MATH 467-3 **Winter 2003**

Dynamical Systems

Homework Set 3 Due Friday, 31 January 2003

Course Web Site: `http://www.math.sfu.ca/~ralfw/math467/`

Homework Problems: One- and two-parameter bifurcations of fixed points

From the textbook by Strogatz:

3.2.5, 3.3.1, 3.4.8, 3.4.12, 3.4.14

Additional problems (Lab assignment): Numerical Bifurcation Diagrams

Consider a one-dimensional dynamical system $\dot{x} = f(x, r)$, describing the evolution of $x(t)$ depending on some parameter r (there may also be other parameters, say b, but we will ignore them for now, treating them as constants). We wish to plot the bifurcation diagram, that is, to compute the set of zeros $x^*(r)$ of f (or equilibrium points of the dynamical system), indicating how they depend on the varying parameter r. In this assignment we will consider an approach to computing the equilibria $x^*(r)$, and use some Matlab code to implement the method.

The idea is quite simple (and beautiful): In general the curve may be multivalued, that is, for a given r there may be more than one fixed point; thus it is not convenient to attempt to compute $x^*(r)$ directly. However, we can obtain it parametrically; we may introduce a parameter s, and suppose that the curve of fixed points is given parametrically by $r = r(s)$, $x^* = x(s)$: that is, $f(x(s), r(s)) = 0$ for all s (in some interval). Taking the total derivative with respect to s, we obtain

$$0 = \frac{df(x(s), r(s))}{ds} = \frac{\partial f(x(s), r(s))}{\partial x} x' + \frac{\partial f(x(s), r(s))}{\partial r} r', \tag{1}$$

where $x' \equiv dx/ds$, $r' \equiv dr/ds$. Now the simplest way to satisfy (1) is to set

$$\begin{aligned} r' &= \frac{\partial f(x(s), r(s))}{\partial x}, \\ x' &= -\frac{\partial f(x(s), r(s))}{\partial r}, \end{aligned} \tag{2}$$

and taking initial data (r_0, x_0) on the curve, so that $f(x_0, r_0) = 0$. (We may choose r' and x' in this way since the actual choice of the parameter s is irrelevant; to compute the curve, only the ratio x'/r' is relevant.) The solution $(r(s), x(s))$ of the initial-value problem (2) is a branch of the bifurcation diagram.

For a saddle-node bifurcation, such a solution integrated for a sufficiently large interval of s-values will yield the complete bifurcation diagram. However, observe that for a transcritical or pitchfork bifurcation, the bifurcation point (r_c, x^*) is a fixed point of (2), since both $\partial f/\partial x$ and $\partial f/\partial r$ vanish at the bifurcation point. Thus we cannot integrate (2) past the bifurcation point (this reflects the fact that two or three curves of zeros of f meet at the bifurcation point, so we cannot continue a single curve uniquely through that point). Hence in order to obtain the complete bifurcation diagram, we need to integrate (2) starting separately on each branch of zeros of f.

An interpretation: We may interpret the system (2) as a Hamiltonian system from classical mechanics. All Hamiltonian systems (where we may think of r as a generalized coordinate, with x its associated, or conjugate, momentum) have an associated conserved quantity, or first integral, often identified with an energy; in this case it is f. Solutions of a Hamiltonian system lie on level sets of f. We are interested in the zero level set of f. Thus we can identify the curves of equilibria of the dynamical system $\dot{x} = f(x, r)$ with orbits of the Hamiltonian system (2).

We shall use the above system (2) to compute the bifurcation diagrams of some dynamical systems using Matlab. To do so, I have supplied the m-file `bifurcationplot.m`. This file computes a branch of the bifurcation diagram for a one-dimensional dynamical system $\dot{x} = f(x, r)$ (specified in a file of the type `func.m`), taking as input a file of the form `vfield.m` which defines the vector field on the right-hand side of (2). We also need to specify values r_0 and x_0 on the curve; to find such values, we solve $f(x, r) = 0$ for a fixed value of r (using the Matlab command `fzero` and the function m-file `func.m`).

An example of using the code `bifurcationplot.m` is given in the script m-file `compute_bifn1.m`, demonstrated in class, with associated function m-files `func1.m` and `vfield1.m`. This script computes the bifurcation diagram for the supercritical pitchfork bifurcation occurring in the system $\dot{x} = -x + r \tanh x$. You should work through this code, run it (hit any key to continue when it pauses), and make sure you understand the different commands. Note that we need to run the script `bifurcationplot` four times with four different initial points to obtain the complete bifurcation picture for this pitchfork bifurcation.

One can also read off the stability of the fixed points from the computed bifurcation diagram. The segments that are integrated forwards in s are plotted as red, solid curves, while those computed backwards in s are blue and dashed. Thus we can see, for instance, that the upper branch of the pitchfork is stable: it is clear that the initial value $r = r_0$, $x = x_0$ occurs where the red and blue segments meet. Since the red part of the upper branch occurs for $r < r_0$, we see that computing forward in time, r decreases with s, that is, $r' < 0$. Looking at the equations (2), this shows that $\partial f/\partial x < 0$ along this branch of fixed points, so by linear stability analysis, the fixed points along this upper branch are stable. Similarly, any branch of equilibria for which $dr/ds < 0$ (the lower branch past the pitchfork bifurcation, and $x^* = 0$ before the bifurcation) is stable, and a branch for which $dr/ds > 0$ ($x^* = 0$ after the bifurcation) is unstable.

Another example which you should work through, also demonstrated in class, is given by the script m-file `compute_bifn2.m`, with associated function m-files `func2.m` and `vfield2.m`. In this case, we are computing the canonical subcritical pitchfork bifurcation diagram occurring in $\dot{x} = rx + x^3 - x^5$; and again we need to use four different initial values (the upper, and lower, branches connected by saddle-node bifurcations do not need to be considered separately).

Homework Assignment

1. Consider the dynamical system

$$\dot{x} = b + a * x - x^3; \tag{3}$$

   this is the imperfect bifurcation problem discussed in Section 3.6 of the text. The function m-files for this problem are also supplied, as `func3.m` and `vfield3.m`. Notice that in this case we use a for the parameter r, and also have a second parameter b; we shall compute the curves $x^*(a)$ for several different, but fixed, values of b. The value of b is initially fixed as `b = 0.1` in the file `func3.m`; to change the value of b, you would change this line.

Let's get started on computing this bifurcation diagram: In Matlab, you could type

```
>> x = -1:0.02:1.5;
>> figure(1);
>> clf;                  % clears the figure 1 window
>> plot(x, func3(x,1), 'b', [-1 1.5], [0 0], ':')
```

This gives us the graph of $\dot{x} = f$, where we have fixed $a = 1$. Looking at the graph, there appear to be three zeros, near $x = -1$, $x = 0$ and $x = 1$ (note: the second part of the `plot` command draws a straight, dotted line at $y = 0$, which makes it easier to identify the zeros). Now we will use `fzero` to find the zeros accurately:

```
>> x1 = fzero('func3(x,1)', [-1 -0.5])
```

This returns the value of the zero in the interval $[-1, -0.5]$; note how we manually specified $a = 1$ by typing `'func1(x,1)'`. We similarly obtain the second and third zeros in the variables `x2` and `x3` (in intervals suggested by our graph) by typing

```
>> x2 = fzero('func3(x,1)', [-0.5 0.5])
>> x3 = fzero('func3(x,1)', [0.5 1.5])
```

Now we can use `bifurcationplot` to integrate the vector field and trace the a-dependence (remember, b is fixed at 0.1). We need to specify the initial values a_0 and x_0; we will take $a_0 = 1$, and for x_0 we use the three equilibrium values computed above. We also specify times $t_i < 0$ and $t_f > 0$ to integrate to. In Matlab, we type

```
>> figure(2)       % open a new window
>> clf;            % clear figure 2
>> hold on;        % allows us to plot several graphs on top of each other
>> bifurcationplot('vfield3',x1,1,-1,1);
```

For our first computation we chose $t_i = -1$ and $t_f = 1$, as we didn't know how far the curves would go. By trial and error, we can choose appropriate values of t_i and t_f to make a sufficiently pretty graph, and eventually use

```
>> bifurcationplot('vfield3',x1,1,-1,4);
```

Note that since a decreases with s on the lower branch (so $\partial f/\partial x < 0$ on this branch, by (2)) the lower branch below the turning point is stable, and similarly the upper branch is unstable. Now let us check the other fixed points:

```
>> bifurcationplot('vfield3',x2,1,-1,1);
```

This did not add anything to the bifurcation diagram, as the fixed points x_1 and x_2 are on the same branch, which undergoes a saddle-node (or turning point) bifurcation. The remaining branch may be found from the fixed point x_3 at $a = 1$:

```
>> bifurcationplot('vfield3',x3,1,-1,3);
```

This gives the complete bifurcation diagram for $b = 0.1$. Now we can consider other values of b. . .

(a) Analytically compute the stability diagram in the a–b plane, that is, compute the bifurcation curves in this plane by analytically solving $f = 0$ and $f_x = 0$. Plot the stability diagram, and explain why $a = 0$ is "special".
[This is a "warm-up" calculation; you can follow Section 3.6 completely for this.]

(b) Fix values $b > 0$, $b = 0$ and $b < 0$, and use the Matlab code as above to produce bifurcation diagrams of x^* versus a. Make sure that you find all the branches, label the branches as stable or unstable, and identify the type of bifurcations. You should reproduce the figures of Section 3.6 of the book by Strogatz.

2. Now we will do similar calculations for more complicated stability diagrams. Consider the dynamical system

$$\dot{x} = b + a + a^2 x + a x^2 - x^3. \qquad (4)$$

(a) Analytically compute and plot the stability diagram for (4) in the a–b plane by solving $f = 0$, $f_x = 0$. Explain why the lines $b = \pm b_c = \pm\frac{2}{\sqrt{5}}$ are "special". You should use this stability diagram as a "road map" to help you find and explain your bifurcation diagrams.

(b) Show that the system is symmetric under the change of variables $b \mapsto -b$, $a \mapsto -a$, $x \mapsto -x$. Hence it is only necessary to consider the bifurcation diagrams for $b \geq 0$.

(c) Now we will map out the details of the bifurcations: The Matlab files `func4.m` and `vfield4.m` are provided. Fix values of b slightly above and below b_c, say 0.89 and 0.895, and use the Matlab codes to produce bifurcation diagrams of x^* as a function of a (you could start at $a = 0$. Make sure that you find all the branches! You should label the branches as stable or unstable, identify the type (saddle-node, transcritical, ...) of any bifurcations you find, and indicate the bifurcations on the stability diagram as well. Also label each connected set of the stability diagram as to number and type of fixed points.

Now repeat this for $b = b_c$. What kind of bifurcation(s) do you find now?

Finally, choose a small value of b, say $b = 0.2$, calculate the bifurcation diagram and identify the bifurcation(s) you find. To which points do they correspond on the stability diagram?

3. Consider

$$\dot{x} = (x - ae^{-b})(ax - b - x^2). \qquad (5)$$

(a) Show analytically that the stability diagram for (5) consists of two curves which intersect at $a = 2\sqrt{\ln 2}$, $b = \ln 2$. These curves are called the bifurcation curves.

(b) Create appropriate m-files `func5.m` and `vfield5.m` for (5), and use Matlab to compute the bifurcation diagrams (x^* as a function of a) for $b = 0.2$, $b = \ln 2$ and $b = 1.2$. Draw the lines $b = 0.2$, $b = \ln 2$ and $b = 1.2$ on your stability diagram, and indicate the nature of the bifurcation which occurs when each of these lines crosses a bifurcation curve.