2}\int_0^1\left\{\frac{|w''|^2}{z^\alpha}-2\alpha(\alpha+1)\frac{|w'|^2}{z^{\alpha+2}}+\tfrac{1}{2}\alpha(\alpha+1)(\alpha+2)(\alpha+3)\frac{|w|^2}{z^{\alpha+4}}\right\}dz$$
$$+\int_0^1\left\{2\frac{|w'|^2}{z^\alpha}-\alpha(\alpha+1)\frac{|w|^2}{z^{\alpha+2}}\right\}dz \quad +k^2\int_0^1\frac{|w|^2}{z^\alpha}\,dz$$
$$-\Psi\int_0^1\left\{\frac{|w'|^2}{z^\alpha}-\frac{(\alpha-2\nu_2)(\alpha+2\nu_2+2)}{4}\frac{|w|^2}{z^{\alpha+2}}\right\}dz$$
$$-\Psi\frac{1}{k^2}\int_0^1\left\{\frac{|w''|^2}{z^\alpha}-\frac{(\alpha-2\nu_2)(\alpha+2\nu_2+2)}{4}\frac{|w'|^2}{z^{\alpha+2}}\right\}dz, \tag{58}$$

where $\nu_2$ is still free, and we have defined

$$\Psi=\frac{1}{2(2\nu_2+1)(\alpha+2)}\frac{R}{\mu}\left(\int_0^\delta[\gamma+\mu g(z)]\,z^{\alpha/2+1}\,dz\right)^2. \tag{59}$$

The dependence of this constraint on the control parameter $R$, on the boundary layer thickness $\delta$ and on the parameters $\gamma$ and $\mu$ of the background—and hence on the Biot number $\eta$—is contained in $\Psi$. Hardy-type estimates similar to those in the Appendix of Ref. 41 are used in the following to derive constraints on $\Psi$ depending on the exponents $\alpha$ and $\nu_2$; the effects of these constraints on the scaling and resultant bounds will be discussed in Sec. V.

Rearranging (58), the positivity of the quadratic form is ensured by requiring

$$0\le\frac{1}{k^2}\left\{\left[1-\Psi\right]\int_0^1\frac{|w''|^2}{z^\alpha}\,dz-\left[2\alpha(\alpha+1)-\Psi\frac{(\alpha-2\nu_2)(\alpha+2\nu_2+2)}{4}\right]\int_0^1\frac{|w'|^2}{z^{\alpha+2}}\,dz\right.$$
$$\left.+\frac{1}{2}\alpha(\alpha+1)(\alpha+2)(\alpha+3)\int_0^1\frac{|w|^2}{z^{\alpha+4}}\,dz\right\}$$
$$+\left\{\left[2-\Psi\right]\int_0^1\frac{|w'|^2}{z^\alpha}\,dz-\left[\alpha(\alpha+1)-\Psi\frac{(\alpha-2\nu_2)(\alpha+2\nu_2+2)}{4}\right]\int_0^1\frac{|w|^2}{z^{\alpha+2}}\,dz\right\}$$
$$+k^2\int_0^1\frac{|w|^2}{z^\alpha}\,dz \tag{60}$$

for all sufficiently smooth $w(z)$ with $w = w' = 0$ at $z = 0, 1$.

We shall constrain $\Psi$ so that each power of $k$ is separately non-negative; for the $\mathcal{O}(k^2)$ term this is immediate. In order to establish the desired estimates for the $\mathcal{O}(1)$ and $\mathcal{O}(1/k^2)$ terms, we shall repeatedly apply the Hardy inequality for $p \in \mathbb{R}$

$$\int_0^1\frac{|g'|^2}{z^{p-2}}\,dz\ge\frac{(p-1)^2}{4}\int_0^1\frac{|g|^2}{z^p}\,dz, \tag{61}$$

which holds for sufficiently smooth $g$ provided $|g| = o(z^{(p\,-\,1)/2})$ as $z \to 0^+$ and (if $p < 1$) $g(1) = 0$.

#### *1. The $\mathcal{O}(1)$ terms*

In order for the $\mathcal{O}(1)$ terms in (60) to be non-negative as a whole, the same must first necessarily hold for the highest derivative term, which gives the first constraint on $\Psi$:

$$\Psi \leq 2. \tag{62}$$

Assuming (62), and applying the Hardy inequality (61) with $g = w$ and $p = \alpha + 2$, we find that a sufficient condition for the $\mathcal{O}(1)$ terms in (60) to be non-negative is

$$\big[2 - \Psi\big]\frac{(\alpha+1)^2}{4} - \left[\alpha(\alpha+1) - \Psi\frac{(\alpha-2\nu_2)(\alpha+2\nu_2+2)}{4}\right] \geq 0,$$

which simplifies to

$$\frac{1}{2}\left[1-\alpha^2\right] - \frac{\Psi}{4}\left[1+4\nu_2^2+4\nu_2\right] \geq 0,$$

yielding the constraint

$$\Psi \leq \frac{2(1-\alpha^2)}{(2\nu_2+1)^2}; \tag{63}$$

note that this can be satisfied by positive $\Psi$ only if $|\alpha| < 1$.

#### *2. The $\mathcal{O}(1/k^2)$ terms*

For the $\mathcal{O}(1/k^2)$ terms in (60), again we need the highest derivative term to be non-negative, so that necessarily

$$\Psi \leq 1. \tag{64}$$

Now it turns out that if we proceed directly as before, applying a Hardy inequality to $\int_0^1 |w''|^2/z^\alpha\, dz$, there is no positive $\Psi$ for which the coefficient of $\int_0^1 |w'|^2/z^{\alpha+2}\, dz$ can be made non-negative. Instead, we first introduce a change of variables $w = \phi/z^\nu$; with this substitution, the non-negativity condition for the $\mathcal{O}(1/k^2)$ terms in (60) is

$$\begin{aligned}
0 \leq{}& \big[1-\Psi\big]\int_0^1 \frac{1}{z^\alpha}\left|\mathrm{D}^2\left(\frac{\phi}{z^\nu}\right)\right|^2 dz \\
&- \left[2\alpha(\alpha+1) - \Psi\frac{(\alpha-2\nu_2)(\alpha+2\nu_2+2)}{4}\right]\int_0^1 \frac{1}{z^{\alpha+2}}\left|\mathrm{D}\left(\frac{\phi}{z^\nu}\right)\right|^2 dz \\
&+ \frac{1}{2}\alpha(\alpha+1)(\alpha+2)(\alpha+3)\int_0^1 \frac{1}{z^{\alpha+4}}\left|\frac{\phi}{z^\nu}\right|^2 dz \\
={}& \big[1-\Psi\big]\Bigg\{\int_0^1 \frac{|\phi''|^2}{z^{2\nu+\alpha}}\,dz - 2\nu(\nu+\alpha+2)\int_0^1 \frac{|\phi'|^2}{z^{2\nu+\alpha+2}}\,dz \\
&\qquad + \nu(\nu+1)(\nu+\alpha+2)(\nu+\alpha+3)\int_0^1 \frac{|\phi|^2}{z^{2\nu+\alpha+4}}\,dz\Bigg\} \\
&- \left[2\alpha(\alpha+1) - \Psi\frac{(\alpha-2\nu_2)(\alpha+2\nu_2+2)}{4}\right] \\
&\qquad \times \left\{\int_0^1 \frac{|\phi'|^2}{z^{2\nu+\alpha+2}}\,dz - \nu(\nu+\alpha+3)\int_0^1 \frac{|\phi|^2}{z^{2\nu+\alpha+4}}\,dz\right\} \\
&+ \frac{1}{2}\alpha(\alpha+1)(\alpha+2)(\alpha+3)\int_0^1 \frac{|\phi|^2}{z^{2\nu+\alpha+4}}\,dz,
\end{aligned} \tag{65}$$

where we used the identities (A7) and (A8) to expand the derivatives.

Subject to (64), at this point we can now apply the Hardy estimate (61) with $g = \phi'$ and $p = 2\nu + \alpha + 2$ to control the first derivative term using the second derivative term; substituting and collecting terms, we find that (65) is satisfied when

$$
\begin{aligned}
0 \le \Bigg\{ & \left[ \frac{(2\nu+\alpha+1)^2}{4} - 2\nu(\nu+\alpha+2) \right] \big[1-\Psi\big] \\
& \quad - \left[ 2\alpha(\alpha+1) - \Psi \frac{(\alpha-2\nu_2)(\alpha+2\nu_2+2)}{4} \right] \Bigg\} \int_0^1 \frac{|\phi'|^2}{z^{2\nu+\alpha+2}}\, dz \\
+ \Bigg\{ & \nu(\nu+1)(\nu+\alpha+2)(\nu+\alpha+3)\big[1-\Psi\big] \\
& \quad + \nu(\nu+\alpha+3) \left[ 2\alpha(\alpha+1) - \Psi \frac{(\alpha-2\nu_2)(\alpha+2\nu_2+2)}{4} \right] \\
& \quad + \frac{1}{2}\alpha(\alpha+1)(\alpha+2)(\alpha+3) \Bigg\} \int_0^1 \frac{|\phi|^2}{z^{2\nu+\alpha+4}}\, dz.
\end{aligned} \tag{66}
$$

The total contribution of $\int_0^1 |w''|^2/z^\alpha\, dz$ to the coefficient to the first derivative term, $\frac{1}{4}(2\nu + \alpha + 1)^2 - 2\nu(\nu+\alpha+2)$, is maximized when $\nu = -(\alpha + 3)/2$. Substituting this value of $\nu$ into (66) and evaluating the coefficients algebraically, (66) simplifies considerably to become

$$
\begin{aligned}
0 \le & \left\{ \frac{1}{2}\Big(5 - 3\alpha^2\Big) - \frac{\Psi}{4}\Big((\alpha+3)^2 + (2\nu_2+1)^2\Big) \right\} \int_0^1 z|\phi'|^2\, dz \\
& + \left\{ \frac{(\alpha^2-1)(\alpha^2-9)}{16} - \Psi \frac{(\alpha+3)^2(2\nu_2+1)^2}{16} \right\} \int_0^1 \frac{|\phi|^2}{z}\, dz.
\end{aligned} \tag{67}
$$

The condition that the coefficient of the $\int_0^1 z|\phi'|^2\, dz$ term in (67) is non-negative is equivalent to

$$
\Psi \le \frac{2(5-3\alpha^2)}{(3+\alpha)^2 + (2\nu_2+1)^2}; \tag{68}
$$

for $0 \le \alpha < 1$ this constraint (68) may readily be satisfied for a range of $\Psi$.

Finally, observing that in (67) no further non-trivial Hardy inequality is available (for the case $p = 1$ in (61)), the non-negativity condition on the coefficient of $\int_0^1 |\phi|^2/z\, dz$ gives the final constraint on $\Psi$:

$$
\Psi \le \frac{(9-\alpha^2)}{(3+\alpha)^2} \frac{(1-\alpha^2)}{(2\nu_2+1)^2}. \tag{69}
$$

While the above calculation used the change of variables $w = \phi/z^\nu$ with exponent $\nu = -(\alpha + 3)/2$, one can show that the final constraint (69) on $\Psi$ is in fact independent of the choice of $\nu$ in (65).

Now clearly (64) is stronger than (62), while for $0 \le \alpha < 1$ (69) is a tighter constraint than (63). The five constraints (62), (63), (64), (68), and (69) can then be summarized as

$$
\Psi \le \min \left\{ 1, \frac{2(5-3\alpha^2)}{(3+\alpha)^2 + (2\nu_2+1)^2}, \frac{3-\alpha}{3+\alpha} \frac{(1-\alpha^2)}{(2\nu_2+1)^2} \right\}. \tag{70}
$$

We have thus shown that if for given $\alpha$ and $\nu_2$ the quantity $\Psi$ given by (59) satisfies (70), then the quadratic form in (58) is non-negative, which verifies the admissibility condition on the background $\tau(z)$ (that is, the spectral constraint) and thus implies the bounds (45) and (46).

## V. FINAL SCALING OF THE CONSTRAINT AND THE RESULTANT BOUNDS

For a fixed Biot number $\eta$, the general admissibility constraint (70) is a condition on $\delta$ for a given $R$, implying that $\delta \to 0$ as $R \to \infty$; we remain free to choose $\lambda = -\mu G(\delta) > 0$, $\nu_2 > -1/2$ and $\alpha$ $(0 \leq \alpha < 1)$.

In the following we discuss optimal choices of these parameters and thereby obtain the asymptotic scaling of $\delta$ and hence the desired bounds on $\beta$, $\Delta T$ and the Nusselt number $Nu$; many of the details are deferred to Appendix B.

### A. Optimal choices and leading-order scaling

#### *1. Choice of* $\lambda$

Recalling the bounds on $\beta$ and $\Delta T$ computed for our piecewise-defined background profile in Sec. III A, the form of $\lambda$ may be deduced from considering these bounds in the fixed temperature and fixed flux limits:

In the fixed temperature case $\eta = 0$, we have $\Delta T = 1$ (so $Nu = \beta$), and from (45)

$$\beta \leq \mathcal{B}_0[\tau] = \frac{(1+2\lambda)^2}{2\delta} + T_B.$$

In this case we are free to choose $\lambda = \mathcal{O}(1)$ at no cost to the overall scaling of the bound. In the fixed flux limit $\eta = \infty$, on the other hand, we have $\beta = 1$ (so $Nu = \Delta T^{-1}$), and (46) gives

$$\Delta T \geq \mathcal{D}_\infty[\tau] = 2\delta - 4\lambda - T_B.$$

Now in order to have a non-negative lower bound on $\Delta T$, we require $\lambda = \mathcal{O}(\delta)$ in this limit.

A form of $\lambda$ that satisfies the conditions $\lambda = \mathcal{O}(1)$ for $\eta \ll \delta$ and $\lambda = \mathcal{O}(\delta)$ for $\delta \ll \eta$ is

$$\lambda = c\frac{\delta(1+2\eta)}{\delta+\eta}, \tag{71}$$

where $c > 0$ is an absolute constant to be specified later.

#### *2. Asymptotic form of bounds and constraint*

Recall now from (42) to (46) that in addition to the terms calculated from the linear boundary layer, the bounds on $\beta$ and $\Delta T$ depend on $T_B$, the contribution to $\int_0^1 \tau'^2\,dz$ from the (nonlinear) bulk part of the background profile. As shown in Appendix B 1, for $\lambda$ chosen as (71) and $0 \leq \alpha < 1$ it turns out that in the asymptotic $R \to \infty$ limit ($\delta \to 0^+$) the boundary layer contribution dominates and we may neglect $T_B$ in the bounds. Thus the bounds (45) and (46) in this limit are (using (71)),

$$\beta \leq \mathcal{B}_\eta[\tau] \sim \frac{(1+2\eta+2\lambda)^2}{2(\delta+\eta)(1+2\eta)} = \frac{1+2\eta}{2(\delta+\eta)}\left(1+\frac{2c\delta}{\delta+\eta}\right)^2, \tag{72}$$

$$\Delta T \geq \mathcal{D}_\eta[\tau] \sim \frac{\delta(1+2\eta)}{\delta+\eta} - \frac{4\eta\lambda(1+2\eta+\lambda)}{(\delta+\eta)(1+2\eta)} = \frac{\delta(1+2\eta)}{\delta+\eta}\left[1-\frac{4c\eta}{\delta+\eta}\left(1+\frac{c\delta}{\delta+\eta}\right)\right]. \tag{73}$$

The admissibility condition (70) is given in terms of the quantity $\Psi$ introduced in (59). For fixed $0 < \delta < \frac{1}{2}$, using (38), (40), and (41) we evaluate the integral appearing in the formula for $\Psi$

as follows:

$$\begin{aligned}\int_0^\delta [\gamma+\mu g(z)]\, z^{\alpha/2+1}\, dz &= \int_0^\delta \left[\gamma - \frac{\lambda}{G(\delta)}\left(\frac{1}{z^\alpha}+\frac{1}{(1-z)^\alpha}\right)\right] z^{\alpha/2+1}\, dz\\ &\le \int_0^\delta \left[\gamma z^{\alpha/2+1} - \frac{\lambda}{G(\delta)} z^{-\alpha/2+1} - \frac{\lambda}{G(\delta)}\frac{z^{\alpha/2+1}}{(1-1/2)^\alpha}\right] dz\\ &= \frac{2}{4+\alpha}\frac{1+2\eta+2\lambda}{2(\delta+\eta)}\delta^{2+\alpha/2}\\ &\qquad + \frac{2(1-\alpha)\lambda}{[(1-\delta)^{1-\alpha}-\delta^{1-\alpha}]}\frac{\delta^{2-\alpha/2}}{4-\alpha}\left(1+\frac{4-\alpha}{4+\alpha}(2\delta)^\alpha\right), \end{aligned} \tag{74}$$

where the first ($\gamma$-dependent) term is due to the boundary layer part of $\tau(z)$, and the remainder derives from the bulk. In Appendix B 1 it is shown that for $0 \le \alpha < 1$ and $\lambda$ given in (71), in this case also the bulk contribution is asymptotically small as $R \to \infty$ ($\delta \to 0^+$). Hence, we find in this limit,

$$\begin{aligned}\Psi &\sim \frac{1}{2(2\nu_2+1)(2+\alpha)}\left(\frac{-G(\delta)}{\lambda}\right) R \left(\frac{2}{4+\alpha}\frac{1+2\eta+2\lambda}{2(\delta+\eta)}\delta^{2+\alpha/2}\right)^2\\ &= \frac{1}{2(2\nu_2+1)(2+\alpha)(4+\alpha)^2}\frac{[(1-\delta)^{1-\alpha}-\delta^{1-\alpha}]}{1-\alpha}\left(\frac{1+2\eta}{\delta+\eta}\right)\left(1+\frac{2c\delta}{\delta+\eta}\right)^2 \frac{1}{c}\delta^{3+\alpha} R. \end{aligned} \tag{75}$$

### 3. Optimal choice of $\nu_2$

It remains to choose the exponent $\alpha$ appearing in the bulk part of the background $\tau(z)$ as in (40), and an associated $\nu_2 > -\frac{1}{2}$.

If $\alpha$ is strictly bounded away from 1 (so $1-\alpha = \mathcal{O}(1)$), we can assume $\alpha = \alpha_0 < 1$; similar considerations hold if $\alpha \to \alpha_0 < 1$ as $R \to \infty$. In this case we can choose any constant $\nu_2$ without affecting the scaling of the bound, and the admissibility constraint (70) is $\Psi \le \mathcal{O}(1)$. For any Biot number $\eta$, the computation of the overall bound (using the results of Appendix B 3 for $\eta > 0$) then yields $Nu \le \mathcal{O}(Ra^{1/(2+\alpha_0)})$. That is, the choice $\alpha \sim \alpha_0 < 1$ allows us to obtain *Nu*-*Ra* bounds scaling with any exponent strictly greater than 1/3, of course with an $\alpha_0$-dependent prefactor. (Note in particular that for $\alpha = 0$, the background profile $\tau(z)$ is piecewise linear, and we obtain the bound $Nu \le \mathcal{O}(Ra^{1/2})$, as for finite Prandtl number (see Ref. 18); to reduce the exponent below 1/2 we need a nonlinear bulk profile with $\alpha > 0$.)

The above considerations suggest that we may optimize the scaling of the *Nu*-*Ra* bound by letting $\alpha \to 1^-$ as $R \to \infty$. For ease of notation, in the remainder let us define $a \equiv 1 - \alpha$, where $0 < a \ll 1$.

The optimal choice of $\nu_2$ is now that which weakens the constraint on the $\nu_2$-independent quantity $\widetilde{\Psi} \equiv (2\nu_2+1)\Psi$, and hence on $\delta$, as much as possible for a given $\alpha$, in order to strengthen the overall bound on *Nu*. This is achieved when (to leading order in $a \ll 1$) the two strongest constraints (68) and (69) coincide, which occurs when $(2\nu_2 + 1) \sim 2a^{1/2}$. A convenient choice of $\nu_2$ with the correct asymptotic behavior is given by

$$(2\nu_2+1)^2 = 2(1-\alpha^2) = 4a - 2a^2 > 0 \qquad \Rightarrow \qquad \nu_2 = \frac{1}{2}\left(-1+\sqrt{2(1-\alpha^2)}\right); \tag{76}$$

for this $\nu_2$, the admissibility condition (70) finally becomes

$$\Psi \le \min\left\{1, \frac{10-6\alpha^2}{11+6\alpha-\alpha^2}, \frac{3-\alpha}{2(3+\alpha)}\right\} = \frac{3-\alpha}{2(3+\alpha)}. \tag{77}$$

### B. Scaling of $\delta$ and bounds on *Nu*

Substituting $\alpha = 1 - a$, and using (76) for $\nu_2$, combining (75) and (77) the spectral constraint on $\delta$ is

$$\frac{[(1-\delta)^a - \delta^a]}{2a(4a-2a^2)^{1/2}(3-a)(5-a)^2}\left(\frac{1+2\eta}{\delta+\eta}\right)\left(1+\frac{2c\delta}{\delta+\eta}\right)^2\frac{1}{c}\delta^{4-a}R\left(1+\mathcal{O}(a)\right) \le \frac{2+a}{2(4-a)},$$

giving

$$\left(\frac{1+2\eta}{\delta+\eta}\right)\left(1+\frac{2c\delta}{\delta+\eta}\right)^2\frac{1}{c}\delta^{4-a} \le \frac{3\cdot 5^2}{[(1-\delta)^a-\delta^a]}a^{3/2}R^{-1}\left(1+\mathcal{O}(a)\right). \tag{78}$$

We interpret this as follows: Let the Biot number $\eta$ and control parameter $R$ be fixed; and assume that $a = a(R)$ and $c$ are specified. The overall constraint (77) is, effectively, a condition on $\delta$ for $\tau(z)$ to be an admissible background (that is, for (35) to be satisfied and hence for the bounds on $\beta$ and $\Delta T$ to hold) which takes the form $\delta \le \delta_c \equiv \delta_c(\eta, R)$. The best bound using this approach is then obtained by letting $\delta = \delta_c$. In the aforegoing discussion leading to (78) we have shown that asymptotically as $R \to \infty$, so $a \to 0$, the optimal $\delta_c$ satisfies

$$\left(\frac{1+2\eta}{\delta+\eta}\right)\left(1+\frac{2c\delta}{\delta+\eta}\right)^2\frac{1}{c}\delta^{4-a} \sim \frac{3\cdot 5^2}{[(1-\delta)^a-\delta^a]}a^{3/2}R^{-1}. \tag{79}$$

To proceed further, we consider the fixed temperature and fixed flux scaling behaviors separately.

#### *1. Fixed temperature boundary conditions*

For perfectly conducting boundaries $\eta = 0$, giving Dirichlet (fixed temperature) thermal BCs, we have $\Delta T = \Delta\tau = 1$, $\lambda = c$, and from (72)

$$Nu = \beta \le \mathcal{B}_0[\tau] \sim \frac{(1+2c)^2}{2\delta}, \tag{80}$$

where by (79) the optimal value of $\delta$ satisfies

$$\delta^{3-a} \sim \left(\frac{3\cdot 5^2}{[(1-\delta)^a-\delta^a]}\frac{c}{(1+2c)^2}\right)\frac{a^{3/2}}{R}. \tag{81}$$

It is clear from (80) that the best bound is obtained by choosing $a$ to maximize $\delta$ satisfying (81).

The details of the maximization are discussed in Appendix B 2: the relation (81) has the form of (B1) with $p_1 = 3$, $p_2 = 3/2$, and we find that the optimal value of $a$ (B3) in this case is $a = 9/(2\ln R)$, so that by (B5) we find $\delta^a \sim \mathrm{e}^{-3/2}$ and hence $[(1-\delta)^a - \delta^a] \sim 1 - \mathrm{e}^{-3/2}$.

Using the result from (B4), the optimal value $\delta_c$ of $\delta$ thus satisfies

$$\begin{aligned}\delta &\sim \left(\frac{3\cdot 5^2}{1-\mathrm{e}^{-3/2}}\frac{c}{(1+2c)^2}\right)^{1/3}\left(\frac{9}{2}\right)^{1/2}\mathrm{e}^{-1/2}R^{-1/3}(\ln R)^{-1/2}\\ &= \frac{3^{4/3}5^{2/3}}{2^{1/2}\mathrm{e}^{1/2}(1-\mathrm{e}^{-3/2})^{1/3}}\frac{c^{1/3}}{(1+2c)^{2/3}}R^{-1/3}(\ln R)^{-1/2},\end{aligned}$$

giving the bound (from (80))

$$Nu = \beta \le \mathcal{B}_0[\tau] \sim \frac{\mathrm{e}^{1/2}(1-\mathrm{e}^{-3/2})^{1/3}}{2^{1/2}3^{4/3}5^{2/3}}\frac{(1+2c)^{8/3}}{c^{1/3}}R^{1/3}(\ln R)^{1/2}.$$

Now choose the constant $c$ to minimize this bound, which occurs at $c = 1/14$. Substituting for $c$, and recalling that in the fixed temperature case $\eta = 0$ we have $Ra = R\Delta T = R$, we obtain an asymptotic bound on the Nusselt number for fixed temperature, infinite Prandtl number convection:

$$Nu \lesssim \frac{2^{47/6}e^{1/2}(1-e^{-3/2})^{1/3}}{3^{4/3}5^{2/3}7^{7/3}} Ra^{1/3}(\ln Ra)^{1/2} = 0.29149\cdots Ra^{1/3}(\ln Ra)^{1/2}. \tag{82}$$

#### 2. Fixed flux boundary conditions

In the limit of perfectly insulating boundaries we have Neumann (fixed flux) BCs $\eta = \infty$, and the scaling leading to the overall $Nu$ bound proceeds somewhat differently: In this case $\beta = 1$, so that $Nu = 1/\Delta T$, where (with $\lambda = 2c\delta$) from (73)

$$\Delta T \geq \mathcal{D}_\infty[\tau] \sim 2\delta(1-4c), \tag{83}$$

and we clearly require $c < 1/4$.

Proceeding as before, we can choose $a$ to estimate the optimal $\delta$ and hence $Nu = \mathcal{O}(1/\delta)$ in terms of the control parameter $R$. To find the bound on $Nu$ in terms of the Rayleigh number $Ra$, following Ref. 19 we further use $Ra = R\Delta T$ to relate $R$ and $Ra$, as described in Appendix B 3. We omit the details here, as they are a special case $\eta = \infty$ of the general Biot number calculation done below.

#### 3. Scaling regimes

For general imperfectly conducting boundaries, we have mixed (Robin) thermal BCs with (fixed) Biot number $\eta > 0$. The nature of the bounds then depends on the relative sizes of $\eta$ and $\delta$, and as in the case of finite Prandtl number convection,[20,33] there are distinct scaling regimes behaving as in the fixed temperature and fixed flux limits.

For sufficiently small Biot number $\eta < 1/2$, when the control parameter $R$ is not too large the boundary layer thickness $\delta$ is much larger than $\eta$, in which case the *fixed temperature scaling* pertains: For $\eta \ll \delta$ (and thus $\eta \ll 1$) we have (from (71) to (73)) $\lambda = \mathcal{O}(1)$, $\beta \leq \mathcal{B}_\eta[\tau] \sim (1+2c)^2/2\delta = \mathcal{O}(\delta^{-1})$ and $\Delta T \geq \mathcal{D}_\eta[\tau] \sim 1$, so that $Ra = R\Delta T \sim R$. In this case the scaling of $a$ and $\delta$ is just as in the fixed temperature limit $\eta = 0$ discussed above.

As the control parameter $R$ increases, however, $\delta$ decreases and eventually falls below the Biot number $\eta$, and the above scaling no longer applies. Instead, for sufficiently large $R$ we enter the *fixed flux scaling regime*, when $\delta \ll \eta$. In this case we find (from (71) to (73)) $\lambda = \mathcal{O}(\delta)$, $\beta \leq \mathcal{B}_\eta[\tau] \sim \mathcal{O}(1)$ and $\Delta T \geq \mathcal{D}_\eta[\tau] \sim \mathcal{O}(\delta)$, as in the fixed flux case. This scaling does not occur for fixed temperature BCs $\eta = 0$, but is the asymptotic behavior for any $\eta > 0$; that is, for any imperfectly conducting surface boundary condition the system will, at sufficiently large Rayleigh numbers, behave as for fixed-flux boundaries.

#### 4. Asymptotic bounds for imperfectly conducting boundaries

We derive the asymptotic scaling behavior as $R \to \infty$ for general imperfectly conducting boundaries, with fixed Biot number $\eta > 0$ (including the fixed flux case $\eta = \infty$), assuming strong enough thermal driving (large enough $R$) that $\delta \ll 1$ and $\delta \ll \eta$:

With $\lambda \sim c\delta(1 + 2\eta)/\eta$, by (72) and (73) in this limit the averaged heat flux and averaged temperature drop are bounded by

$$\beta \leq \mathcal{B}_\eta[\tau] \sim \frac{1+2\eta}{2\eta} \tag{84}$$

and

$$\Delta T \geq \mathcal{D}_\eta[\tau] \sim \frac{\delta(1+2\eta)}{\eta}(1-4c), \tag{85}$$

so that the bound on the Nusselt number is

$$Nu = \frac{\beta}{\Delta T} \leq \frac{\mathcal{B}_\eta[\tau]}{\mathcal{D}_\eta[\tau]} \sim \frac{1}{2\delta}(1-4c)^{-1}. \tag{86}$$

Using (79), for $\delta \ll \eta$ the optimal $\delta$ is given by

$$\delta^{4-a} \sim \left(\frac{3\cdot 5^2}{[(1-\delta)^a-\delta^a]}\frac{\eta}{1+2\eta}c\right)\frac{a^{3/2}}{R}. \tag{87}$$

Comparing (87) with (B1), we may use the conclusions of Appendix B 2 with $p_1 = 4$, $p_2 = 3/2$: by (B3) and (B5) the asympotically optimal choice is $a = 6/\ln R$, in which case $[(1 - \delta)^a - \delta^a] \sim 1 - e^{-3/2}$. By (B4) we thus find that the critical value $\delta_c$ satisfying the admissibility constraint scales as

$$\begin{aligned}\delta &\sim \left(\frac{3\cdot 5^2}{1-e^{-3/2}}\frac{\eta}{1+2\eta}c\right)^{1/4} 6^{3/8}e^{-3/8}R^{-1/4}(\ln R)^{-3/8}\\ &= \frac{2^{3/8}3^{5/8}5^{1/2}}{e^{3/8}(1-e^{-3/2})^{1/4}}\left(\frac{\eta}{1+2\eta}\right)^{1/4} c^{1/4}R^{-1/4}(\ln R)^{-3/8};\end{aligned}$$

hence the lower bound on $\Delta T$ becomes

$$\Delta T \geq \mathcal{D}_\eta[\tau] \sim \frac{2^{3/8}3^{5/8}5^{1/2}}{e^{3/8}(1-e^{-3/2})^{1/4}}\left(\frac{1+2\eta}{\eta}\right)^{3/4} c^{1/4}(1-4c)R^{-1/4}(\ln R)^{-3/8},$$

with a corresponding upper bound on $Nu$. The maximum of $c^{1/4}(1 - 4c)$ occurs at $c = 1/20$; substituting, we find the lower bound for $\Delta T$

$$\Delta T \geq \mathcal{D}_\eta[\tau] \sim \frac{2^{15/8}3^{5/8}}{5^{3/4}e^{3/8}(1-e^{-3/2})^{1/4}}\left(\frac{1+2\eta}{\eta}\right)^{3/4} R^{-1/4}(\ln R)^{-3/8}, \tag{88}$$

and hence obtain the asymptotic upper bound on the Nusselt number in terms of the control parameter $R$, valid for $\delta \ll \eta$:

$$Nu = \frac{\beta}{\Delta T} \lesssim \frac{1+2\eta}{2\eta}\frac{1}{\mathcal{D}_\eta[\tau]} \sim \frac{5^{3/4}e^{3/8}(1-e^{-3/2})^{1/4}}{2^{23/8}3^{5/8}}\left(\frac{1+2\eta}{\eta}\right)^{1/4} R^{-1/4}(\ln R)^{-3/8}. \tag{89}$$

In order to find the bound on $Nu$ in terms of the usual Rayleigh number $Ra$, we relate $Ra$ and $R$ using $Ra = R\Delta T$ (6) and hence estimate $R$ in terms of $Ra$. This calculation is performed in the general case in Appendix B 3, and we can read off those results directly: The asymptotic bound (88) on $\Delta T$ has the form of (B6) with exponents $p_1 = 1/4$ and $p_2 = 3/8$, so $p_1/(1 - p_1) = 1/3$ and $p_2/(1 - p_1) = 1/2$. With (B8) giving $R$ in terms of $Ra$, we can then bound $\Delta T$ and hence $Nu$ as a function of $Ra$.

Using (12), (84), and (B9) (with the parameters for (B6) given in (88)), the final result for the asymptotic bound on the Nusselt number $Nu$ in terms of $Ra$ for imperfectly conducting boundaries $\eta > 0$ (including the fixed flux $\eta = \infty$ case) is

$$\begin{aligned}Nu &= \frac{\beta}{\Delta T} \lesssim \frac{1+2\eta}{2\eta}\Delta T^{-1}\\ &\lesssim \frac{1+2\eta}{2\eta}\left[\frac{2^{15/8}3^{5/8}}{5^{3/4}e^{3/8}(1-e^{-3/2})^{1/4}}\left(\frac{1+2\eta}{\eta}\right)^{3/4}\right]^{-4/3}\left(\frac{3}{4}\right)^{-1/2} Ra^{1/3}(\ln Ra)^{1/2}\\ &= \frac{5e^{1/2}(1-e^{-3/2})^{1/3}}{2^{5/2}3^{4/3}} Ra^{1/3}(\ln Ra)^{1/2} = 0.30962\cdots Ra^{1/3}(\ln Ra)^{1/2}.\end{aligned} \tag{90}$$

Note that while the dependence (89) of $Nu$ on the control parameter $R$ depends on the Biot number $\eta$, the asymptotic bound (90) on $Nu$ in terms of the Rayleigh number $Ra$ is independent of the Biot number for $\eta > 0$. Comparing this bound with that in the fixed temperature case (82), we see that the scaling with $Ra$ is the same, although the prefactor for nonzero Biot number is slightly larger (in this bounding formalism), as was observed also for finite Prandtl number.[20]

## VI. CONCLUSIONS AND OUTLOOK

The bound obtained here in (82) and (90) is not as tight as the fixed temperature bounds of either Ref. 28 or Ref. 29, which have weaker logarithmic corrections. The significance of the present result, however, is that it holds more generically for imperfectly conducting boundaries. The persistence of the (logarithmically corrected) $\frac{1}{3}$ scaling at infinite Prandtl number for a no-slip fluid supports, in concert with similar results for finite Pr (see Refs. 19, 20, 33, and 34), the understanding that changes in the thermal boundary conditions do not materially affect the scaling of the heat transport. Our methods indicate how to make use of the slaving of vertical velocity to temperature even when there is no direct control on the thermal fluctuations $\theta$ near the boundary. Other than the higher-order Hardy inequalities used here, the primary difference between the current result and that obtained previously via the background method for fixed temperature boundaries is in the choice of the background temperature profile. As discussed in Sec. III A, without being able to fix the temperature at the top and bottom plates we are unable to use the purely logarithmic background profile[28] ($\alpha = 1$ in (40)); however, since the optimal choice is $\alpha = 1 - \mathcal{O}(1/\ln R) \to 1$ as $R \to \infty$, the optimal asymptotic background profile in general agrees qualitatively with that in the fixed temperature case.

As described in Ref. 28, the successful use of a non-monotonic background profile indicates that the marginally stable boundary layer argument proposed by Malkus (see Ref. 10) is valid for infinite Prandtl number convection so long as the bulk temperature profile can be modified to weaken the constraint on the size of the boundary layer as $R \to \infty$. The cost of balancing the size of the boundary layer with the bulk is then the logarithmic correction to the final bound. While generalizations of the background method as employed in Refs. 26 and 29 have yielded bounds with weaker logarithmic dependence, it has not yet been possible to remove the logarithm entirely.

In the light of recent results on slippery convection,[24,25] it is worthwhile to consider the effect that variations from a fixed temperature boundary condition may have in that context. To date the no-slip boundary condition has proved essential for bounding the heat transport when the temperature is not fixed at the top and bottom plates. In addition, the best known bounds for slippery convection have not required a nonlinear stably stratifying bulk background profile, even at infinite $Pr$,[25] and indeed, the careful numerical and asymptotic calculations of[23] indicate that such stable stratification is not necessary. It is worth considering whether such differences in the implementation of the background method may have physical ramifications beyond the scaling of the Nusselt number with $Ra$ for no-slip or stress-free convection.

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## APPENDIX A: SOME USEFUL INTEGRAL IDENTITIES

### 1. Weighted integration by parts

Evaluating $\int g(z)\mathrm{Re}[w\theta^*]\,dz$ in (47), where $\theta$ and $w$ are related through (48) and $g(z)$ takes the power law form given by (40), requires multiple integrations by parts for functions weighted by a power of $z$. For ease of reference we record the relevant integral identities here.

The boundary terms depend on values of $w$ and its derivatives at the endpoints of the domain of integration. For generality, the identities below are stated for the case that $w$ vanishes at the endpoints, but with no assumptions on $w'$ or $w''$, as would be relevant for a fluid between rigid impermeable walls under either no-slip or free-slip BCs.

Thus, integrating over an interval $[c, d]$ with $w(c) = w(d) = 0$, for sufficiently smooth functions $w(z)$ and for $\alpha \in \mathbb{R}$, we have

$$\int_c^d \frac{w' w^* + \text{c.c.}}{z^\alpha}\, dz = \alpha \int_c^d \frac{|w|^2}{z^{\alpha+1}}\, dz, \tag{A1}$$

$$\int_c^d \frac{w'' w^* + \text{c.c.}}{z^\alpha}\, dz = -2 \int_c^d \frac{|w'|^2}{z^\alpha}\, dz + \alpha(\alpha+1) \int_c^d \frac{|w|^2}{z^{\alpha+2}}\, dz, \tag{A2}$$

$$\int_c^d \frac{w'' w'^* + \text{c.c.}}{z^\alpha}\, dz = \alpha \int_c^d \frac{|w'|^2}{z^{\alpha+1}}\, dz + \left(\frac{|w'|^2}{z^\alpha}\right)\Bigg|_{z=c}^{d}, \tag{A3}$$

$$\int_c^d \frac{w''' w^* + \text{c.c.}}{z^\alpha}\, dz = -3\alpha \int_c^d \frac{|w'|^2}{z^{\alpha+1}}\, dz + \alpha(\alpha+1)(\alpha+2) \int_c^d \frac{|w|^2}{z^{\alpha+3}}\, dz - \left(\frac{|w'|^2}{z^\alpha}\right)\Bigg|_{z=c}^{d}, \tag{A4}$$

$$\int_c^d \frac{w''' w'^* + \text{c.c.}}{z^\alpha}\, dz = -2 \int_c^d \frac{|w''|^2}{z^\alpha}\, dz + \alpha(\alpha+1) \int_c^d \frac{|w'|^2}{z^{\alpha+2}}\, dz + \left(\frac{\left(|w'|^2\right)'}{z^\alpha} + \alpha \frac{|w'|^2}{z^{\alpha+1}}\right)\Bigg|_{z=c}^{d}, \tag{A5}$$

$$\int_c^d \frac{w'''' w^* + \text{c.c.}}{z^\alpha}\, dz = 2 \int_c^d \frac{|w''|^2}{z^\alpha}\, dz - 4\alpha(\alpha+1) \int_c^d \frac{|w'|^2}{z^{\alpha+2}}\, dz + \alpha(\alpha+1)(\alpha+2)(\alpha+3) \int_c^d \frac{|w|^2}{z^{\alpha+4}}\, dz - \left(\frac{\left(|w'|^2\right)'}{z^\alpha} + 2\alpha \frac{|w'|^2}{z^{\alpha+1}}\right)\Bigg|_{z=c}^{d}. \tag{A6}$$

Note that if $w'(c) = w'(d) = 0$ (for instance, for the vertical velocity component with no-slip BCs), then all boundary terms vanish. In contrast, if stress-free BCs are considered, these quantities become relevant, indicating part of the reason that these results are not easily extended to slippery (stress-free) convection.

If one of the limits of integration is at $z = 0$ (say $c = 0$), then there are constraints on $\alpha$ for these integral identities to hold, depending on the local growth of $w$ and its derivatives. Specifically, assume $|w(z)| = \mathcal{O}(z^{\alpha_1})$ as $z \to 0$ with $w(0) = w'(0) = 0$; then the boundary terms in all the above identities vanish provided $\alpha < 2\alpha_1 - 3$ ((A1)–(A4) can be established under somewhat weaker conditions). This condition guarantees that the necessary integrals converge.

### 2. Weighted quadratic forms

The following integral identities for (weighted) quadratic forms of derivatives of functions rescaled by a power of $z$ are helpful in our estimates on the boundary term and the conditions for admissibility of the background. As above, we state the identities under the assumption that $w(c) = w(d) = 0$, for sufficiently smooth functions $w$ and for $\alpha, \beta \in \mathbb{R}$.

By the product rule and the above weighted integration by parts identities,

$$\begin{aligned}
\int_c^d \frac{1}{z^\alpha} \left| \mathrm{D}\left(\frac{w}{z^\beta}\right) \right|^2 dz &= \int_c^d \frac{1}{z^\alpha} \left| \frac{w'}{z^\beta} - \beta \frac{w}{z^{\beta+1}} \right|^2 dz \\
&= \int_c^d \frac{1}{z^\alpha} \left[ \frac{|w'|^2}{z^{2\beta}} - \beta \frac{w' w^* + \text{c.c.}}{z^{2\beta+1}} + \beta^2 \frac{|w|^2}{z^{2\beta+2}} \right] dz \\
&= \int_c^d \frac{|w'|^2}{z^{2\beta+\alpha}}\, dz - \beta(\beta+\alpha+1) \int_c^d \frac{|w|^2}{z^{2\beta+\alpha+2}}\, dz,
\end{aligned} \tag{A7}$$

where we used (A1). If $c = 0$, then with $|w(z)| = \mathcal{O}(z^{\alpha_1})$ as $z \to 0$, we have convergence with vanishing boundary term at the origin provided $2\beta + \alpha + 1 < 2\alpha_1$.

The corresponding second derivative identity is, similarly,

$$\begin{aligned}\int_c^d \frac{1}{z^\alpha}\left|\mathrm{D}^2\left(\frac{w}{z^\beta}\right)\right|^2 dz &= \int_c^d \frac{1}{z^\alpha}\left|\frac{w''}{z^\beta} - 2\beta\frac{w'}{z^{\beta+1}} + \beta(\beta+1)\frac{w}{z^{\beta+2}}\right|^2 dz \\ &= \int_c^d \left[\frac{|w''|^2}{z^{2\beta+\alpha}} + 4\beta^2\frac{|w'|^2}{z^{2\beta+\alpha+2}} + \beta^2(\beta+1)^2\frac{|w|^2}{z^{2\beta+\alpha+4}} - 2\beta\frac{w''w'^* + \text{c.c.}}{z^{2\beta+\alpha+1}}\right. \\ &\qquad \left. +\beta(\beta+1)\frac{w''w^* + \text{c.c.}}{z^{2\beta+\alpha+2}} - 2\beta^2(\beta+1)\frac{w'w^* + \text{c.c.}}{z^{2\beta+\alpha+3}}\right] dz \\ &= \int_c^d \frac{|w''|^2}{z^{2\beta+\alpha}}\,dz - 2\beta(\beta+\alpha+2)\int_c^d \frac{|w'|^2}{z^{2\beta+\alpha+2}}\,dz \\ &\qquad + \beta(\beta+1)(\beta+\alpha+2)(\beta+\alpha+3)\int_c^d \frac{|w|^2}{z^{2\beta+\alpha+4}}\,dz - 2\beta\left(\frac{|w'|^2}{z^{2\beta+\alpha+1}}\right)\Bigg|_{z=c}^{d},\end{aligned} \tag{A8}$$

where we used (A3), (A2), and (A1) before simplifying, and the remaining boundary term vanishes when $w'(c) = w'(d) = 0$. The convergence condition for the integrals and the corresponding boundary terms in this case, for $c = 0$ and $|w(z)| = \mathcal{O}(z^{\alpha_1})$, is $2\beta + \alpha + 3 < 2\alpha_1$.

## APPENDIX B: DETAILS OF THE OPTIMAL SCALING CALCULATIONS

### 1. Dominant boundary layer contribution to constraint and bounds

Motivated by Ref. 28, the background field $\tau(z)$ (given in Sec. III, with $\lambda = -\mu G(\delta)$ as in (71)) is defined piecewise to be linear in the "boundary layer" regions $[0, \delta]$ and $[1 - \delta, 1]$, and to have a profile in the interior, or "bulk," $\delta < z < 1 - \delta$, which is in general nonlinear (for $\alpha > 0$). The bulk profile is crucial for providing control of the indefinite term to allow satisfaction of the spectral constraint, thereby permitting improvement in the overall bounds from the $Nu \le \mathcal{O}(Ra^{1/2})$ scaling available for finite Prandtl number. However, we show in this appendix that in the limit of large $R$ (so $\delta \ll 1$) the bulk contributions are negligible, relative to those from the boundary layer, in the bounds on $\beta$ and $\Delta T$, and in the term $\Psi$ defined in (59), which appears in the admissibility condition (70).

This conclusion is valid for all $0 \le \alpha < 1$; with $a = 1 - \alpha$, we distinguish between the following cases:

- Exponent $\alpha$ bounded away from 1, so $a = \mathcal{O}(1)$, $\delta^a \ll 1$, $[(1 - \delta)^a - \delta^a] \sim 1$.
- Exponent $\alpha \to 1$ as $R \to \infty$ ($\delta \to 0$), so $a \ll 1$. Here we need to assume that $[(1 - \delta)^a - \delta^a] = \mathcal{O}(1)$, remaining strictly bounded away from 0; this is verified in Appendix B 2 below, where we find that for the optimal $a(R)$ and $\delta(R)$, we have $\delta^a \sim \mathrm{e}^{-p_2} = \mathcal{O}(1)$ for some $p_2 > 0$.

The scaling behavior also depends on the relative sizes of $\eta$ and $\delta$; we consider separately the "fixed temperature scaling" limit $\eta \ll \delta$, when $\lambda = \mathcal{O}(1)$, and the "fixed flux" limit $\delta \ll \eta$, in which case $\lambda = \mathcal{O}(\delta)$.

The relevant formulas for the bounds on $\beta$ and $\Delta T$ are given for the power law background by (45) and (46). Here the bulk contribution is encapsulated in the term $T_B = \int_\delta^{1-\delta} \tau'^2\,dz$ (evaluated in (43) with (44)), which scales as $\mathcal{O}(a^2\lambda^2\delta^{2a-1})$ (for $a < \frac{1}{2}$).[42] For the bound $\mathcal{B}_\eta[\tau]$ on $\beta$ given in (45), we thus need to compare the boundary layer (BL) term $(1 + 2\eta + 2\lambda)^2/2(\delta + \eta)$ with the bulk contribution $T_B$; while for the bound $\mathcal{D}_\eta[\tau]$ on $\Delta T$, from (46) the comparison is between the BL term $\delta(1 + 2\eta)/(\delta + \eta)$ and the bulk term $2\eta T_B/(1 + 2\eta)$.

The integral $\int_0^\delta [\gamma + \mu g(z)]\, z^{\alpha/2+1}\,dz$ appearing in $\Psi$ (59) is evaluated in (74), again revealing distinct terms due to the boundary layer and bulk parts of the background profile. Recalling from (71) that $\lambda \le \mathcal{O}(1)$, the BL contribution scales as $\mathcal{O}\big(\delta^{(5-a)/2}/(\delta + \eta)\big)$, while the bulk term is $\mathcal{O}\big(a\lambda\delta^{(3+a)/2}\big)$.

The asymptotic scaling of these terms is summarized in Table I, where we see that in each case the bulk contributions to $\mathcal{B}_\eta[\tau]$, $\mathcal{D}_\eta[\tau]$, and $\Psi$ are asymptotically negligible relative to the

TABLE I. Scaling of boundary layer (BL) and bulk terms in bounds and $\Psi$; in each case the bulk term is asymptotically small relative to the corresponding BL term.

| | Fixed temperature scaling $\eta \ll \delta$ | | Fixed flux scaling $\delta \ll \eta$ | |
|---|---|---|---|---|
| | BL term | Bulk term | BL term | Bulk term |
| Bound $\mathcal{B}_\eta[\tau]$ on $\beta$ | $\mathcal{O}(\delta^{-1})$ | $\mathcal{O}(a^2\delta^{2a-1})$ | $\mathcal{O}(1)$ | $\mathcal{O}(a^2\delta^{2a+1})$ |
| Bound $\mathcal{D}_\eta[\tau]$ on $\Delta T$ | $\mathcal{O}(1)$ | $\mathcal{O}\big((\eta/\delta)a^2\delta^{2a}\big)$ | $\mathcal{O}(\delta)$ | $\mathcal{O}(a^2\delta^{2a+1})$ |
| Integral in $\Psi$ | $\mathcal{O}(\delta^{(3-a)/2})$ | $\mathcal{O}(a\delta^{(3+a)/2})$ | $\mathcal{O}(\delta^{(5-a)/2})$ | $\mathcal{O}(a\delta^{(5+a)/2})$ |

corresponding boundary layer terms as $\delta \to 0$. It follows that to leading order, the bounds on $\beta$ and $\Delta T$ are given by (72) and (73), while (75) gives an asymptotically valid formula for the expression $\Psi$.

### 2. Optimal choice of $a = 1 - \alpha$ and asymptotic scaling of $\delta$ with $R$

As shown in Sec. V, the admissibility condition on the background $\tau(z)$ eventually reduces to a constraint on $\delta$ of the form $\delta \le \delta_c$, where the optimal choice $\delta_c$ is given asymptotically as $R \to \infty$ as the solution of

$$\delta^{p_1 - a} \sim \frac{k'}{[(1-\delta)^a - \delta^a]}\frac{a^{p_2}}{R}. \tag{B1}$$

Here $p_1 > 0$, $p_2 > 0$, and $k' > 0$ is an $R$- and $a$-independent constant. Under the assumption $0 < a \ll 1$, with $a \to 0$ as $R \to \infty$, we are free to choose $a$ to maximize $\delta$ satisfying (B1), and wish to derive the optimal choice of $a$ and the ensuing scaling of $\delta$ with $R$.

Since both $\delta \to 0$ and $a \to 0$ as $R \to \infty$, we must assume that the relationship between $\delta$ and $a$ gives $\lim_{R\to\infty}\delta^a < 1$, so that the denominator $[(1-\delta)^a - \delta^a] = \mathcal{O}(1)$ is strictly bounded below away from zero. We shall proceed under this assumption, and then verify it below for the optimal $a$ and $\delta$.

Thus from (B1) we have, for some constant $k > 0$,

$$\delta \sim k^{1/(p_1-a)}\left(\frac{a^{p_2}}{R}\right)^{1/(p_1-a)} = k^{1/(p_1-a)}\exp\left[\frac{1}{p_1-a}(p_2\ln a - \ln R)\right]. \tag{B2}$$

The maximum of $(p_2 \ln a - \ln R)/(p_1 - a)$ over $a$ occurs when $p_1p_2/a = \ln R - p_2\ln a + p_2$, or $a \sim p_1p_2/\ln R$ to leading order as $R \to \infty$. To find the optimal scaling of $\delta$ with $R$, we thus choose

$$a = \frac{p_1p_2}{\ln R}. \tag{B3}$$

For this $a$, we compute (as $R \to \infty$)

$$R^{-\frac{1}{p_1-a}} = R^{-\frac{1}{p_1}\left(1+\frac{a}{p_1}+\mathcal{O}(a^2)\right)} = R^{-\frac{1}{p_1}-\frac{p_2}{p_1\ln R}+\mathcal{O}\left(\frac{1}{(\ln R)^2}\right)} \sim \mathrm{e}^{-p_2/p_1}R^{-1/p_1},$$

$$a^{\frac{p_2}{p_1-a}} = (p_1p_2)^{\frac{p_2}{p_1}+\mathcal{O}\left(\frac{1}{\ln R}\right)}(\ln R)^{-\frac{p_2}{p_1}+\mathcal{O}\left(\frac{1}{\ln R}\right)} \sim (p_1p_2)^{p_2/p_1}(\ln R)^{-p_2/p_1},$$

and

$$k^{\frac{1}{p_1-a}} = k^{\frac{1}{p_1}+\mathcal{O}\left(\frac{1}{\ln R}\right)} \sim k^{1/p_1},$$

where for $b \in \mathbb{R}$ we used $x^{b/\ln x} = \mathrm{e}^b$, while $k^{b/\ln x} \sim 1$, $(\ln x)^{b/\ln x} = \exp\big[b\ln(\ln x)/\ln x\big] \sim 1$ and $x^{b/(\ln x)^2} = \mathrm{e}^{b/\ln x} \sim 1$ as $x \to \infty$. Using these results in (B2), we find that when $a$ is given as in (B3), the optimal $\delta$ scales as

$$\delta \sim k^{1/p_1}(p_1p_2)^{p_2/p_1}\mathrm{e}^{-p_2/p_1}R^{-1/p_1}(\ln R)^{-p_2/p_1}. \tag{B4}$$

Finally, from (B3) and (B4) we compute

$$\delta^a = \delta^{p_1p_2/\ln R} \sim \left[k^{1/p_1}(p_1p_2)^{p_2/p_1}\mathrm{e}^{-p_2/p_1}\right]^{p_1p_2/\ln R}R^{-p_2/\ln R}(\ln R)^{-p_2^2/\ln R} \sim \mathrm{e}^{-p_2} \tag{B5}$$

as $R \to \infty$, and so

$$(1-\delta)^a - \delta^a \sim 1 - \mathrm{e}^{-p_2} > 0$$

for $p_2 > 0$, verifying the previous assumption. The optimal solution of (B1) is thus given by (B4) with $k = k'/(1-\mathrm{e}^{-p_2})$.

### 3. Relating control parameter *R*- and Rayleigh number *Ra*-dependence

For fixed flux boundary conditions $\eta = \infty$, or for general Biot number $\eta > 0$ whenever $\delta \ll \eta$, the averaged boundary heat flux $\beta$ is uniformly bounded from above, and upper bound on the Nusselt number $Nu$ is derived from the lower bound on the averaged temperature drop $\Delta T$ (with $\Delta T = \mathcal{O}(\delta)$, $Nu = \beta/\Delta T = \mathcal{O}(\delta^{-1})$). In this case, the Rayleigh number $Ra$ is related to the control parameter $R$ via $Ra = R\Delta T$ (6), and we obtain the $Nu$-$Ra$ bound by estimating both $Nu$ and $Ra$ in terms of $R$.

In the following we derive a general formula for the relationships between the relevant scaling exponents. Since $\beta \le \mathcal{O}(1)$ in the fixed flux scaling regime, we concentrate on the scaling of $\Delta T$:

Assume that, asymptotically as $R \to \infty$, we have

$$\Delta T \gtrsim k R^{-p_1} (\ln R)^{-p_2} \tag{B6}$$

for some $R$-independent constant $k > 0$ and exponents $p_1, p_2 > 0$. Then by (6)

$$Ra = R\Delta T \gtrsim k R^{1-p_1} (\ln R)^{-p_2},$$

and we have

$$R \lesssim \left[k^{-1} Ra (\ln R)^{p_2}\right]^{1/(1-p_1)} = k^{-1/(1-p_1)} Ra^{1/(1-p_1)} (\ln R)^{p_2/(1-p_1)}, \tag{B7}$$

so that

$$\ln R \lesssim \frac{1}{1-p_1} \ln Ra + \frac{p_2}{1-p_1} \ln(\ln R) - \frac{1}{1-p_1} \ln k \sim \frac{1}{1-p_1} \ln Ra$$

to leading order as $R \to \infty$ ($Ra \to \infty$). Substituting into (B7), we estimate $R$ in terms of $Ra$ as

$$R \lesssim k^{-1/(1-p_1)} \left(\frac{1}{1-p_1}\right)^{p_2/(1-p_1)} Ra^{1/(1-p_1)} (\ln Ra)^{p_2/(1-p_1)}. \tag{B8}$$

Using this relationship in (B6), we find an asymptotic lower bound for $\Delta T$ as a function of $Ra$,

$$\begin{aligned}\Delta T &\gtrsim k \left[k^{-1/(1-p_1)} \left(\frac{1}{1-p_1}\right)^{p_2/(1-p_1)} Ra^{1/(1-p_1)} (\ln Ra)^{p_2/(1-p_1)}\right]^{-p_1} \left[\frac{1}{1-p_1} \ln Ra\right]^{-p_2} \\ &= k^{1/(1-p_1)} (1-p_1)^{p_2/(1-p_1)} Ra^{-p_1/(1-p_1)} (\ln Ra)^{-p_2/(1-p_1)},\end{aligned} \tag{B9}$$

which implies the desired upper bound for $Nu = \beta/\Delta T$ in terms of $Ra$. In particular, a bound on $Nu$ of the form $Nu \le \mathcal{O}\big(R^{p_1} (\ln R)^{p_2}\big)$ corresponds to $Nu \le \mathcal{O}\big(Ra^{p_1/(1-p_1)} (\ln Ra)^{p_2/(1-p_1)}\big)$ in the fixed flux scaling regime.


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