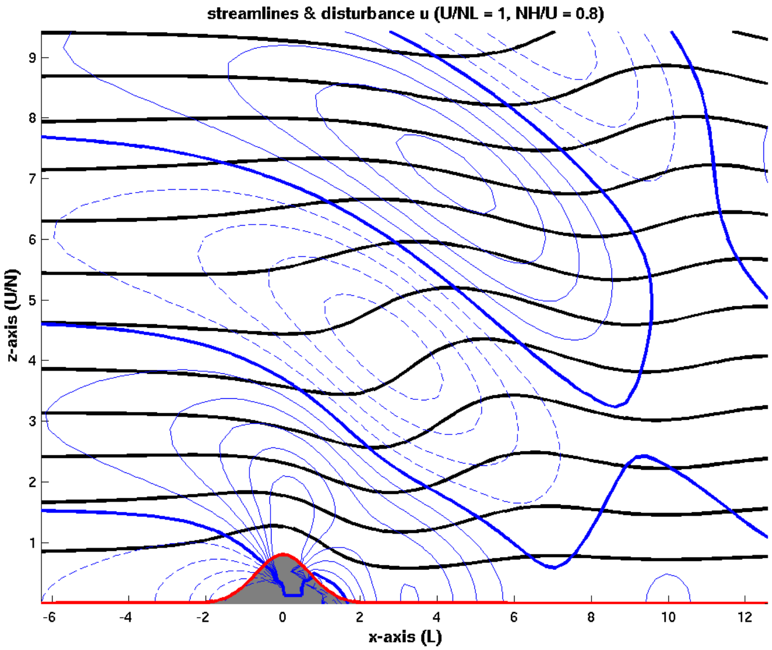
Waves Generated by Airflow over a Mountain Ridge

- ▷ Long's 1953 theory for steady, density-stratified flow over obstacles
- ▷ integral equation approaches for a Helmholtz problem



http://www.fridgeproductions.pwp.blueyonder.co.uk/

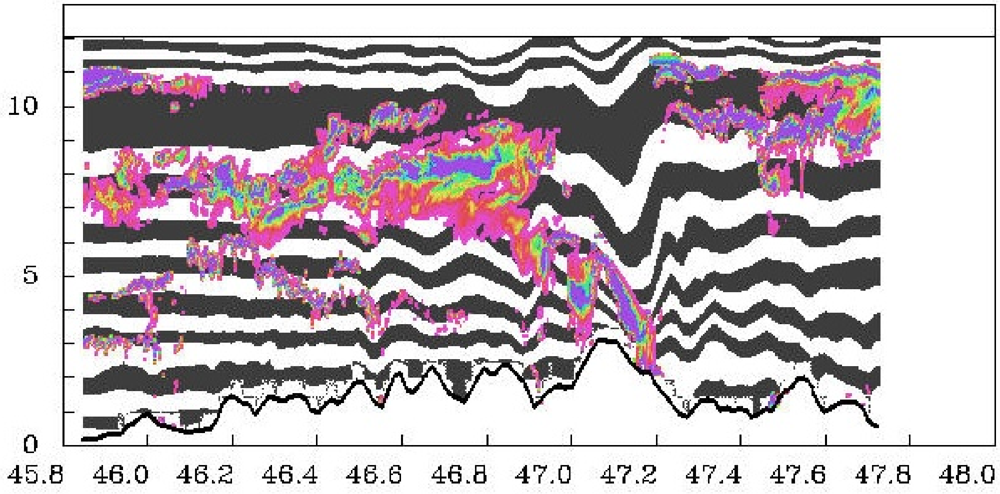
- ▷ Dave Muraki, Simon Fraser University
- ▷ Youngsuk Lee & David Alexander, SFU
- ▷ Craig Epifanio, Texas A&M

Topographic Gravity Waves

Atmospheric Concerns

http://www.casa.gov.au/avreg/fsa/download/02jan/ATSB.pdf

http://users.snowcrest.net/weshawk/LayeredLentic.jpeg



Volkert, et.al. 2002

▷ mathematical story: idealized steady 2D flows & their stability

Atmospheric Fluid Dynamics

Fluid Dynamics & Thermodynamics

- incompressible 2D Euler equations with Boussinesq buoyancy

$$
\begin{aligned}
u_x + w_z &= 0 \\
\frac{Du}{Dt} &= -\phi_x \\
\frac{Dw}{Dt} - B &= -\phi_z \\
\frac{DB}{Dt} &= 0
\end{aligned}
$$

- adiabatic buoyancy, B (buoyant $\leftrightarrow$ light) & geopotential, ϕ (pressure)
- 2D advection: $\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + u\,\dfrac{\partial}{\partial x} + w\,\dfrac{\partial}{\partial z}$

Streamfunction & Vorticity

- streamfunction, $\Psi \longrightarrow u = \Psi_z \;;\; w = -\Psi_x$
- vorticity, $\eta \longrightarrow \eta = u_z - w_x = \nabla^2\Psi$

Stratified Potential Flow

Vorticity/Buoyancy Formulation

- vorticity inversion: $\nabla^2\Psi = \eta$

$$\begin{aligned} \frac{D\eta}{Dt} + B_x &= \eta_t + J(\eta,\Psi) + B_x = 0 \\ \frac{DB}{Dt} &= B_t + J(B,\Psi) = 0 \end{aligned}$$

- 2D streamfunction advection $\longrightarrow$ Jacobian determinant

$$J(f,\Psi) = \begin{vmatrix} f_x & \Psi_x \\ f_z & \Psi_z \end{vmatrix} = u f_x + w f_z$$

Steady Flow

- zero Jacobian condition: $J(B,\Psi) = 0 \longrightarrow B$ is constant along streamlines
- upstream/mean conditions (uniform wind & constant stratification):

$$\left.\begin{aligned} \Psi &= \mathcal{U}\, z + \psi \\ B &= \mathcal{N}^2 z + b \end{aligned}\right\} \longrightarrow B = \frac{\mathcal{U}}{\mathcal{N}^2}\,\Psi$$

- vorticity condition for disturbance streamfunction, $\psi(x,z)$

$$J(\eta,\Psi) + \frac{\mathcal{N}^2}{\mathcal{U}}\Psi_x = J\left(\nabla^2\psi - \frac{\mathcal{N}^2}{\mathcal{U}} z, \mathcal{U}z + \psi\right) = 0$$

Long's 1953 Theory

Helmholtz Equation

- linear Helmholtz equation for steady 2D streamfunction, $\psi(x, z)$

$$\nabla^2\psi + \left(\frac{\mathcal{N}}{\mathcal{U}}\right)^2 \psi = 0$$

- special nonlinear solutions for constant stratification & uniform incident wind

Scales

- simple topographic case: three length scales

$$L = \text{mountain width} \;\; ; \;\; \mathcal{U}/\mathcal{N} = \text{wave height} \;\; ; \;\; H = \text{mountain height}$$

- two dimensionless parameters

$$\sigma \equiv \frac{\mathcal{U}}{\mathcal{N}L}, \text{ nonhydrostatic parameter} \;\; ; \;\; \mathcal{A} \equiv \frac{\mathcal{N}H}{\mathcal{U}}, \text{ height parameter}$$

Nondimensionalized Problem

- Helmholtz equation ($\sigma \longrightarrow 0$, hydrostatic case)

$$\sigma^2\psi_{xx} + \psi_{zz} + \psi = 0$$

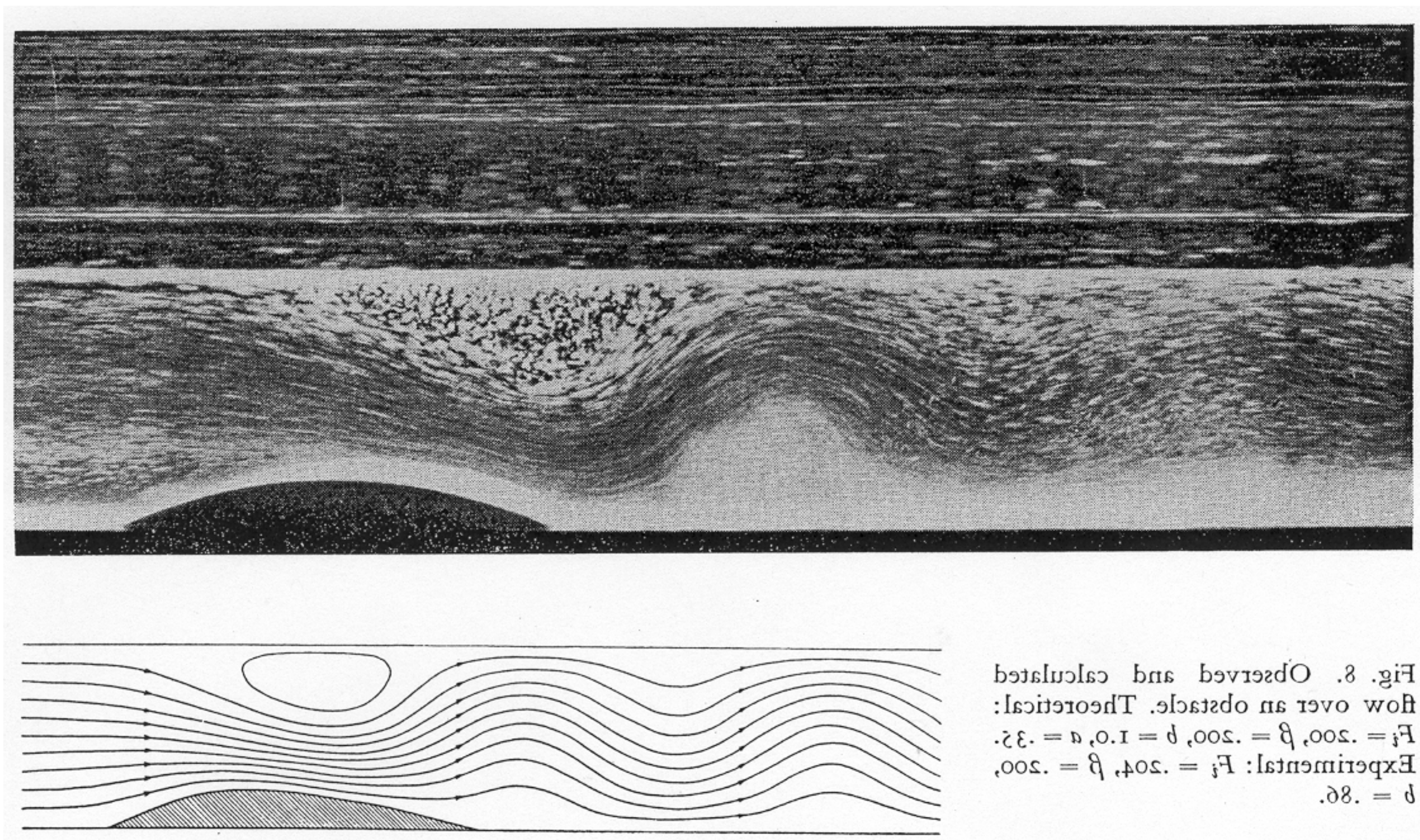
- zero surface streamfunction: $\Psi(x, \mathcal{A}h(x)) = \mathcal{A}h(x) + \psi(x, \mathcal{A}h(x)) = 0$

Long 1955: Theory & Experiment

$$\sigma^2 \psi_{xx} + \psi_{zz} + \psi = 0$$

Finite Amplitude Topography

- ▷ on streamline boundaries: $\psi = \mathcal{A}h(x) + \psi(x, \mathcal{A}h(x)) = \text{constant}$



Long 1953

Fourier Modes, $e^{i(kx+mz)}$

- Helmholtz dispersion relation: $m^2 = 1 - \sigma^2 k^2$
- sign choice $\longrightarrow$ far-field conditions: upward group velocity or decay (Lyra, 1940)

$$m(k) = \begin{cases} \operatorname{sign}(k)\sqrt{1-\sigma^2k^2} & \text{for } |\sigma k| \le 1 \quad \text{(long scale radiation)} \\ i\sqrt{\sigma^2k^2-1} & \text{for } |\sigma k| \ge 1 \quad \text{(short scale decay)} \end{cases}$$

General Helmholtz Solution

- Fourier integral representation with far-field conditions

$$\psi(x,z) \;=\; -\mathcal{A}\int_{-\infty}^{+\infty} \hat{c}(k)\; e^{i(kx+m(k)z)}\; dk$$

- $z = \mathcal{A}h(x)$ surface condition: $\mathcal{A}h(x) + \psi(x, \mathcal{A}h(x)) = 0$

$$\boxed{h(x) \;-\; \int_{-\infty}^{+\infty} \hat{c}(k)\; e^{i(kx+m(k)\mathcal{A}h(x))}\; dk \;=\; 0}$$

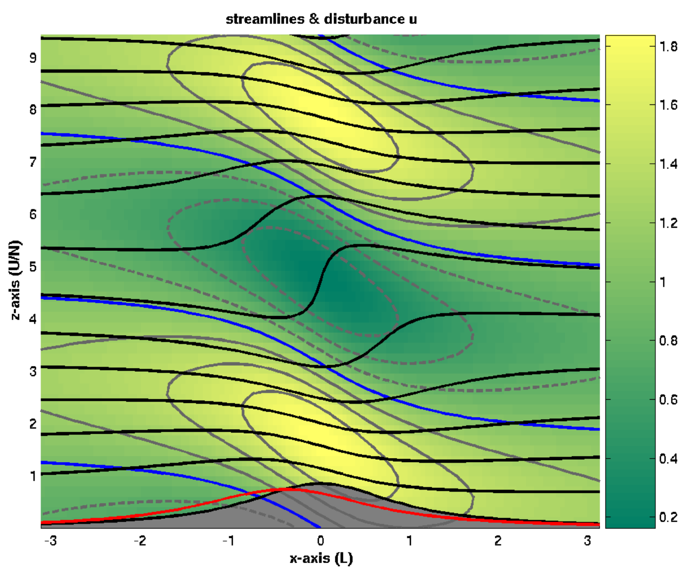
- linear integral operator on $\hat{c}(k)$ $\longrightarrow$ Fredholm integral equation of first-kind
- numerically equivalent to a matrix inversion

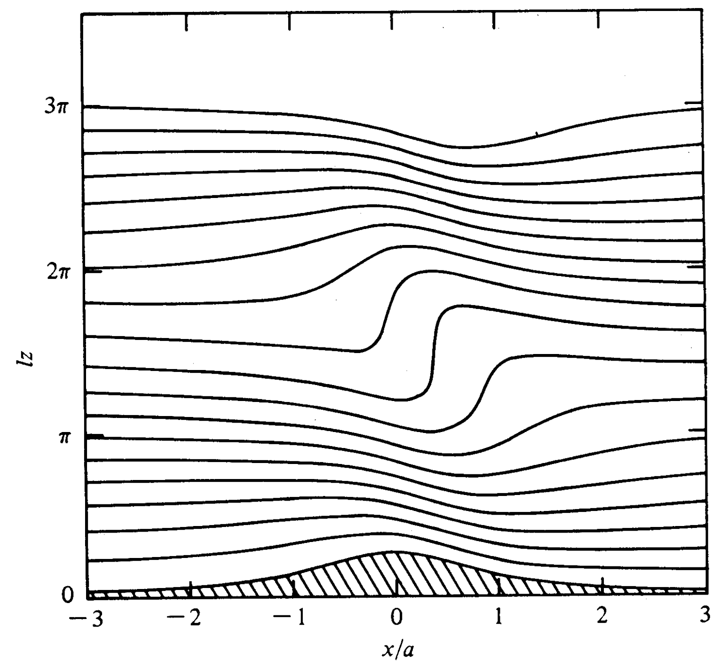
Weak Topographic Approximation

$$h(x) \; - \; \int_{-\infty}^{+\infty} \hat{c}(k)\; e^{ikx}\; dk \; = \; 0$$

$\mathcal{A} \to 0$ "Linear" Limit is Fourier Inversion: $\hat{c}(k) = \hat{h}(k)$

- ▷ hydrostatic ($\sigma = 0$) & critical overturning ($\mathcal{A} = 0.85$); computed via FFTs





Lilly/Klemp 1979

- ▷ bottom (zero) streamline does not match topographic surface: $h(x) = 1/(1+x^2)$
- ▷ FFT-based iterative solvers for exact surface condition (Raymond, 1972; Laprise/Peltier, 1988)

Direct Steady Solve

$$h(x) \; - \; \int_{-\infty}^{+\infty} \hat{c}(k) \; e^{i(kx+m(k)\mathcal{A}h(x))} \; dk \; = \; 0$$

Numerical Discretization

- collocation points: $\{x_1 \; \ldots \; x_\alpha \; \ldots \; x_N\}$ & N knowns: $h_\alpha = h(x_\alpha)$
- wavenumbers: $\{k_1 \; \ldots \; k_\beta \; \ldots \; k_N\}$ & N unknowns: $\hat{c}_\beta \approx \hat{c}(k_\beta)$
- approximate integral for each x_α by trapezoidal rule over $\beta = 1 \ldots N$

$$h_\alpha \; - \; \sum_{\beta=1}^{N} \hat{c}_\beta \; \underbrace{e^{i(k_\beta x_\alpha + m(k_\beta)\mathcal{A}h(x_\alpha))} \; w_\beta \, \Delta k}_{\mathbf{K}_{\alpha,\beta}} \; = \; 0$$

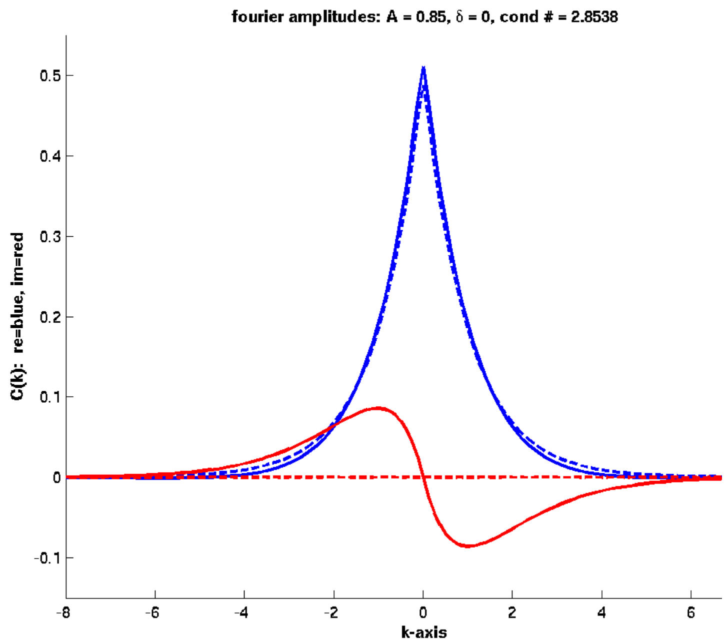
Matrix Inversion

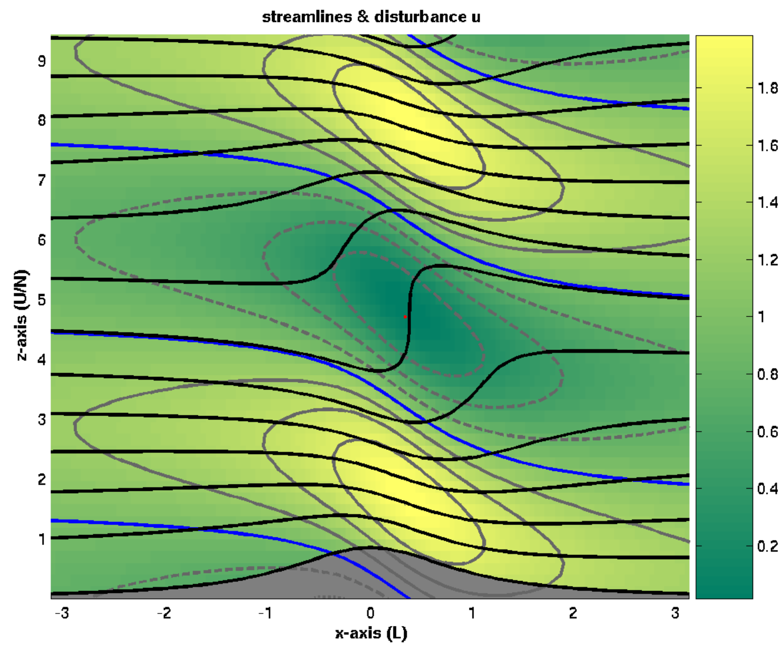
- N linear equations in N unknowns: $(\vec{h_\alpha}) = [\mathbf{K}_{\alpha,\beta}] \; (\vec{c_\beta})$
- $m(k)$ is discontinuous at $k = 0 \longrightarrow$ half-line integrals
- full matrix $\mathbf{K}$ can be ill-conditioned $\longrightarrow$ catastrophic loss of precision as N increases

Numerical Implementation

Fourier Conditioning

- for $\mathcal{A} = 0$ linear theory, discrete Fourier transform is well-conditioned
- equi-spaced discretizations with $\Delta k \, \Delta x = 2\pi/N$ is <u>essential</u>



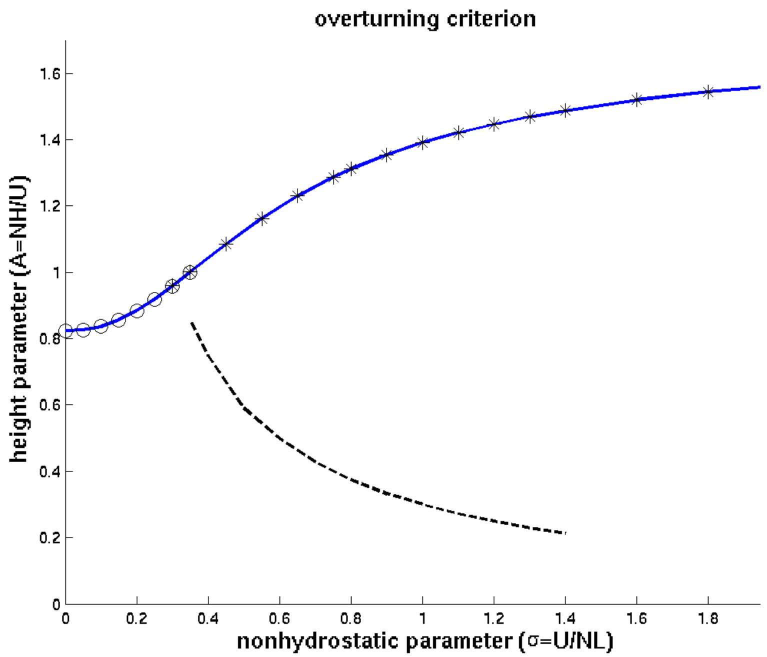


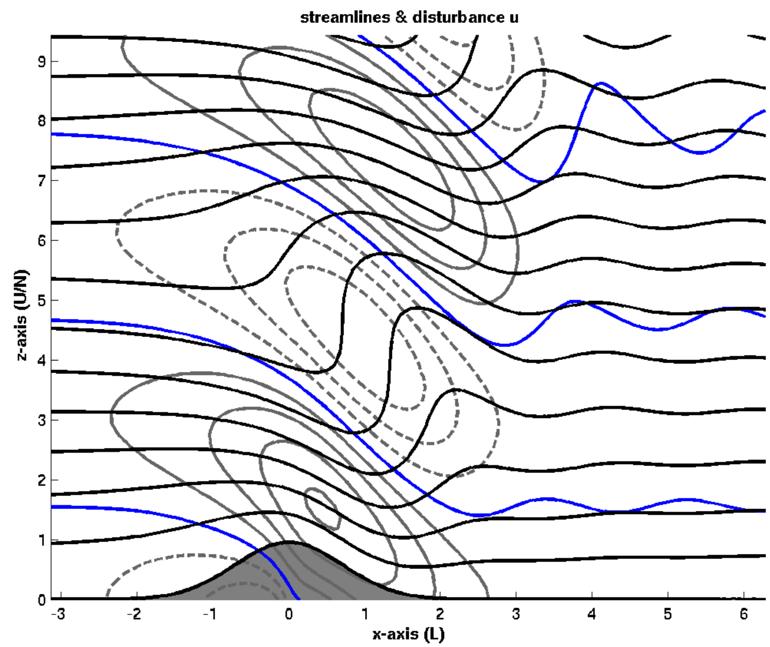
- hydrostatic critical overturning case (Lilly/Klemp 1979)
  - $N = 256, x_\infty = 8\pi$: 1.1s to solve & 2.0s to plot, log-condition number = 2.85
- Fourier representation allows periodic wrap-around $\longrightarrow$ large computational domains

Critical Overturning I

Gaussian Topography

- critical height $\mathcal{A}$ as a function of nonhydrostatic parameter σ
- *wavebreaking* limit for stable (density) stratification



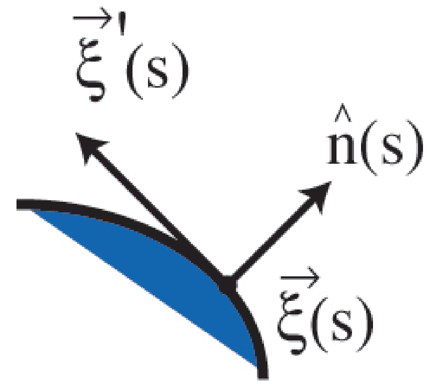


- Fourier formulation (o) limited by large condition numbers
  - ill-conditioning edge: $\sim$7 digits lost
  - $\sigma = 0.35$ flow

$$\mathcal{G}_{xx} + \mathcal{G}_{zz} + \mathcal{G} = \delta(\vec{x} - \vec{\xi})$$

Helmholtz Free-Space Green's Function ($\sigma = 1$)

- radiating solution for a delta-function source at $\vec{\xi}$: $\mathcal{G}(\vec{x} - \vec{\xi})$
- classical, time-harmonic scattering problem in electromagnetics/acoustics
  - delta-function response in 2D involves Hankel functions: $J_0(r) \pm i\,Y_0(r)$
  - sign choice determined by far-field radiation condition (implied by time-harmonicity)



Boundary Integral Method

- $\mu(s)$, weighted surface distribution of Green's functions
- $\vec{\xi}(s)$, parametrization of surface boundary (clockwise)

$$\psi(\vec{x}) = -\mathcal{A} \int_{\mathcal{S}} \mu(s)\ \ 2\,\frac{\partial \mathcal{G}}{\partial n}(\vec{x} - \vec{\xi}(s))\ ds$$

- need topographic Green's function $\mathcal{G}(\vec{x} - \vec{\xi})$ & weights $\mu(s)$

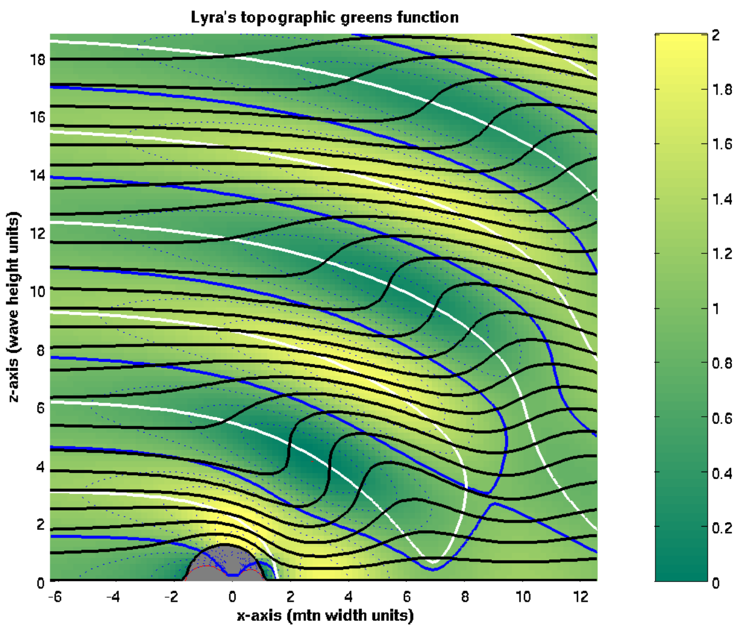
Lyra's Topographic Green's Function

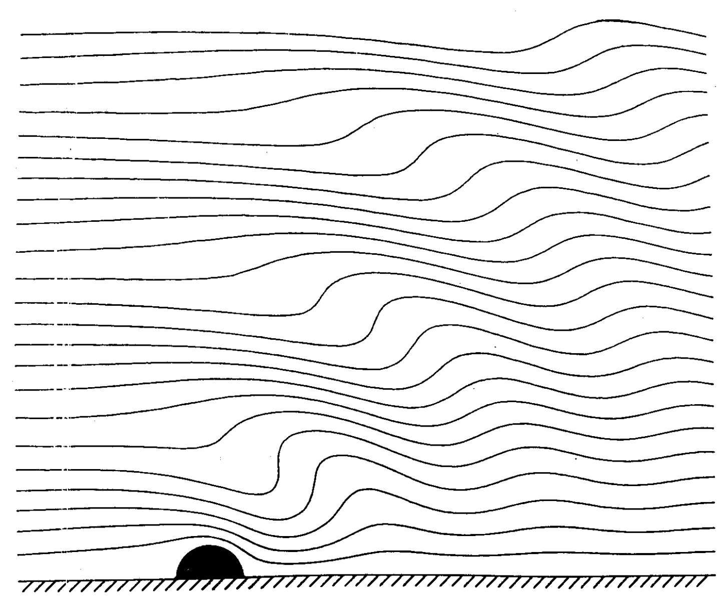
Delta-Function Topography (linear theory)

- from Lyra 1940 & 1943 (via Alaka 1960) for $\sigma = 1$ as Bessel series

$$\mathcal{G}_z(r,\theta) \;=\; \frac{1}{2}\,Y_1(r)\sin\theta \;+\; \frac{1}{\pi}\sum_{1}^{\infty}\frac{4n}{4n^2-1}\,J_{2n}(r)\sin 2n\theta$$

- Lyra's critical overturning solution: $\Psi = z + 4.06\;\mathcal{G}_z(r,\theta)$





Miles/Huppert 1968

- left/right asymmetric Greens function: <u>waves must be downstream</u> (Miles/Huppert 1968)

Fredholm Integral Equation of Second-Kind

Singular Integral Representation

- Plemelj formula for surface values, $\vec{x}_{\mathcal{S}}$

$$\psi(\vec{x}_{\mathcal{S}}) = -\mathcal{A}\;\mu(\vec{x}_{\mathcal{S}}) - \mathcal{A}\int_{\mathcal{S}} \mu(s)\;\; 2\,\frac{\partial \mathcal{G}}{\partial n}(\vec{x}_{\mathcal{S}} - \vec{\xi}\,(s))\; ds$$

- surface boundary condition $\longrightarrow$ <u>second-kind</u> integral equation for $\mu(\vec{x}_{\mathcal{S}})$

$$\boxed{\mu(\vec{x}_{\mathcal{S}}) + \int_{\mathcal{S}} \mu(s)\;\; 2\,\frac{\partial \mathcal{G}}{\partial n}(\vec{x}_{\mathcal{S}} - \vec{\xi}\,(s))\; ds \quad = \quad h(\vec{x}_{\mathcal{S}})}$$

- kernel function is continuous at $\vec{x}_{\mathcal{S}} = \vec{\xi}\,(s) \longrightarrow$ curvature contribution
- discretized quadrature gives diagonally-dominant matrix $\longrightarrow$ <u>well-conditioned inversion</u>

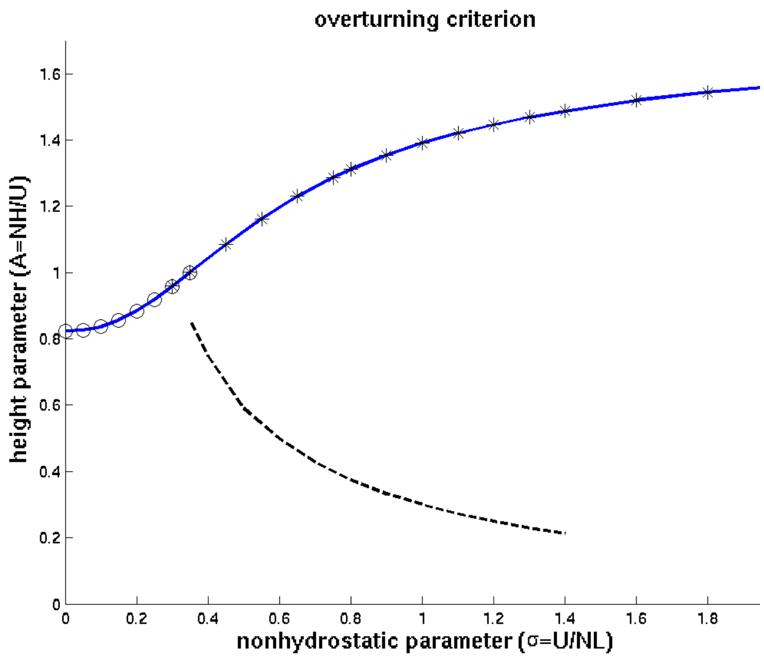
- amplitude parameter, $\mathcal{A}$, enters through surface parametrization: $\vec{\xi}\,(s) = \begin{pmatrix} x(s) \\ \mathcal{A}h(x(s)) \end{pmatrix}$
- small $\mathcal{A}$ limit: $\mu(\vec{x}_{\mathcal{S}}) \longrightarrow h(\vec{x}_{\mathcal{S}})$

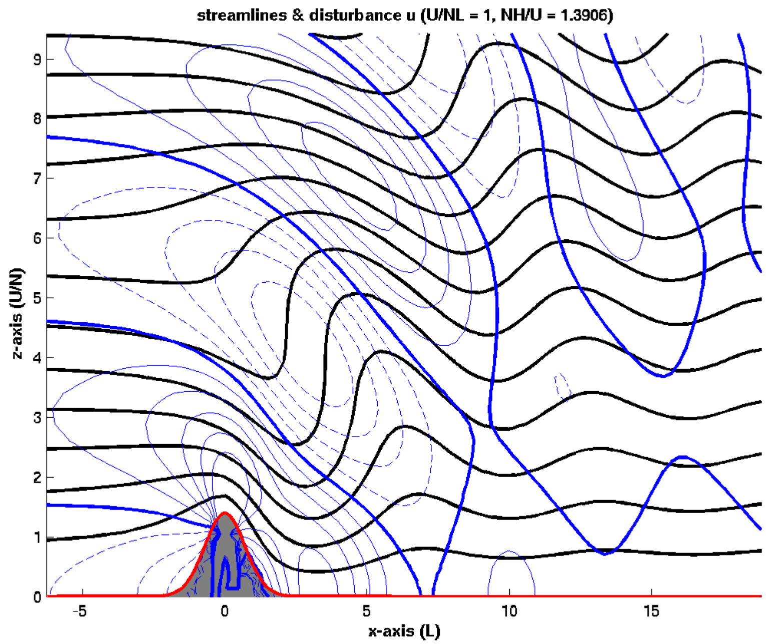
- nonhydrostatic parameter, σ, handled by rescaling in x (singular as $\sigma \longrightarrow 0$)

Critical Overturning II

Strongly Nonhydrostatic ($\sigma \geq 0.3$)

- boundary integral method ($*$) remains well-conditioned



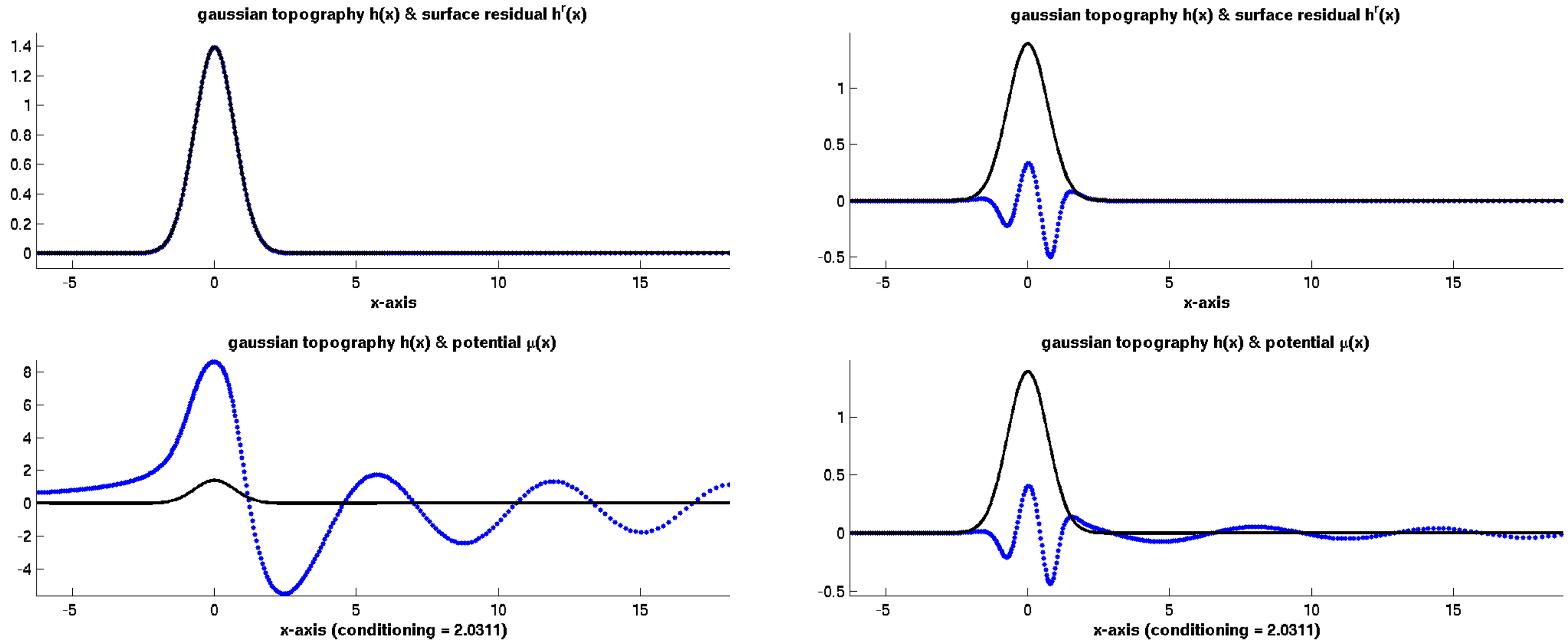


- accurate inversion for deviation from Lyra's overturning solution
  - $\sigma = 1.0$ & $\mathcal{A} = 1.39$ shown above
  - near-surface extrapolations for plotting contours

Large Amplitude Solutions

Slow Decay

- boundary integral method limited by downstream wake in $\mu(x)$



- use Lyra's analytical solution as first guess

$$\psi(\vec{x}) = \Lambda\, \mathcal{G}_z(\vec{x}) - \mathcal{A} \int_{\mathcal{S}} \mu(s) \;\; 2\, \frac{\partial \mathcal{G}}{\partial n}(\vec{x} - \vec{\xi}\,(s))\; ds$$

  - accurate computation based on surface residual: $h^r(x) = h(x) + \Lambda\, \mathcal{G}_z(x, h(x))$
  - Λ obtained by good guesswork (4.06 for critical overturning)

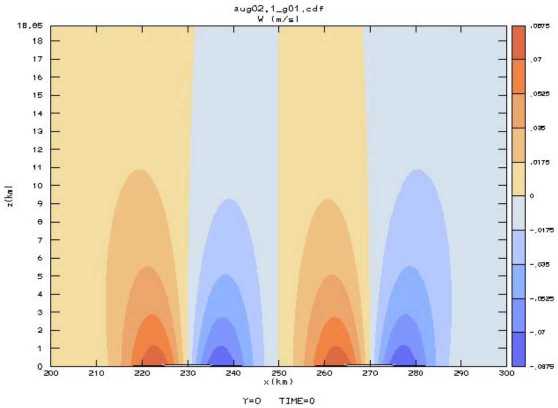
Question of Stability

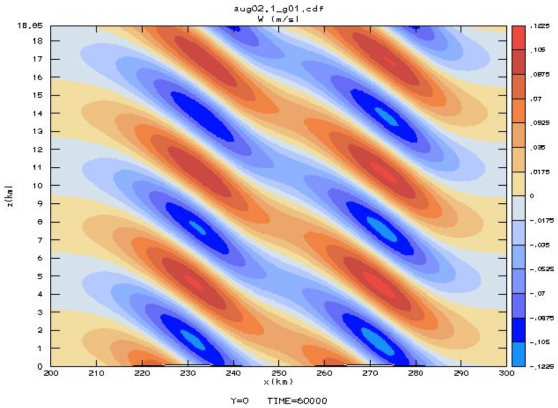
Gravity Wave Instability

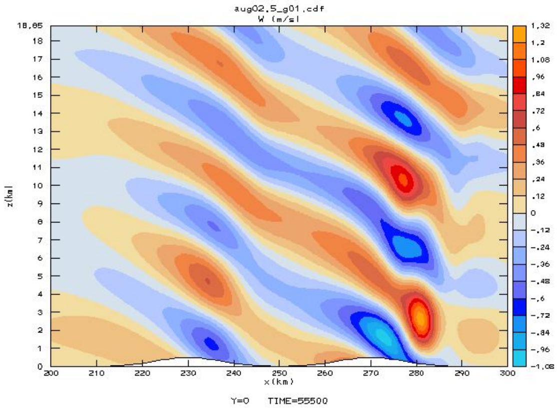
- ▷ Mied (1976), plane gravity waves are parametrically unstable
- ▷ Lilly/Klemp (1979), instability observed for sinusoidal topography
- ▷ Scinocca/Peltier (1994), unstable dynamics from critical overturning

Time-Dependent Simulations (Craig Epifanio, Texas A&M)

- ▷ twin peaks, nearly-hydrostatic ($\sigma = 0.1$), vertical motion w plots
  - ▷ initialized from potential flow
  - ▷ small height $\longrightarrow$ stability to Long's solution
  - ▷ medium height $\longrightarrow$ oscillatory instability to blow-up







Linear Stability of Long's Steady Solutions

Hydrostatic ($\sigma = 0$) Disturbance Equations (David Alexander & Youngsuk Lee, SFU)

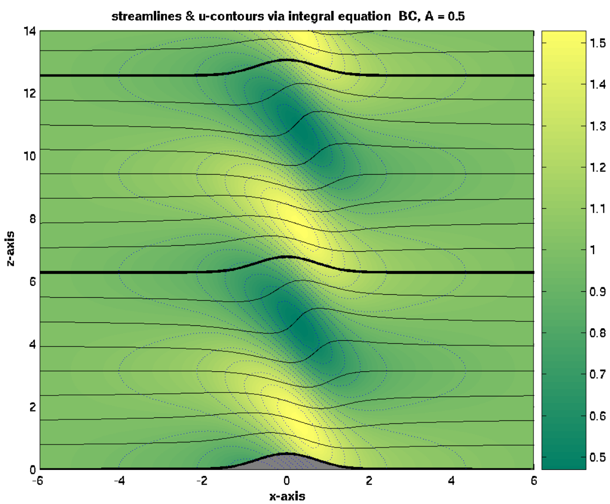
- ▷ non-constant coefficients from Long's streamfunction $\psi(x, z)$

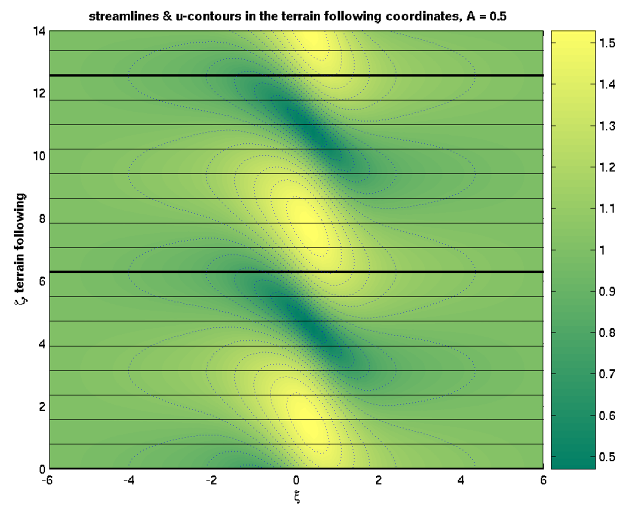
$$\begin{aligned}
\tilde{\psi}_{zzt} - \tilde{\psi}_x + \tilde{b}_x \quad &+ \quad J(\tilde{\psi}_{zz} + \tilde{\psi}, z + \psi) &= \quad 0 \\
\tilde{b}_t \quad &+ \quad J(\tilde{b} - \tilde{\psi}, z + \psi) &= \quad 0
\end{aligned}$$

- ▷ 2D PDE eigenvalue problem for $\tilde{\psi} \longrightarrow \tilde{\psi}(x, z)e^{\lambda t}$ & $\tilde{b} \longrightarrow \tilde{b}(x, z)e^{\lambda t}$

Numerical Linear Algebra

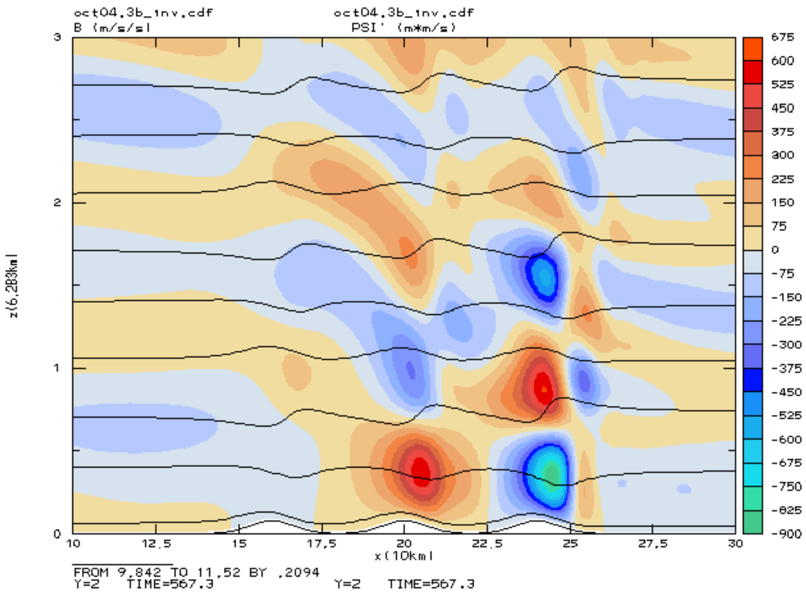
- ▷ buoyancy coordinates for regular lower boundary: $(x, z) \longrightarrow (x, B(x, z))$
- ▷ self-adjoint formulation $\longrightarrow$ Arnoldi iterative search for eigenvalues

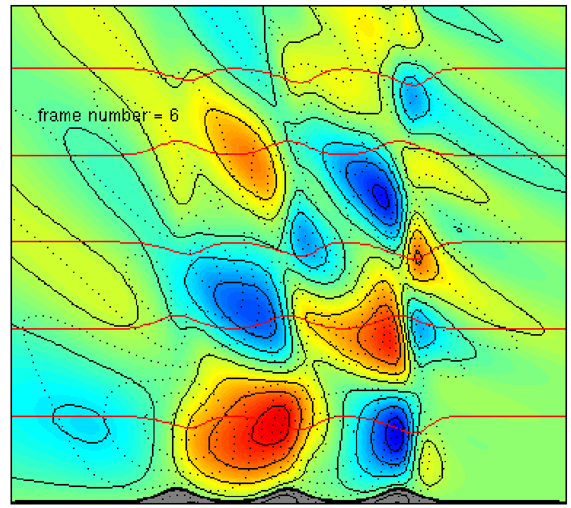




A Search for Eigenvalues . . .

Simulated Instability vs Unstable Eigenfunction



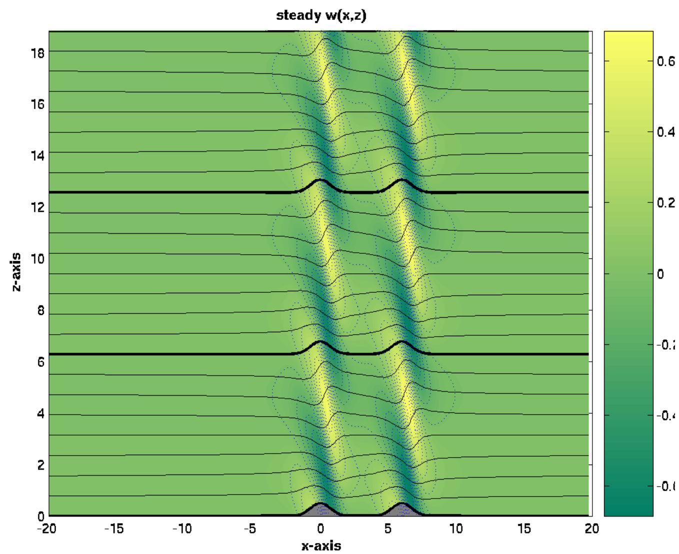


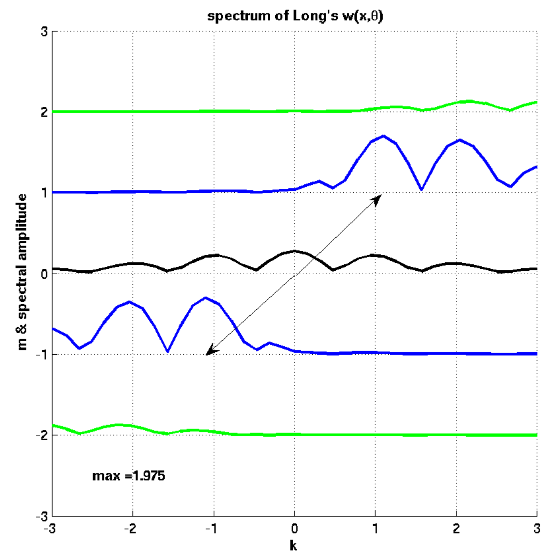
Analytical & Computational Issues

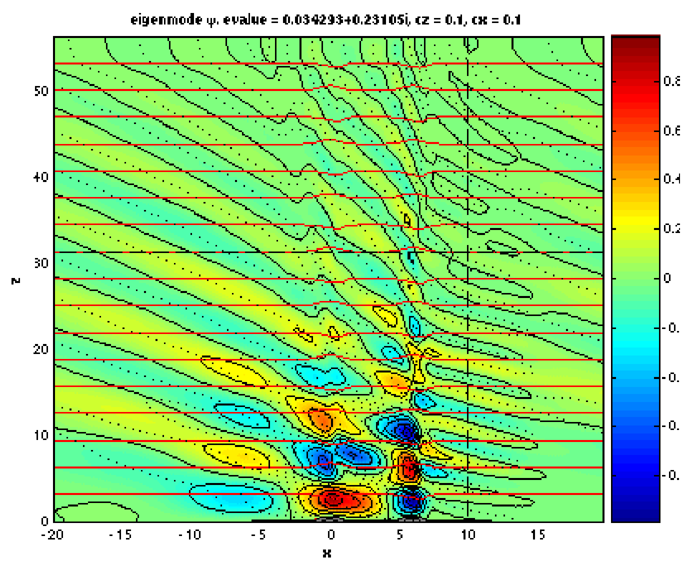
- ▷ 1, 2 & 3-mountain instabilities
- ▷ parametric dependences: mountain heights & spacings
- ▷ fine-tuning damping layers (aloft & lateral)
- ▷ unstable spectrum: multiple discrete or continuous?
- ▷ physical & mathematical mechanisms for instability?

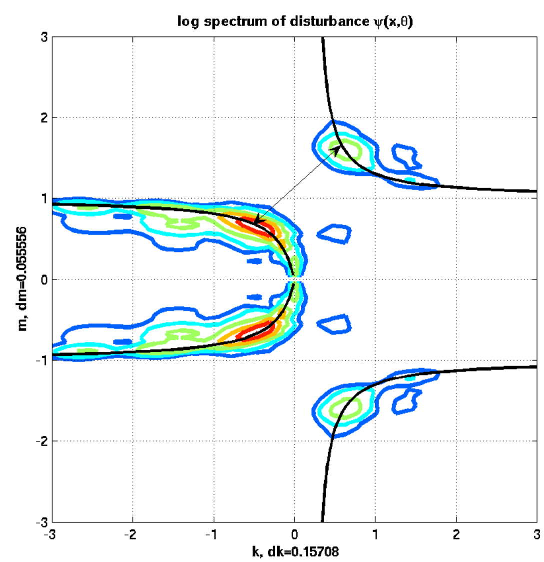
A Resonant Triad Mechanism?

Fourier Spectral Plots (low frequency branch, $\omega = 0.23$)



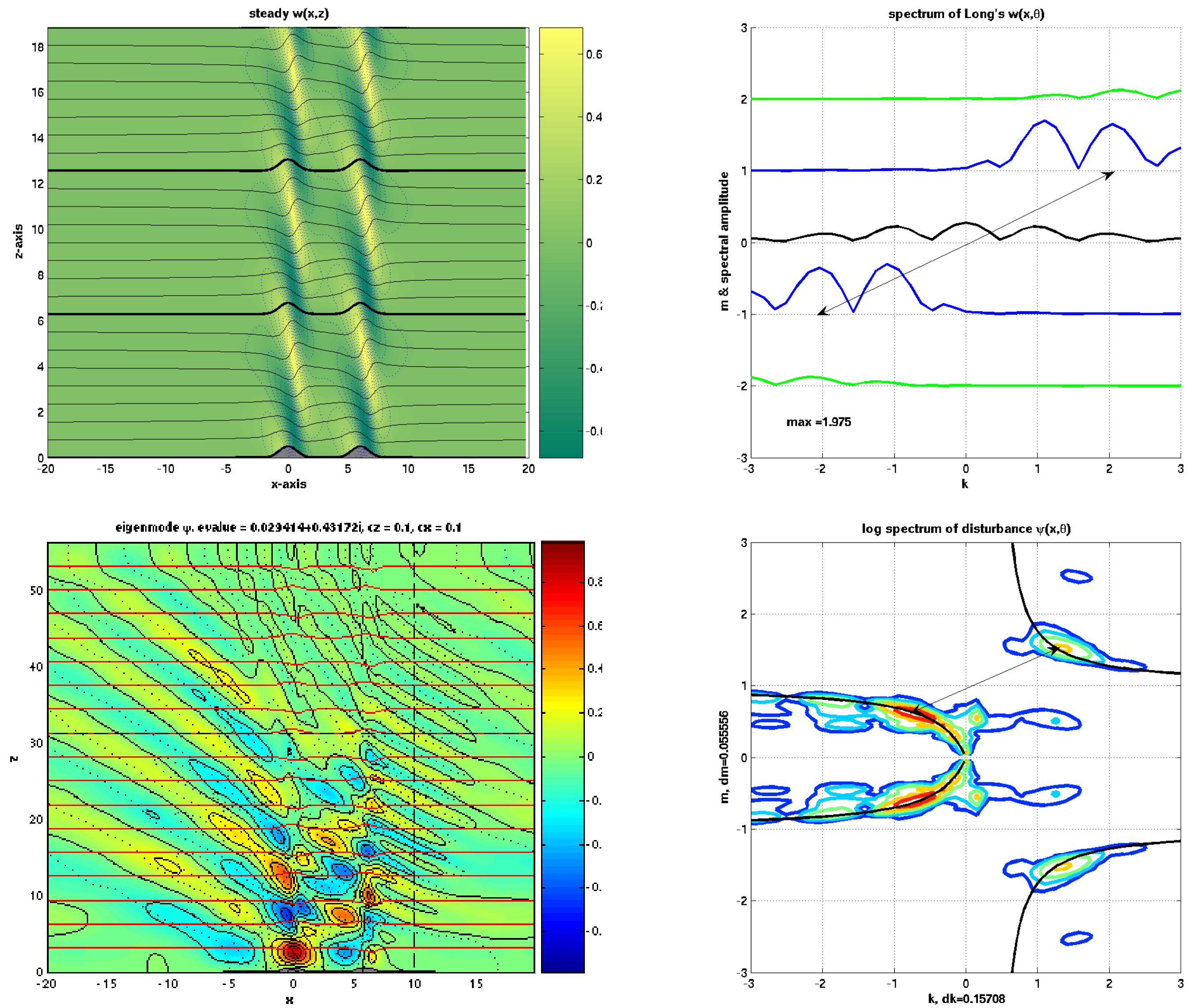






A Resonant Triad Mechanism?

Fourier Spectral Plots (high frequency branch, $\omega = 0.43$)



In Closing

Direct Steady Solve

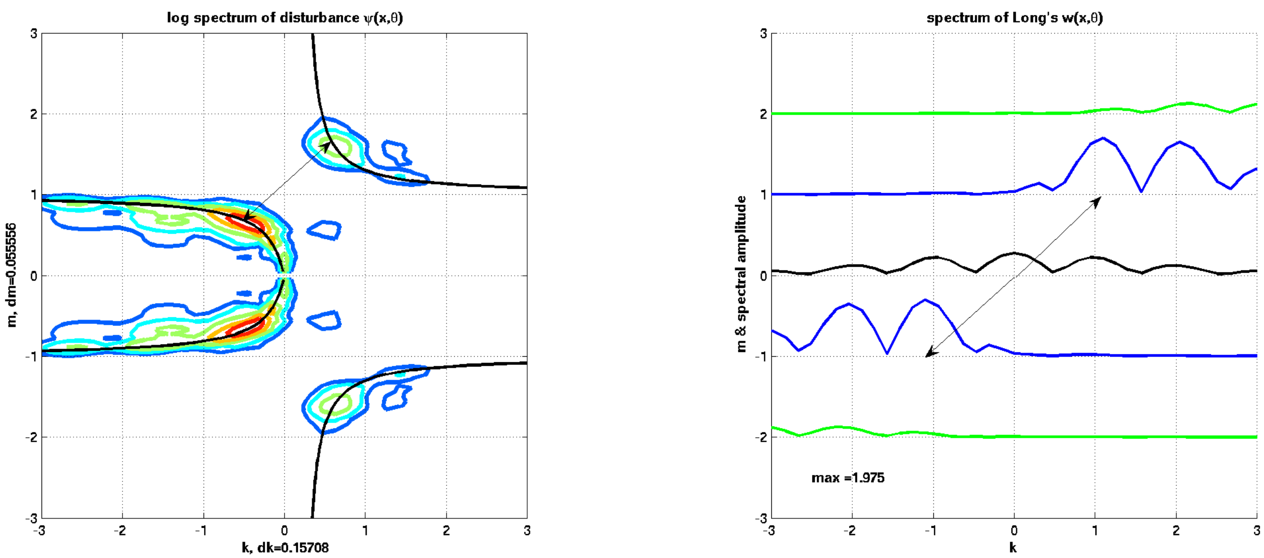
- non-iterative formulations for exact topographic surface condtion
  - Fourier-based 1^{st}-kind solver: near-hydrostatic regime $(0 \leq \sigma < 0.5)$
  - Green's function-based 2^{nd}-kind solver: hydrostatic regime $(0.3 \geq \sigma < 4^{+})$
- overturning criterion to strongly nonhydrostatic regime
- accurate solutions for linear stability analysis

Linear Stability

- identification of linear instabilities for multiply-peaked terrain
  - benchmark against time-dependent simulations
  - triad resonance mechanism
  - height & separation criterion for instability
- <u>implications for atmospheric wave drag estimates/parametrizations?</u>

Eigenfunction Spectra

- ▷ Fourier spectra computed in isentropic coordinates
  - ▷ Long's $w(x,B)$ is 2π-periodic in B; $\tilde{\psi}(x,B)$ has zero BCs at top & bottom
- ▷ $\mathcal{F}[\tilde{\psi}](k,m)$ spectral peaks concentrated on linear dispersion relation: $\omega(k,m) = \mathrm{Im}(\lambda)$
  - ▷ wavevectors of largest peaks related by a Long wavevector



Triad Resonance

- ▷ non-constant coefficients involve multiplications of $\tilde{\psi}(x,B)$ by Long's $u(x,B)$ & $w(x,B)$
  - ▷ multiplication of Fourier modes $\leftrightarrow$ addition of wavevectors