A PV Dynamics for Rotating Shallow Water on the Sphere
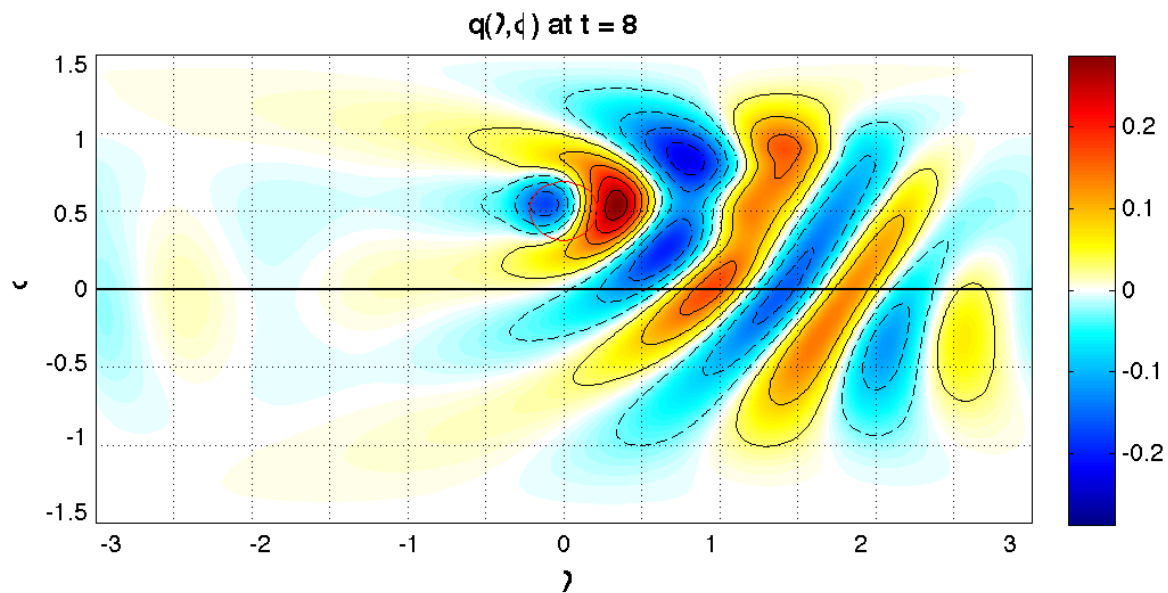
▷   search for a balance dynamics on the full sphere

▷   potential vorticity inversion & advection

▷   ray theory for short scale waves

▷   David J Muraki & Andrea Blazenko, Simon Fraser University

▷   Chris Snyder, NCAR

Midlatitude Balanced Dynamics

Rotating Shallow Water (rSW), β-Scaled

  ▷   winds, $\vec{u}$ & height field, $H = 1 + \epsilon\, \eta$

  ▷   longitude-latitude <u>local</u> coordinates, $(\lambda, \phi) \approx (\lambda_0, \phi_0)$

$$\epsilon^2\, \frac{D\vec{u}}{Dt} \quad + \quad (1 + \beta(\phi - \phi_0))(\hat{r}_0 \times \vec{u}) \quad \sim \quad -\epsilon\, \nabla\eta$$

$$\epsilon\, \frac{D\eta}{Dt} \quad + \quad (1 + \epsilon\, \eta)\ (\nabla \cdot \vec{u}) \quad = \quad 0$$

  ▷   dimensionless parameters: Rossby number & Coriolis-β

$$\epsilon = \frac{\sqrt{gH_0}}{(2\Omega \sin \phi_0)\, r} \ll 1 \quad ; \quad \beta = \cot \phi_0$$

Midlatitude Quasigeostrophy (QG)

  ▷   *balanced* dynamics: slow Rossby waves & NO fast gravity waves

  ▷   restrict to short deformation scales: $\lambda - \lambda_0 = \epsilon x$; $\phi - \phi_0 = \epsilon y$

    ▷ geostrophy: $\hat{r}_0 \times \vec{u} \sim -\epsilon\, \nabla\eta \quad \Rightarrow \quad$ non-divergent winds

    ▷ limit as $\epsilon \to 0$, geostrophic degeneracy

    ▷ advection & inversion of potential vorticity (PV)

  ▷   geometric obstacle: <u>local Rossby number singular at Equator, $\epsilon \to \infty$</u>

rSW on the Full Sphere

Spherical Coordinates

▷ longitude-latitude global coordinates, (λ, ϕ)

$$\epsilon^2 \left\{ \frac{Du}{Dt} + v\hat{\lambda} \cdot \frac{D\hat{\phi}}{Dt} \right\} \quad - \quad v \sin \phi \quad = \quad -\epsilon \frac{1}{\cos \phi} \eta_\lambda$$

$$\epsilon^2 \left\{ \frac{Dv}{Dt} + u\hat{\phi} \cdot \frac{D\hat{\lambda}}{Dt} \right\} \quad + \quad u \sin \phi \quad = \quad -\epsilon \quad \eta_\phi$$

$$\epsilon \quad \frac{D\eta}{Dt} + \{1 + \epsilon\,\eta\} \frac{u_\lambda + (v \cos \phi)_\phi}{\cos \phi} \quad = \quad 0$$

▷ Lamb parameter

$$\epsilon = \left(\frac{4\Omega^2 r^2}{gH_0} \right)^{-1/2} \ll 1$$

Balanced Dynamics

▷ *PV Inversion on a Hemisphere*, McIntyre/Norton 1999

   ▷ landmark for PV as prognostic variable

   ▷ non-divergent winds (at leading-order)

   ▷ dynamical oddity: mirror symmetry across Equator

▷ on midlatitude scales, limit as $\epsilon \to 0$ is now inconsistent!

A Case for Global Balanced Dynamics

Grose & Hoskins (1979)

  ▷   rSW flow past mountain at 30°N

     ▷   steady, super-rotation zonal wind

  ▷   steady waves by Rayleigh-damped perturbations

     ▷   perturbation vorticity after 17 days (with damping) & $\epsilon \approx 0.33$
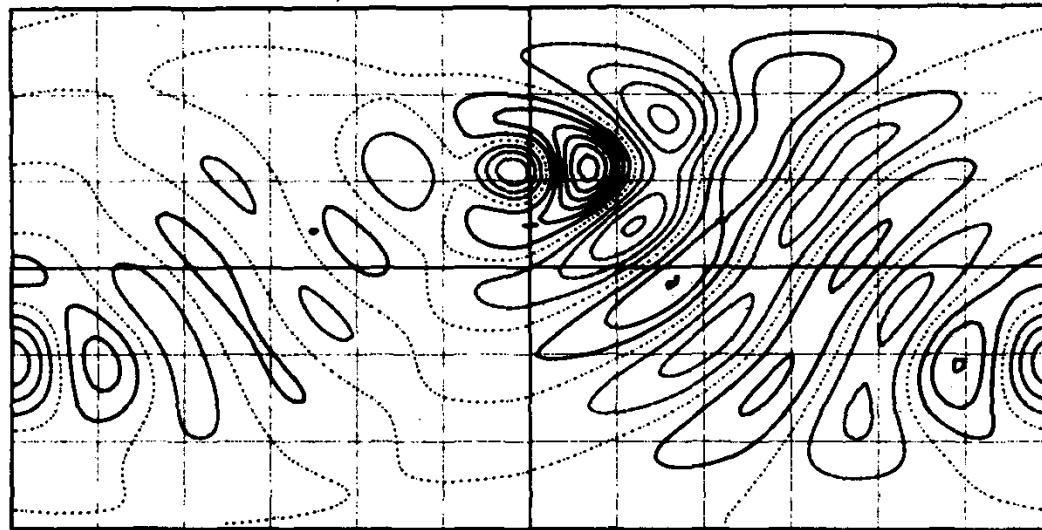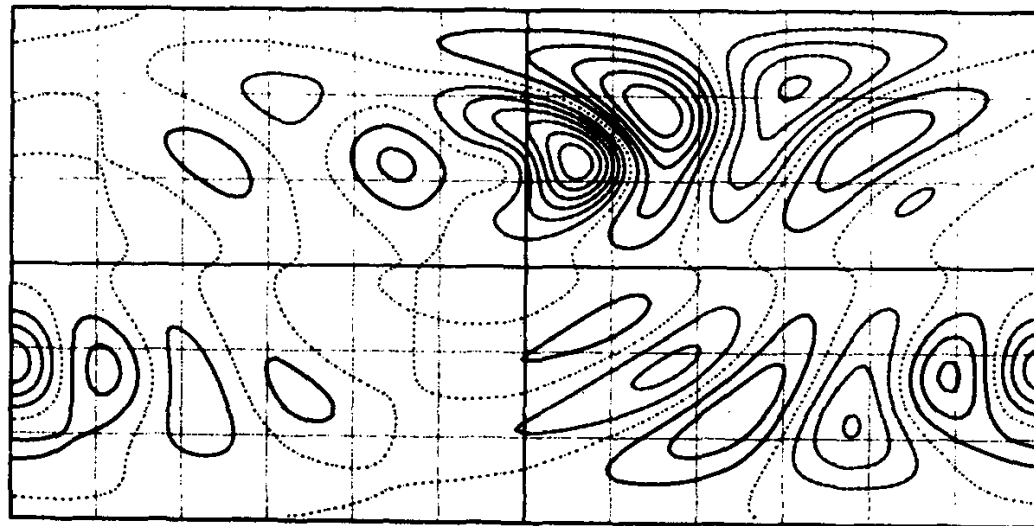

  ▷   waves propagate smoothly across Equator

  ▷   nearly non-divergent flow

A Case for Global Balanced Dynamics

Grose & Hoskins (1979)

    ▷   rSW flow past mountain at 30°N

       ▷ steady, super-rotation zonal wind

    ▷   steady waves by Rayleigh-damped perturbations

       ▷ perturbation height after 17 days (with damping) & $\epsilon \approx 0.33$

    ▷   essentially zero height disturbance at Equator

    ▷   non-divergent balance relations, with streamfunction ψ

$$u = -\,\epsilon\,\psi_\phi \qquad ; \qquad v = \epsilon\,\frac{1}{\cos\phi}\,\psi_\lambda \qquad ; \qquad \eta = \psi\sin\phi$$

$$\epsilon^2 \frac{Du}{Dt} \qquad\qquad - \ v \sin \phi \quad \sim \quad - \epsilon \frac{1}{\cos \phi} \left(\frac{\eta}{\sin \phi} \right)_\lambda \sin \phi$$

$$\epsilon^2 \frac{Dv}{Dt} \qquad\qquad + \ u \sin \phi \quad \sim \quad - \epsilon \left(\frac{\eta}{\sin \phi} \ \sin \phi \right)_\phi$$

$$\epsilon \ \frac{D\eta}{Dt} \ + \ \{1 + \epsilon \, \eta\} \ \frac{u_\lambda + (v \cos \phi)_\phi}{\cos \phi} \quad = \quad 0$$

▷   non-divergent balance relations, with streamfunction ψ

$$u = - \epsilon \, \psi_\phi \qquad ; \qquad v = \epsilon \, \frac{1}{\cos \phi} \, \psi_\lambda \qquad ; \qquad \eta = \psi \, \sin \phi$$

rSW on the Sphere Redone

$$\epsilon^2 \frac{Du}{Dt} \qquad\qquad - v \sin\phi \quad \sim \quad - \epsilon \frac{1}{\cos\phi} \psi_\lambda \sin\phi$$

$$\epsilon^2 \frac{Dv}{Dt} \qquad\qquad + u \sin\phi \quad \sim \quad - \epsilon \psi_\phi \sin\phi \qquad - \epsilon \psi \cos\phi$$

$$\epsilon \frac{D\eta}{Dt} + \{1 + \epsilon \eta\} \frac{u_\lambda + (v \cos\phi)_\phi}{\cos\phi} \quad = \quad 0$$

Geostrophic Degeneracy Restored

▷ on midlatitude scales, limit as $\epsilon \to 0$ is consistently degenerate

▷ non-divergent balance relations

▷ β-effect displaced: meridional advection of planetary PV

Potential Vorticity

▷ total PV is advected quantity: $DQ/Dt = 0$

$$Q = \sin\phi + \epsilon\, q = \frac{\sin\phi + \epsilon^2\, \hat{r} \cdot (\nabla \times \vec{u})}{1 + \epsilon\eta}$$

PV Dynamics on the Sphere (sPV)

Inversion, Streamfunction & Advection

▷ PV-streamfunction relation with topography, $b(\lambda, \phi)$:

$$\epsilon^2 \, \nabla^2 \, \psi - (\sin^2 \phi) \, \psi = q - b \, \sin \phi$$

▷ balance relations for non-divergent winds:

$$u = - \, \epsilon \, \psi_\phi \qquad ; \qquad v = \epsilon \, \frac{1}{\cos \phi} \, \psi_\lambda \qquad ; \qquad \eta = \psi \, \sin \phi$$

▷ disturbance PV advection:

$$\epsilon \left\{ \frac{\partial q}{\partial t} + \frac{u}{\cos \phi} \frac{\partial q}{\partial \lambda} + v \frac{\partial q}{\partial \phi} \right\} + v \, \cos \phi = 0$$

Mountain Waves, Disturbance PV ($t = 4$)
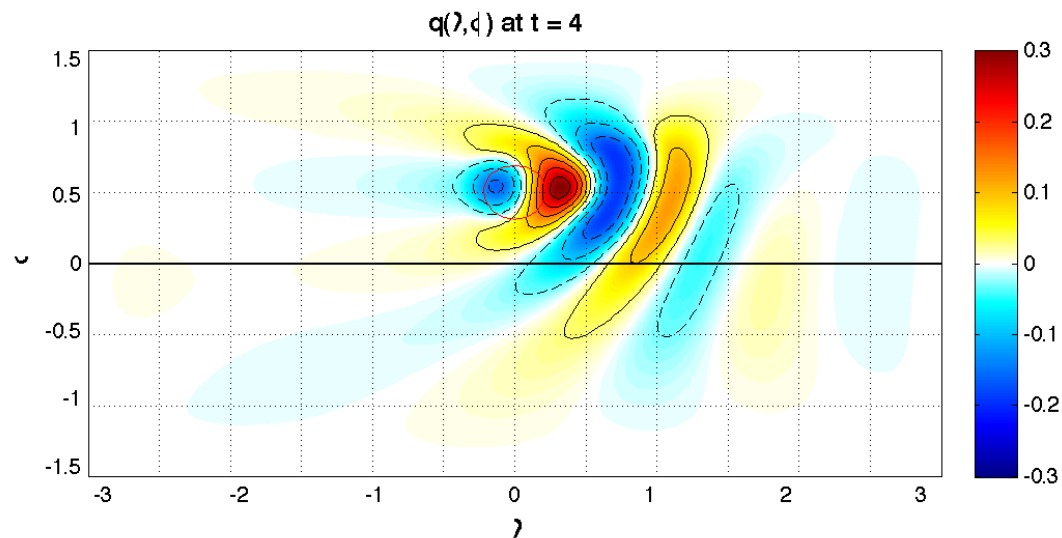
$q(\lambda, \phi)$ at t = 4

PV Dynamics on the Sphere (sPV)

Inversion, Streamfunction & Advection

▷ PV-streamfunction relation with jet; $\bar{u}(\phi), \epsilon\bar{\eta}(\phi) = O(1)$ & topography; $b(\lambda, \phi)$:

$$\epsilon^2 \left(1 + \epsilon\bar{\eta}\right) \nabla^2 \psi - (\sin^2 \phi)\, \psi = (1 + \epsilon\bar{\eta})^2\, q - b\, \sin\phi$$

▷ balance relations for non-divergent winds:

$$u = -\epsilon\, \psi_\phi \qquad ; \qquad v = \epsilon\, \frac{1}{\cos\phi}\, \psi_\lambda \qquad ; \qquad \eta = \psi\, \sin\phi$$

▷ disturbance PV advection:

$$\epsilon \left\{ \frac{\partial q}{\partial t} + \frac{\bar{u} + u}{\cos\phi}\, \frac{\partial q}{\partial\lambda} + v\, \frac{\partial q}{\partial\phi} \right\} + v \left(\frac{\sin\phi}{1 + \epsilon\bar{\eta}} \right)_\phi = 0$$

Mountain Waves, Disturbance PV ($t = 4$)

$q(\lambda, \phi)$ at t = 4
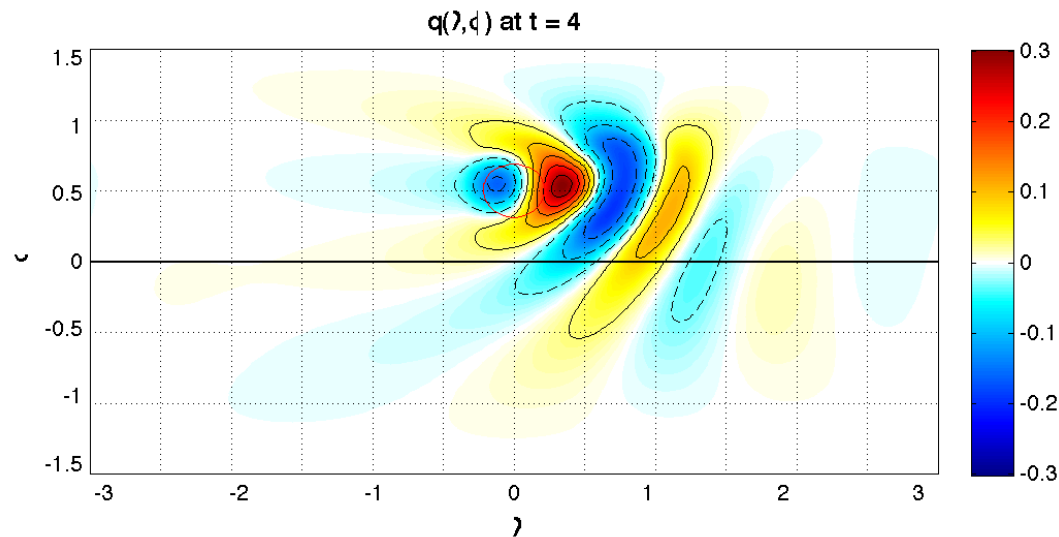
Mountain Waves, Disturbance Height ($t = 8$)
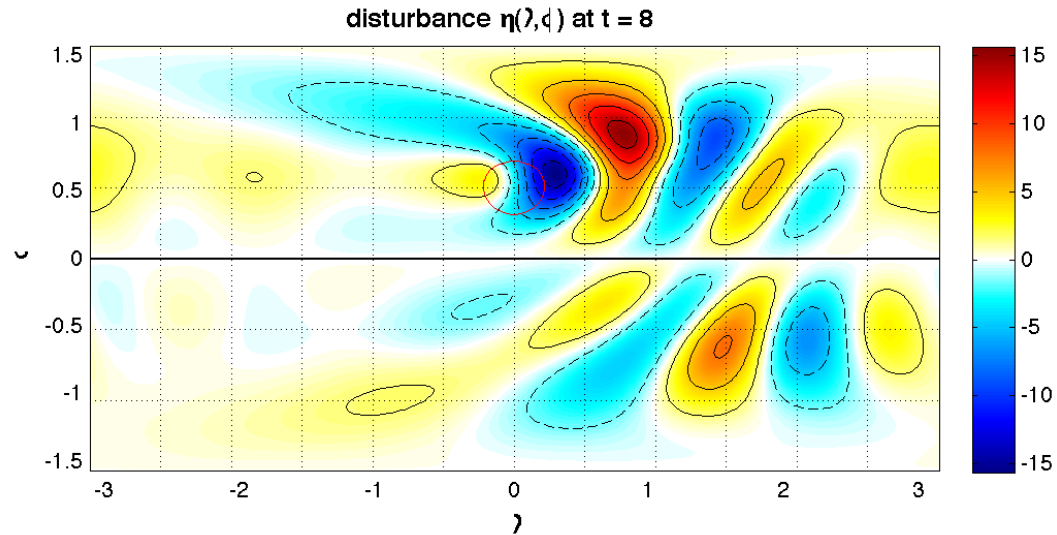
disturbance $\eta(\lambda, \phi)$ at t = 8

Mountain Waves, Disturbance PV ($t = 8$)

$q(\lambda, \phi)$ at t = 8
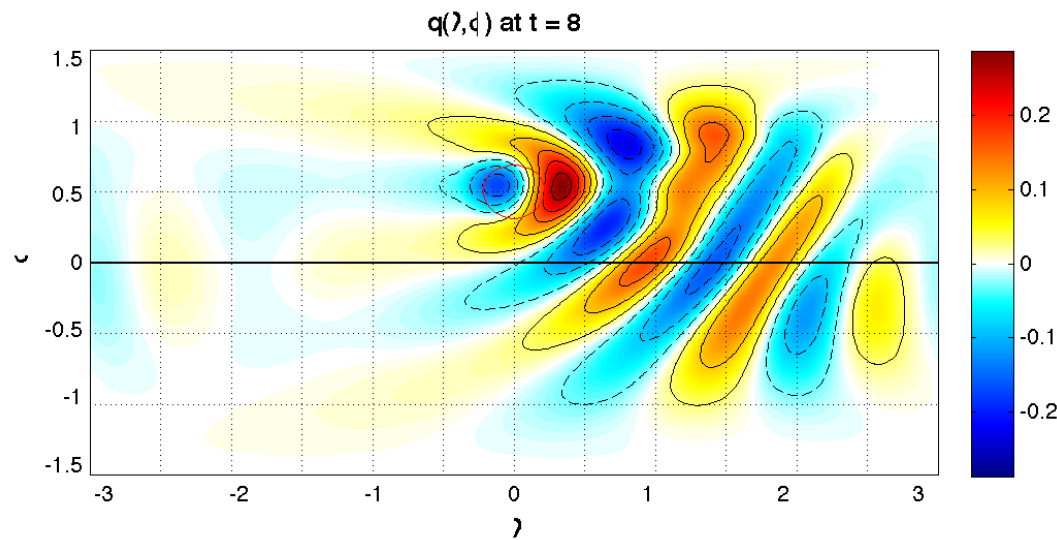
sPV on the Sphere: Comparison with rSW

Disturbance Height: sPV (colour) & rSW (contour)
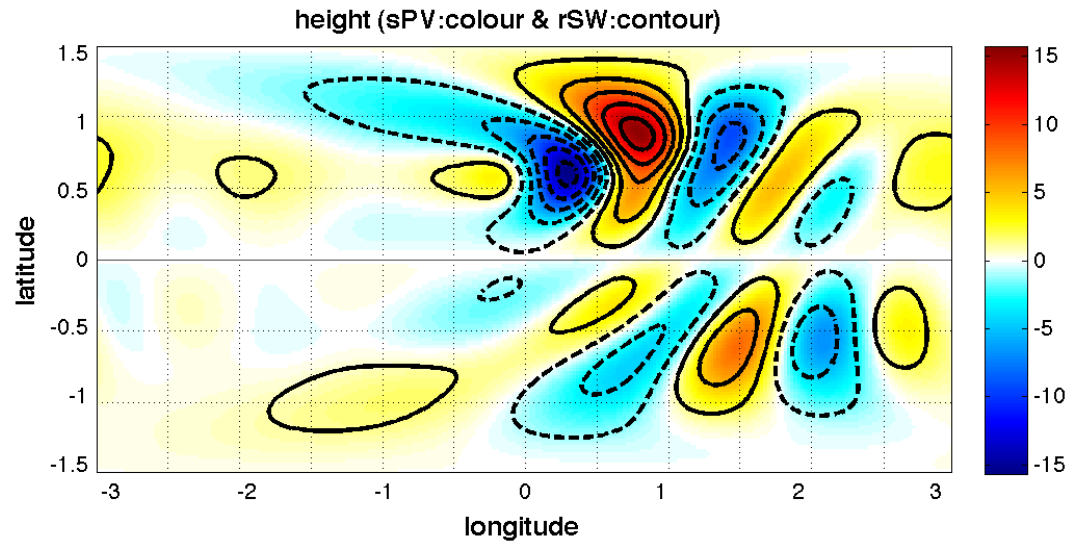
Disturbance PV: $\epsilon \approx 0.337$ — thanks to Joe Klemp, NCAR
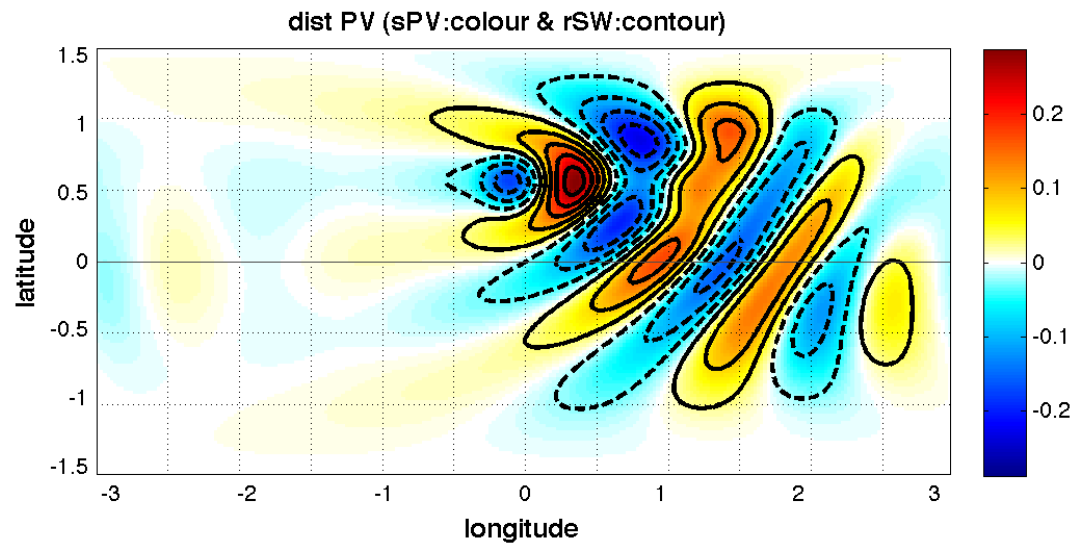
## sPV is not a Global Asymptotic Theory?

$$\epsilon^2 \frac{Du}{Dt} \qquad\qquad - v \sin\phi \quad \sim \quad - \epsilon \frac{1}{\cos\phi} \psi_\lambda \sin\phi$$

$$\epsilon^2 \frac{Dv}{Dt} \qquad\qquad + u \sin\phi \quad \sim \quad - \epsilon \, \psi_\phi \sin\phi \qquad - \epsilon \, \psi \cos\phi$$

$$\epsilon \, \frac{D\eta}{Dt} + \{1 + \epsilon \, \eta\} \, \frac{u_\lambda + (v \cos\phi)_\phi}{\cos\phi} \quad = \quad 0$$

Midlatitude Synoptic-Scale Truncation

▷ leading-order balance assumptions

  ▷ on deformation scales $\Rightarrow \quad \partial/\partial\lambda, \partial/\partial\phi = O(1/\epsilon)$

  ▷ at midlatitudes $\Rightarrow \quad \sin\phi \neq 0$

▷ there is NO expectation of asymptotic validity at the Equator

  ▷ sPV is not *globally* accurate, but is well-posed

  ▷ PV inversion & velocity-streamfunction relationship are non-singular

Q: So, Why are Equatorial Crossings Faithfully Represented?
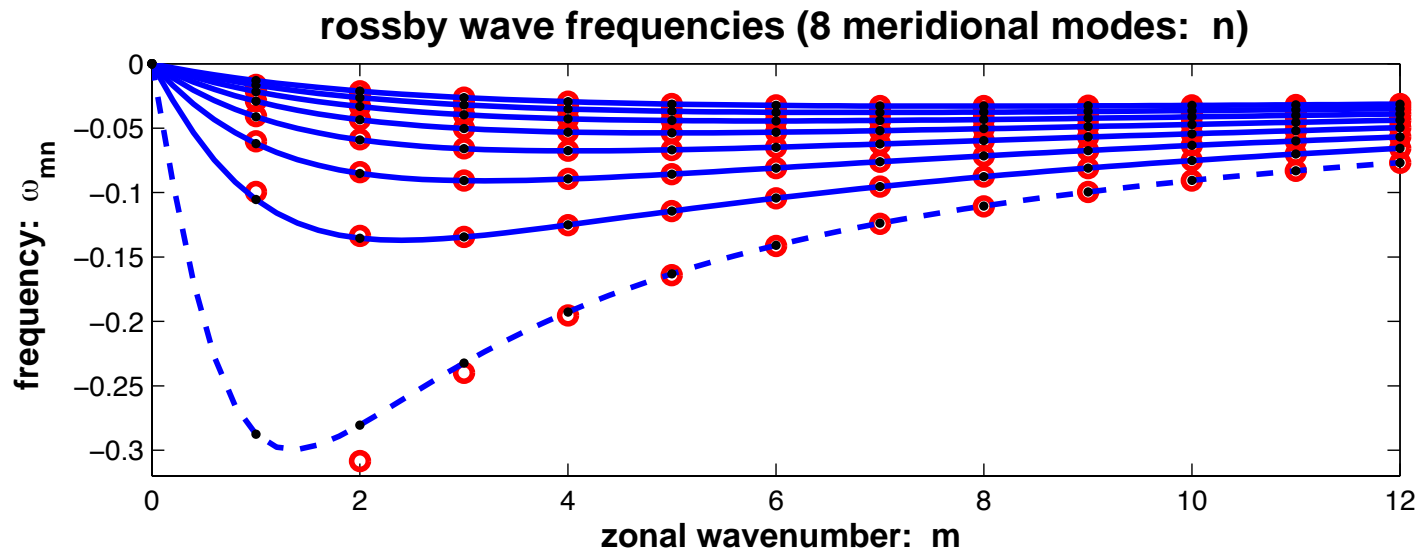
**rossby wave frequencies (8 meridional modes: n)**

Rotating Waves, $\psi(\lambda - ct, \phi)$

▷ ODE eigenfunction [Verkley (2007), Schubert (2008)] $\rightarrow$ exact nonlinear sPV solutions ($\epsilon \approx 0.337$)

$$\epsilon^2 \, \nabla^2 \, \psi - (\sin^2 \phi) \, \psi + (1/c) \, \psi = 0 \quad ; \quad \psi(\pm \pi/2) = 0$$

▷ compare to modes for rSW on sphere (Margules, Hough, Longuet-Higgins, . . .)

▷ only longest planetary waves have $O(1)$ wavespeed errors

sPV Rossby Waves ━━━━━━━━━━━━━
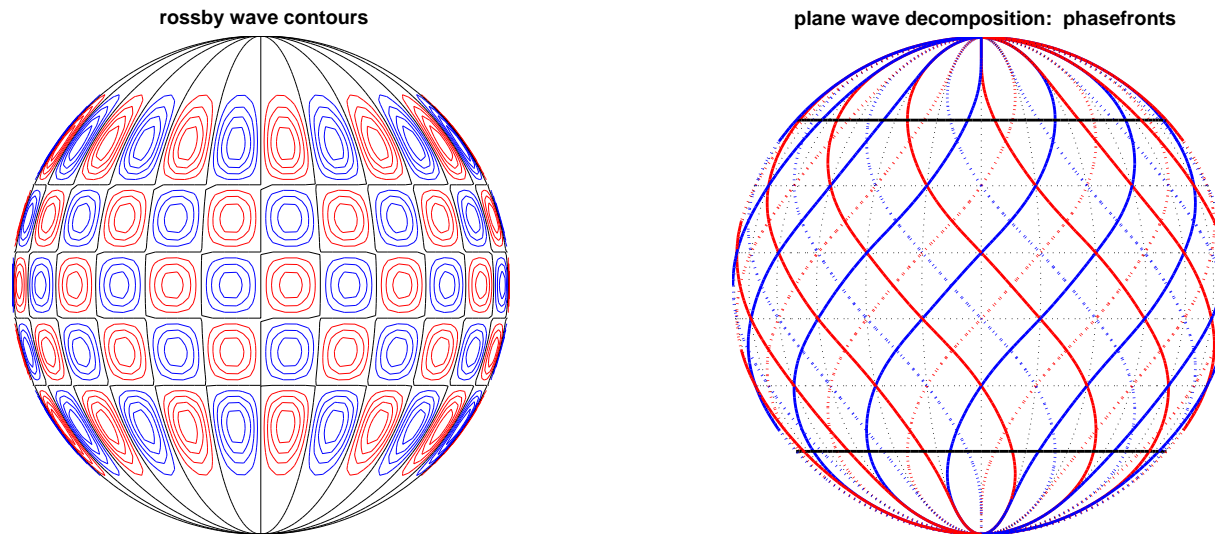
rossby wave contours

plane wave decomposition: phasefronts

Rotating Waves, $\psi(\lambda - ct, \phi)$

▷   ODE eigenfunction [Verkley (2007), Schubert (2008)] $\rightarrow$ exact nonlinear sPV solutions ($\epsilon \approx 0.337$)

$$\epsilon^2 \, \nabla^2 \, \psi - (\sin^2 \phi) \, \psi + (1/c) \, \psi = 0 \quad ; \quad \psi(\pm \pi/2) = 0$$

▷   compare to modes for rSW on sphere (Margules, Hough, Longuet-Higgins, . . .)

▷   only longest planetary waves have $O(1)$ wavespeed errors

Wavepacket Limit of Short-Scale Waves: $\epsilon = \frac{1}{2}\,\epsilon_{gh}$ & $\frac{1}{4}\,\epsilon_{gh}$ ———
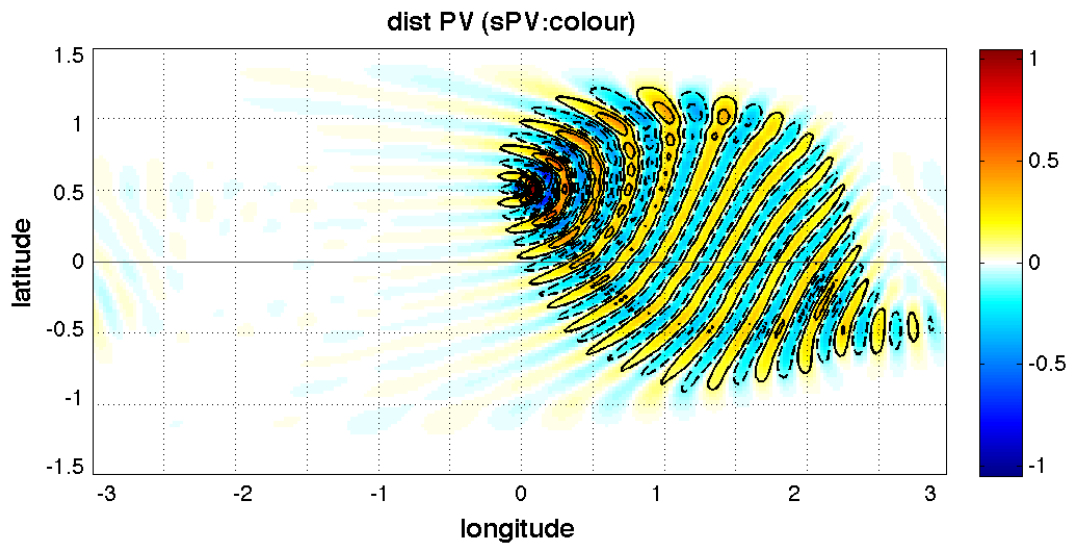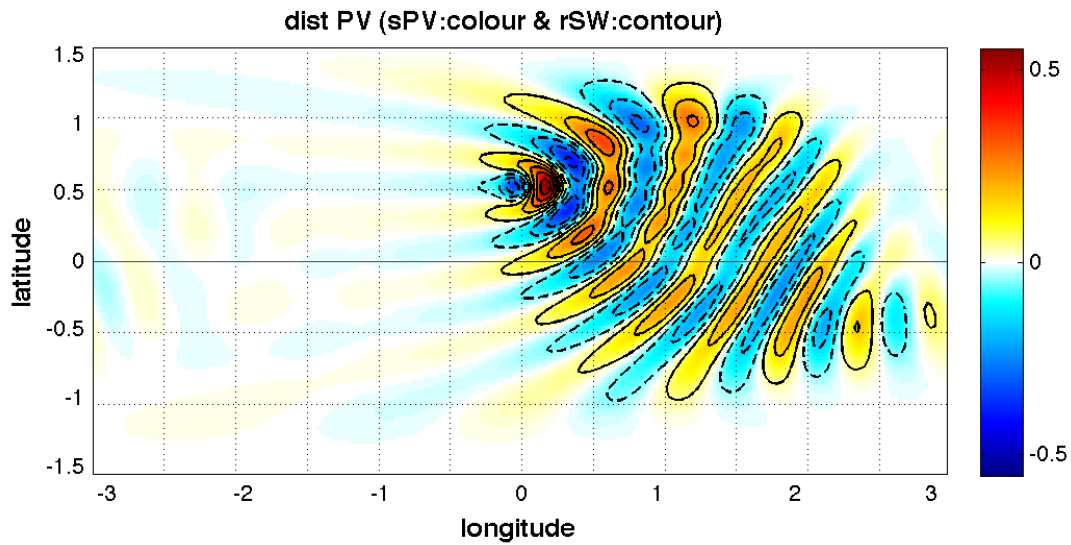
Ray Theory

Geometrical Optics Limit as $\epsilon \to 0$

▷   (flow) = (steady, zonal $\bar{u}(\phi)$) + (wave amplitude) $e^{iS(\lambda,\phi,t)/\epsilon}$

▷   phase $S(\lambda, \phi, t)$ satisfies ϵ-independent Hamilton-Jacobi PDE

$$\left(S_t + \frac{\bar{u}}{\cos\phi}\, S_\lambda \right) \left(\frac{S_\lambda^2}{\cos^2\phi} + S_\phi^2 + \frac{\sin^2\phi}{1 + \epsilon\,\bar{\eta}} \right) - S_\lambda \left(\frac{\sin\phi}{1 + \epsilon\,\bar{\eta}} \right)_\phi (1 + \epsilon\,\bar{\eta}) = 0$$

▷   midlatitude replacements . . .

$$S_t \to -\omega \;\; ; \;\; \frac{S_\lambda}{\cos\phi} \to k \;\; ; \;\; S_\phi \to l \;\; ; \;\; \sin\phi \to \tilde{f} \;\; ; \;\; \cos\phi \to \tilde{\beta} \;\; ; \;\; \epsilon\,\bar{\eta} \to 0$$

▷   . . . give midlatitude Rossby wave dispersion (Hoskins & Karoly, 1981)

$$\omega = \bar{u}\, k - \frac{\tilde{\beta} k}{k^2 + l^2 + \tilde{f}^2}$$

Q: So, Why are Equatorial Crossings Faithfully Represented?

A: Both sPV & rSW have the same global ray theory!

$q(\lambda, \phi)$ at t = 24
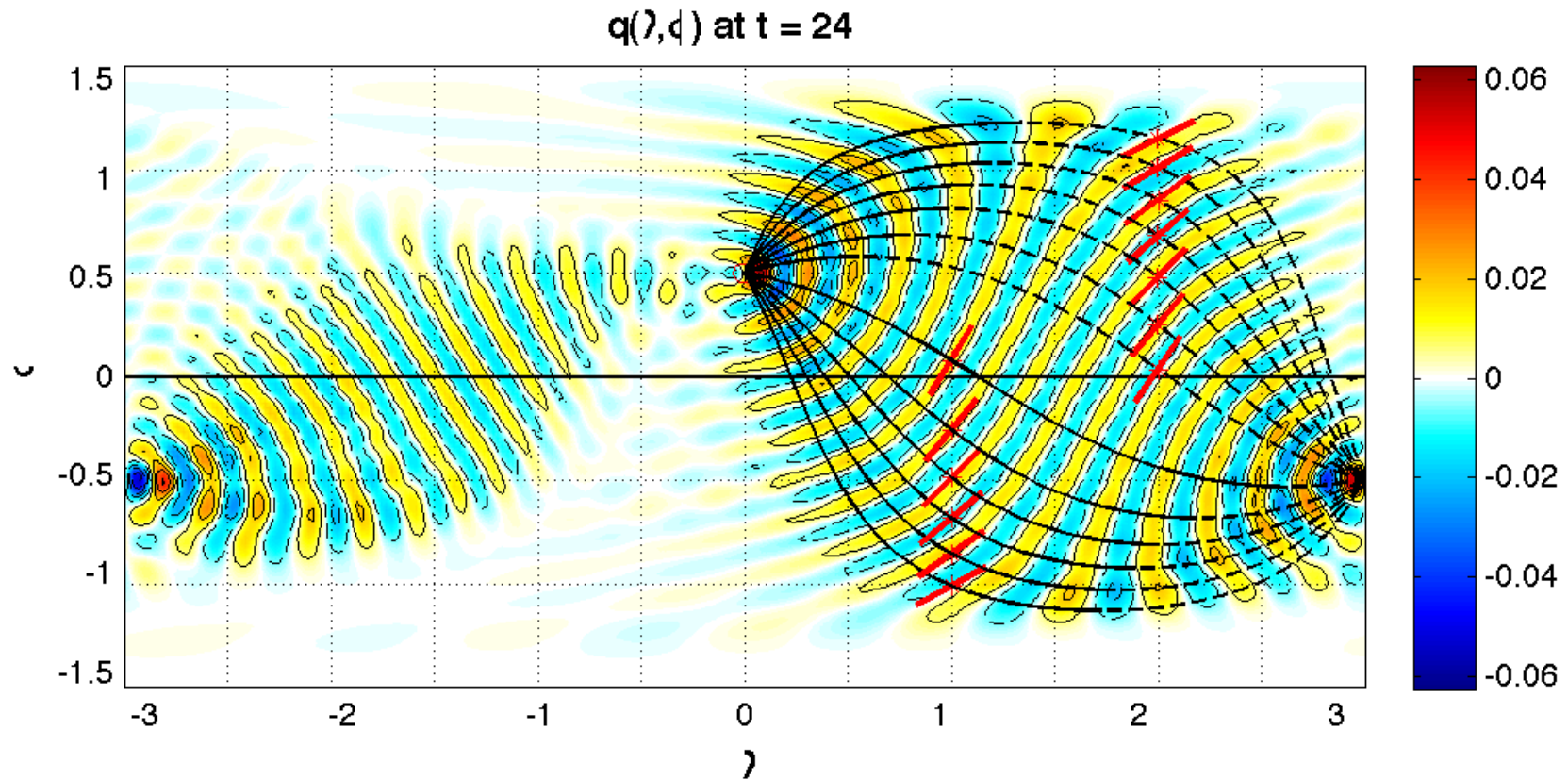
▷   sPV dynamics & steady group rays with phasefronts

NCEP Operational Analysis

▷ 250mB wind vectors (5-day average) — thanks to Mel Shapiro, NCAR

▷ Rossby wave flow in Equatorial Pacific

250mb WINDS (m/s)   05−DAY MEAN FOR:
Mon APR 20 2009 − Fri APR 24 2009

NCEP OPERATIONAL DATASET
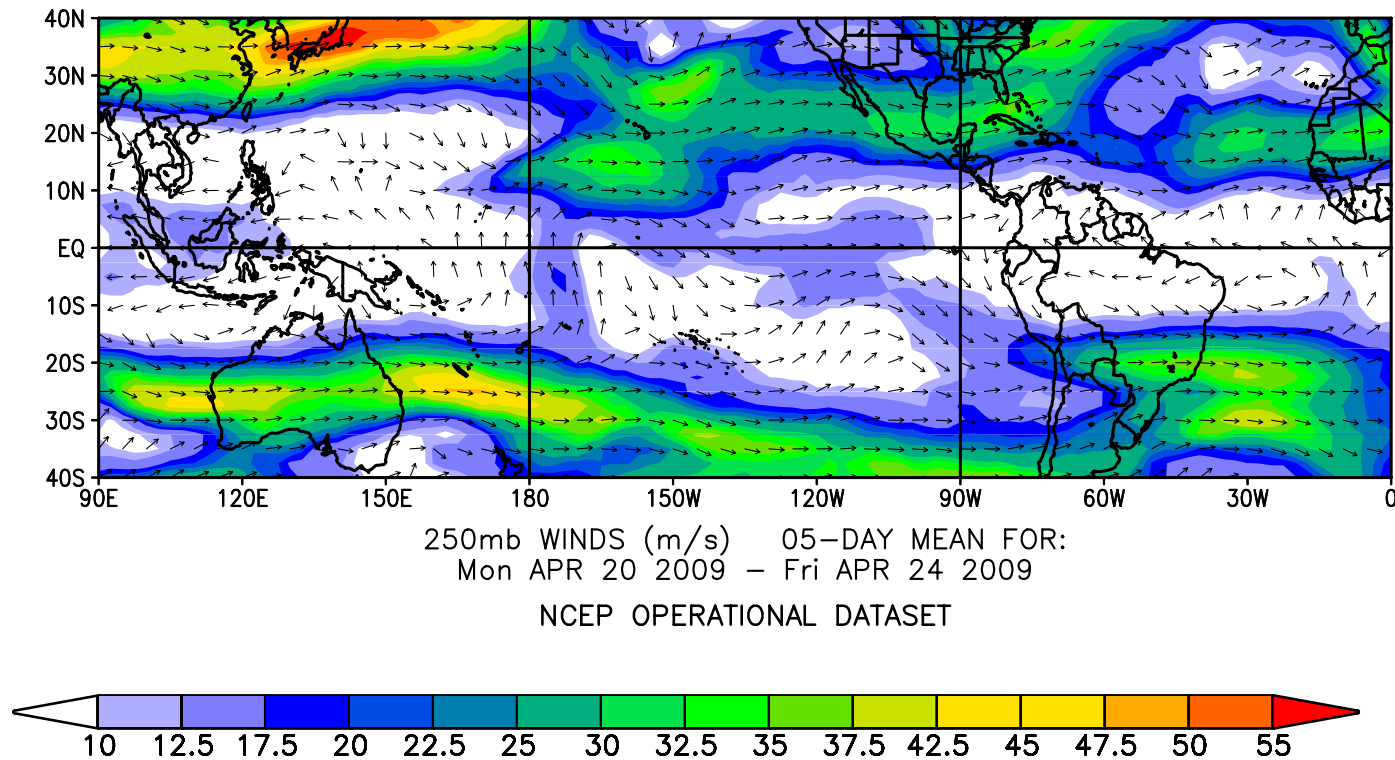
20-24 April 2009

NCEP Operational Analysis

▷ 250mB meridional wind anomaly (5-day average)

▷ Rossby wave flow in Equatorial Pacific

250mb V-WIND COMPONENT (m/s)   05-DAY ANOMALY FOR:
Mon APR 20 2009 - Fri APR 24 2009

NCEP OPERATIONAL DATASET
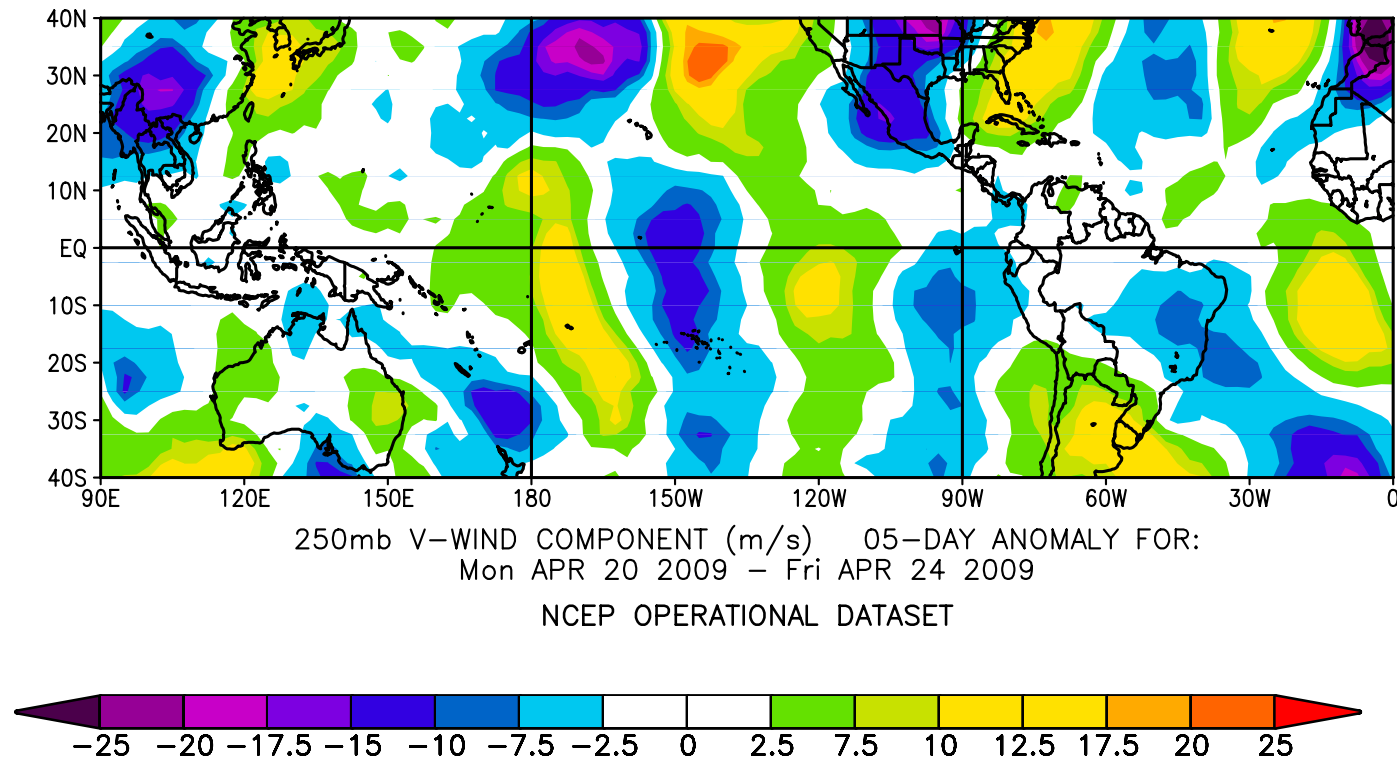
20-24 April 2009

NCEP Operational Analysis

▷ 250mB geopotential height anomaly (5-day average)

▷ Rossby wave flow in Equatorial Pacific

250mb GEOPOTENTIAL HEIGHTS (dam)   05−DAY ANOMALY FOR:
Mon APR 20 2009 − Fri APR 24 2009

NCEP OPERATIONAL DATASET
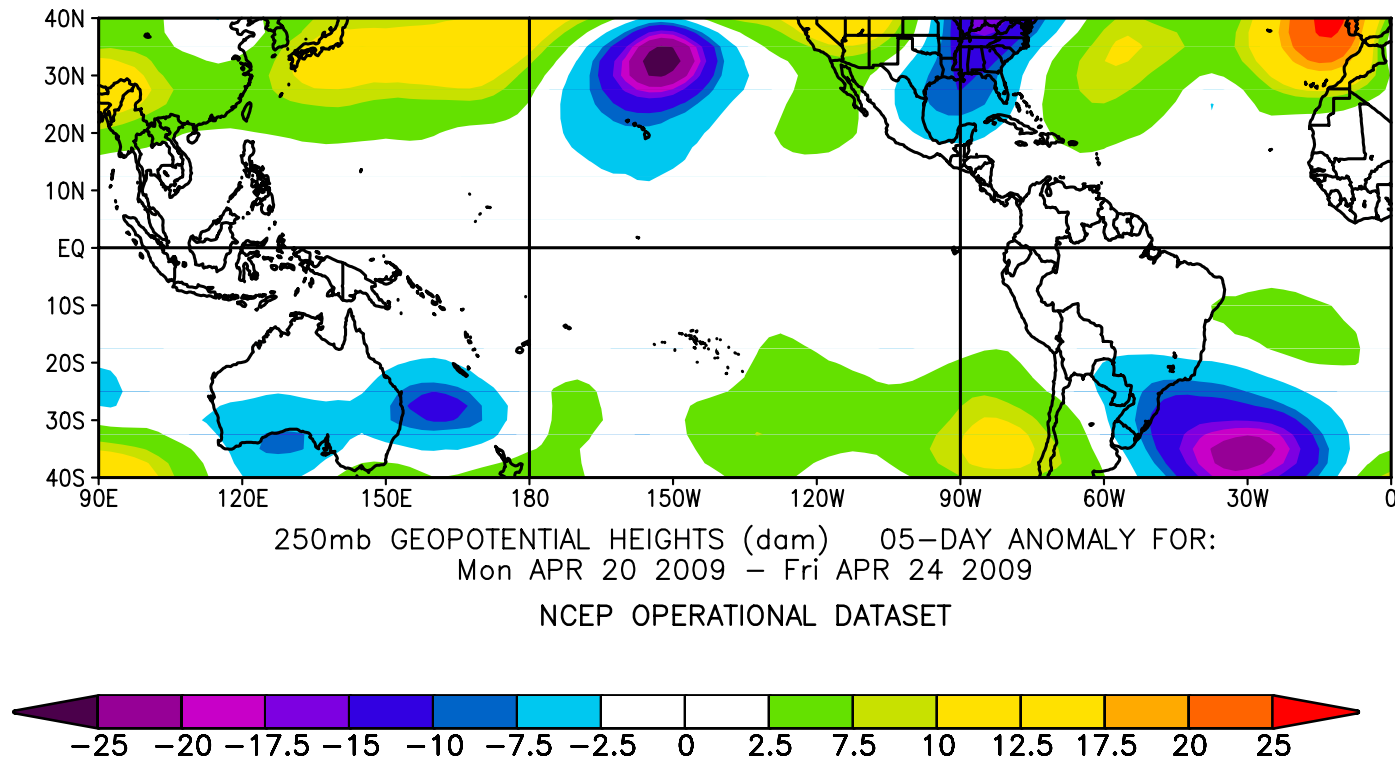
PV-Streamfunction: 2 coupled elliptic equations, $q_1, q_2 \longrightarrow \psi_1, \psi_2$ $(k = 1, 2)$

$$\left[\epsilon^2 \gamma_1 \nabla^2 - \left(\frac{1}{1-\delta} \right) \sin^2 \phi \right] \psi_1 + \left(\frac{1}{1-\delta} \right) \sin^2 \phi \; \psi_2 = \gamma_1^2 \, q_1$$

$$\left[\epsilon^2 \gamma_2 \nabla^2 - \left(\frac{1}{1-\delta} \right) \sin^2 \phi \right] \psi_2 + \left(\frac{\delta}{1-\delta} \right) \sin^2 \phi \; \psi_1 = \gamma_2^2 \, q_2$$

▷   fractional depths

$$\gamma_1 = \left(\frac{\alpha}{1+\alpha} \right) + \epsilon \left(\bar{\eta}_1 - \bar{\eta}_2 \right) \quad ; \quad \gamma_2 = \left(\frac{1}{1+\alpha} \right) + \epsilon \, \bar{\eta}_2$$

sPV Balance Relations: $\psi_k \longrightarrow u_k, v_k, \eta_k$

$$u = -\epsilon \frac{\partial \psi_k}{\partial \phi} \quad ; \quad v = \frac{\epsilon}{\cos \phi} \frac{\partial \psi_k}{\partial \lambda} \quad ; \quad \eta = \psi_k \sin \phi.$$

Disturbance PV Advection: time evolution

$$\frac{DQ_k}{Dt} = \epsilon \left\{ \frac{\partial q_k}{\partial t} + \frac{\bar{u}_k + u_k}{\cos \phi} \frac{\partial q_k}{\partial \lambda} + v \frac{\partial q_k}{\partial \phi} \right\} + v_k \left(\frac{\sin \phi}{\gamma_k} \right)_\phi = 0$$

2-Layer rSW Baroclinic Instability

Linearized rSW, Warn (1976)

- identify unstable eigenvalues; expect sPV results to compare closely for small ϵ

## Jet Profile

- mean jet in upper layer

$$\bar{u}_1 = \frac{1}{8} \sin^2 \phi \cos^3 \phi$$

- max jet speed ≈ 22 m/s at 40°
- no flow in lower layer

$$\bar{u}_2 = 0$$

Parameter values
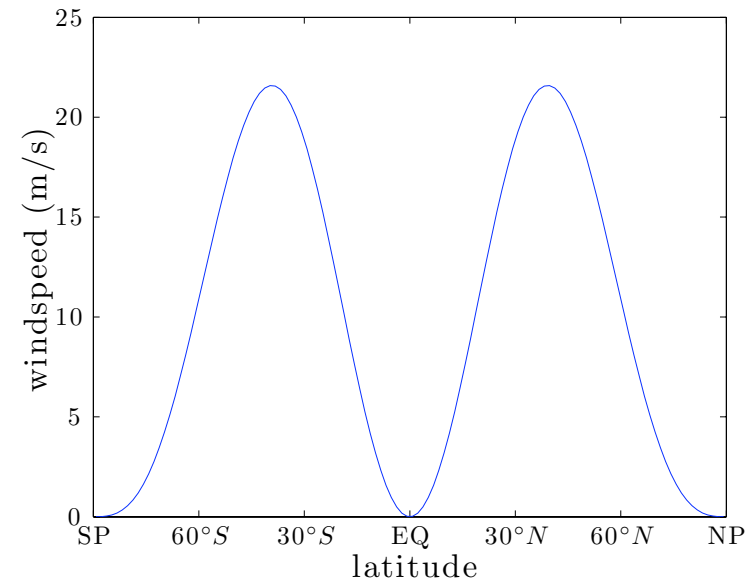
- $\epsilon = \sqrt{2/13} \approx 0.4$
- $\alpha \equiv H_1/H_2 = 1$
- $\delta \equiv \rho_1/\rho_2 = 0.9$

2-Layer rSW Baroclinic Instability

Linearized rSW, Warn (1976)

- identify unstable eigenvalues; expect sPV results to compare closely for small ϵ

Jet Profile

- disturbance height in upper layer

$$\epsilon\,\bar{\eta}_1 = \frac{1}{8}\left(\frac{1}{6}\sin^6\phi - \frac{1}{4}\sin^4\phi + \frac{11}{420}\right)$$

$$\epsilon\,\bar{\eta}_2 = -\frac{\delta}{1-\delta}\,\epsilon\,\bar{\eta}_1$$

- mean depth: $1 + (\epsilon\,\bar{\eta}_1 - \epsilon\,\bar{\eta}_2)$

Parameter values

- $\epsilon = \sqrt{2/13} \approx 0.4$
- $\alpha \equiv H_1/H_2 = 1$
- $\delta \equiv \rho_1/\rho_2 = 0.9$
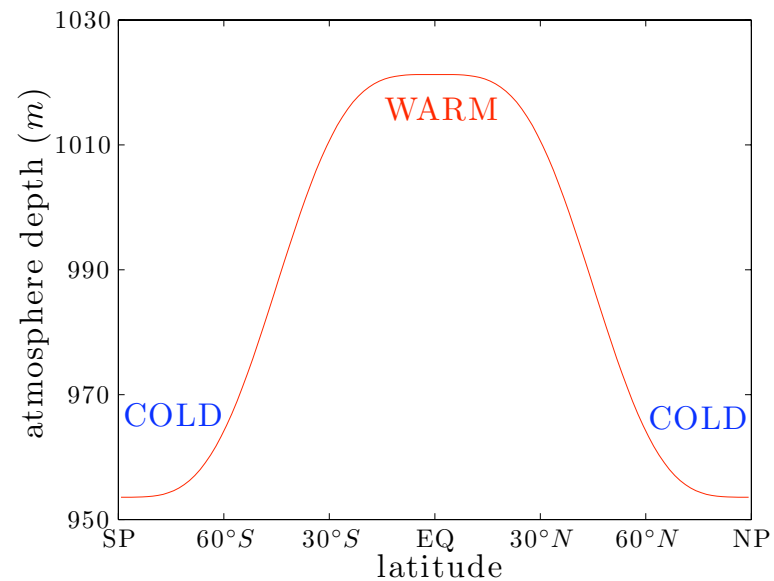
Unstable Growth Rates: ($\epsilon \approx 0.4$)

				m			
	1	2	3	4	5	6	7
sPV	0.014404	0.029759	0.049603	0.084604	0.20412	0.21476	0.074311
rSW	0.0173	0.0352	0.0527	0.1135	0.2054	0.2373	0.1511*

Maximum Growth Rates (day^{-1})

- most unstable mode: wavenumber $m = 6$
- short wave cutoff [Charney (1947), Eady (1949)]

48Hr GFS Issued: 12Z03MAR2009 Valid: 12Z Thu 05 MAR 2009 500mb Geopotential Heights (dam), Vorticity (1e-4/sec)
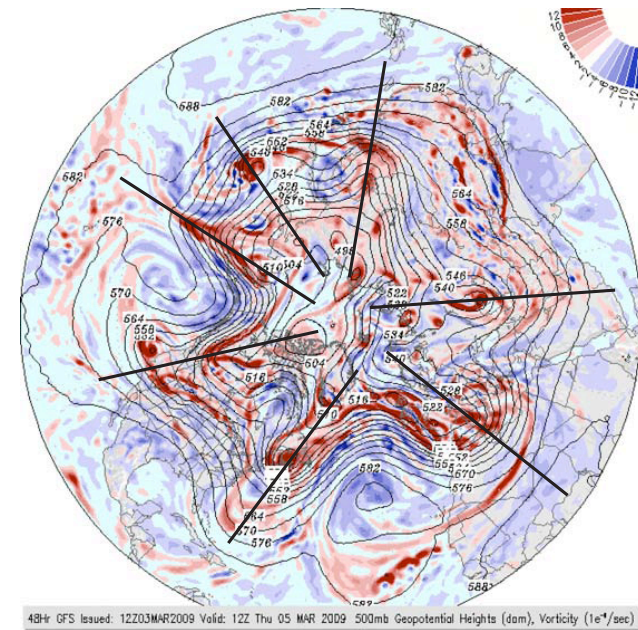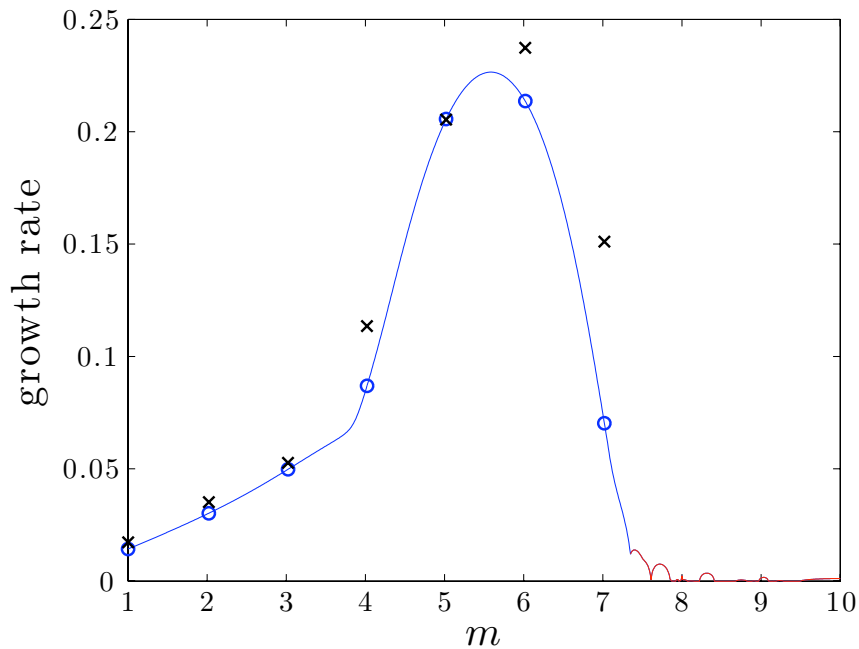
Most Unstable Mode ($m = 6$)

Upper Layer $PV\ (q_1)$ streamfunction (ψ_1)
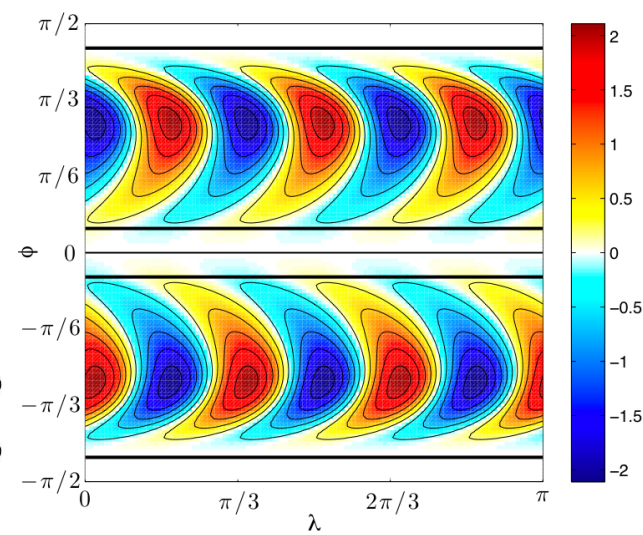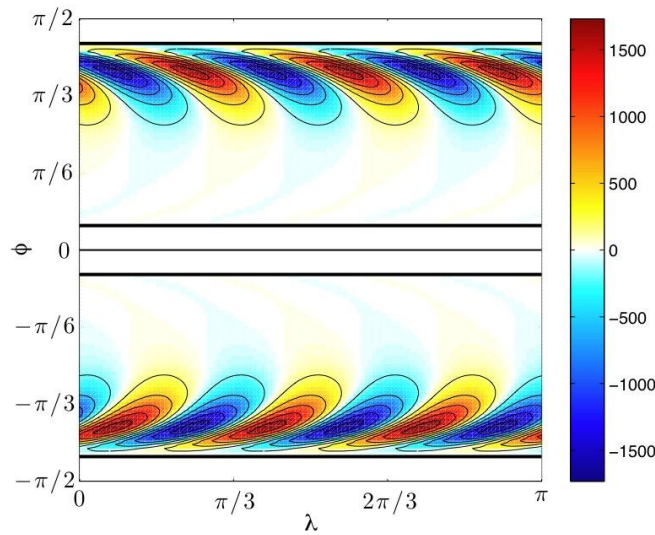
Lower Layer $PV\ (q_2)$ streamfunction (ψ_2)

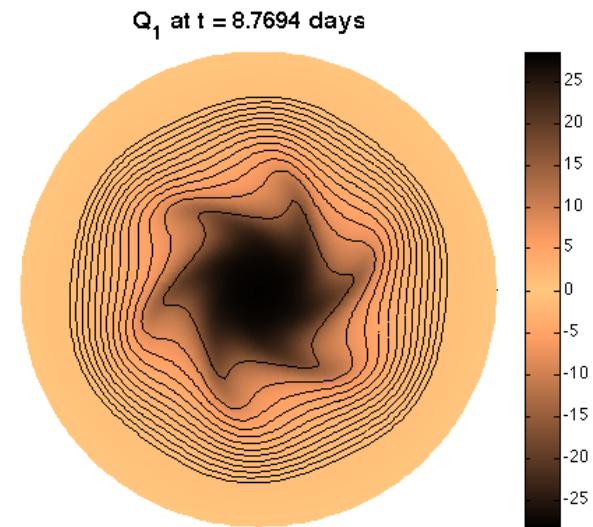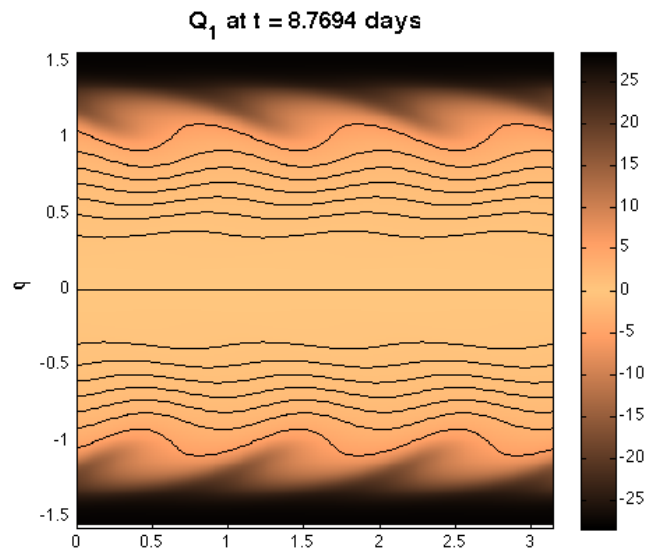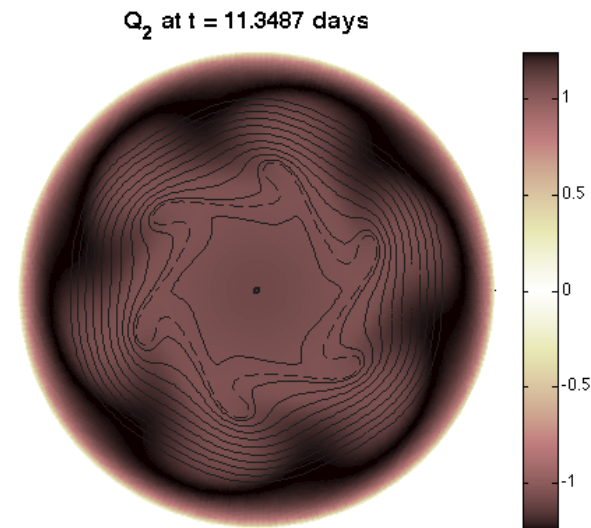## Total PV & Streamfunction/Height
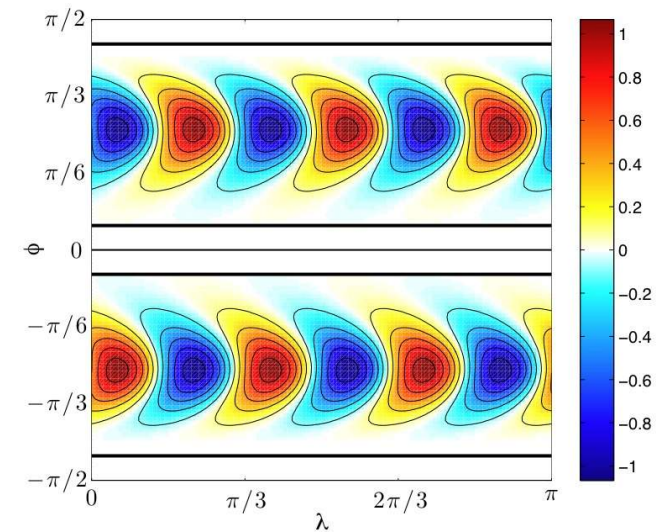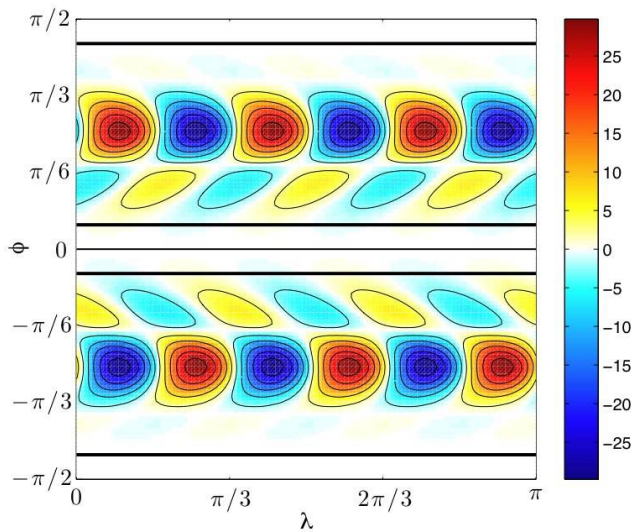
Nonlinear Evolution: $m = 6$ & $t = 8.77$ days
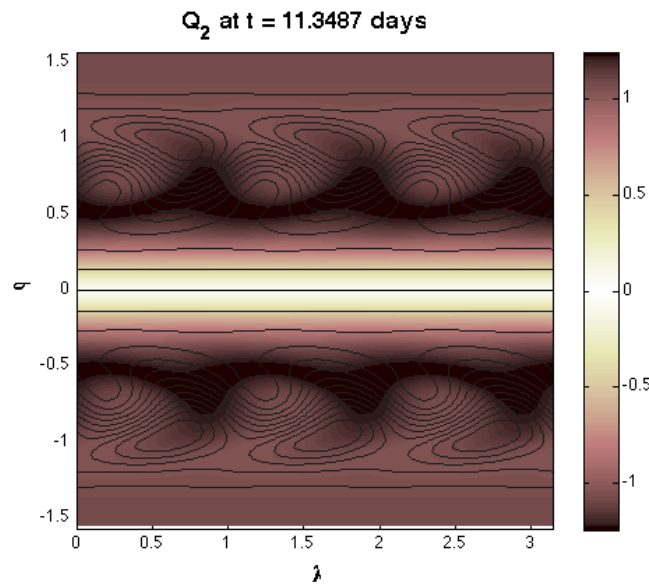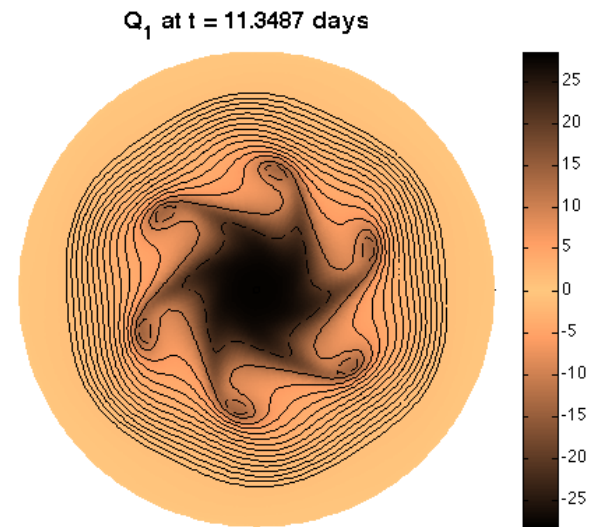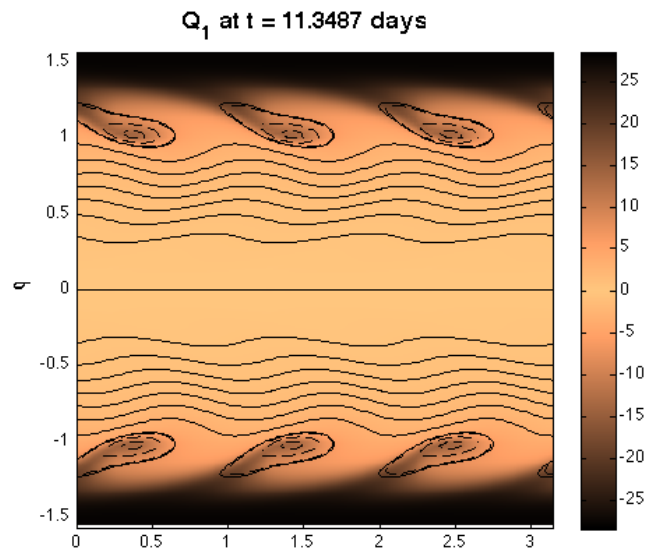
Total PV & Streamfunction/Height

Total PV & Streamfunction/Height

In Closing

PV Dynamics for the rSW Sphere

- ▷ local asymptotic validity in midlatitudes

- ▷ accurate for global Rossby waves at scales smaller than planetary

- ▷ recreates Grose & Hoskins (1979) equatorial wave crossing

- ▷ shares identical (linear) ray theory with rSW

  - ▷ rSW ray theory is degenerate theory, but globally valid

- ▷ sPV includes <u>nonlinear</u> PV advection