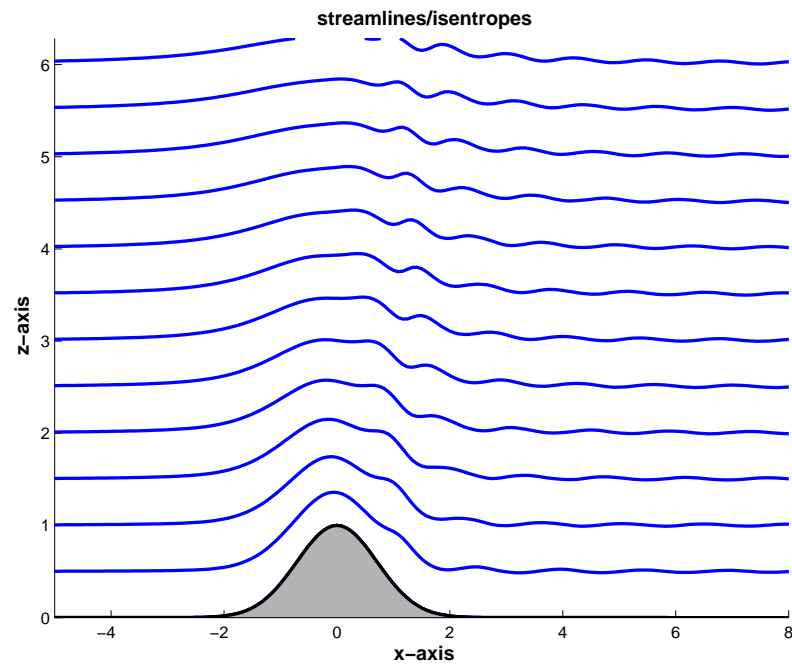
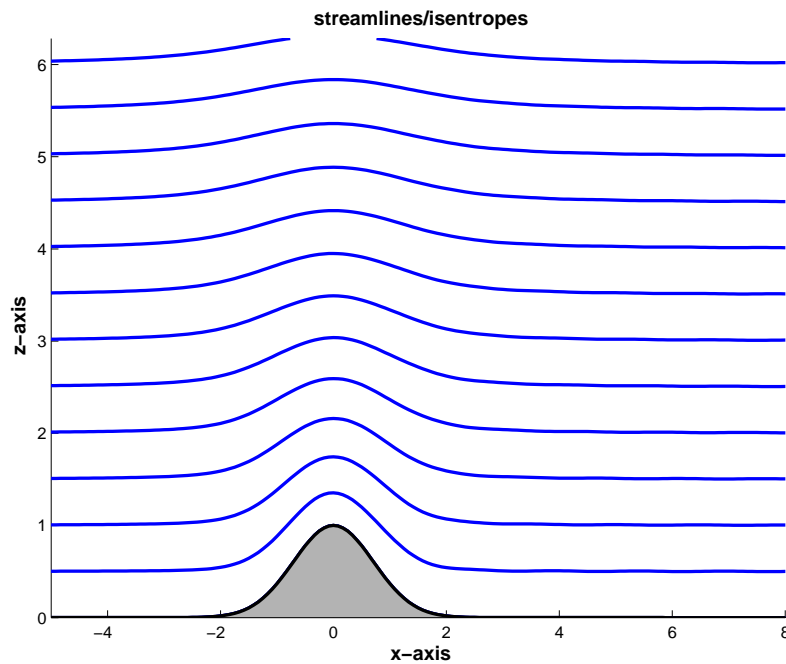


Tales of the Nonlinear: a simple illustration of a spectral cascade

- ▷ Fourier properties of the simplest nonlinear PDE: $u_t + u u_x = 0$
- ▷ nonlinear generation of small scales



- ▷ Dave Muraki (Simon Fraser University)

Topographic Waves in Rotating, Stratified Flow

Uplifted Streamlines & Downstream Wake



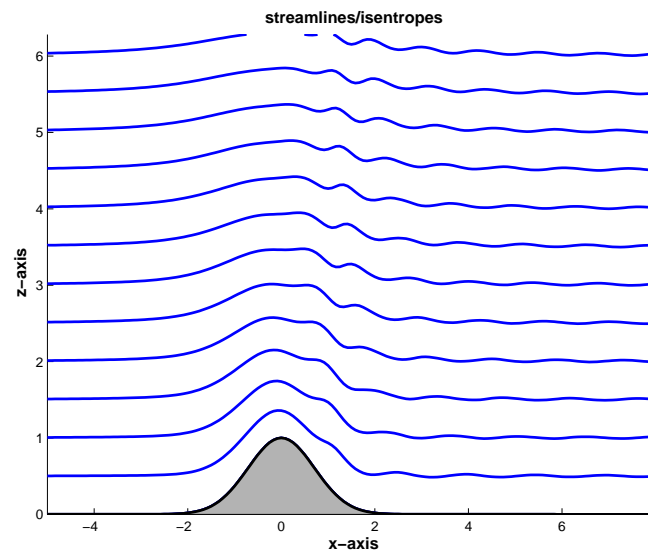
- ▷ primary issue: nonlinear coupling of large & wave-scales at small Rossby number

A New Equation for Topographic Waves

$$b_{xx} + b_{\theta\theta} + \mathcal{R}^2 \left\{ u^3 b_{\theta\theta} \right\}_{xx} = 0 \quad ; \quad u = \frac{1}{1 - \mathcal{R}\mathcal{A}b_{\theta}}$$

Rotating, Hydrostatic & Stratified Flow

- ▷ stratification allows isentropic “density” coordinates in vertical: θ (potential temperature)
- ▷ buoyancy anomaly from constant stratification: $b(x, \theta) = \theta / (\mathcal{R}\mathcal{A}) - z$
- ▷ surface condition: $b(x, 0) = -h(x)$ with decay/radiation BCs
- ▷ topographic height scale ($\mathcal{A} = NH/U$) & Rossby number ($\mathcal{R} = U/fL$)



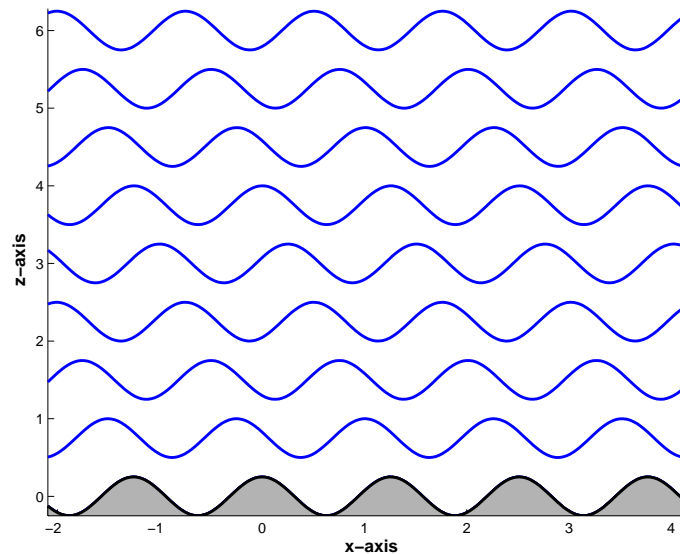
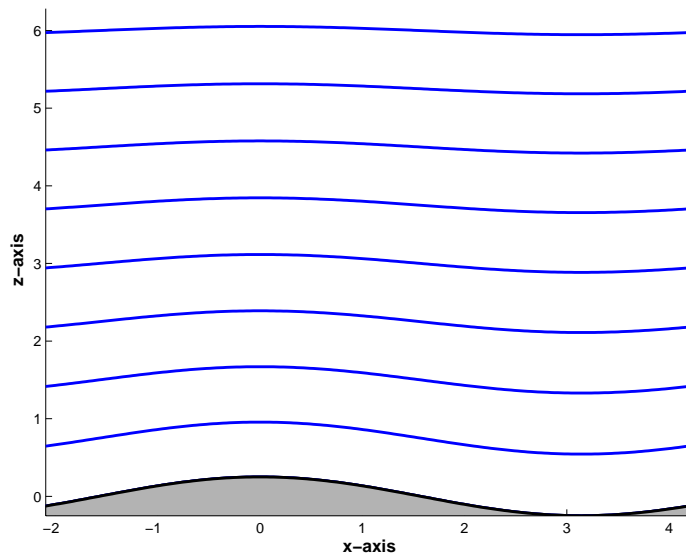
- ▷ a small Rossby ($\mathcal{R} = 0.25$) number mystery; nonlinear ($\mathcal{A} = 1.0$) waves

Waves at Small Scales

$$b_{xx} + b_{\theta\theta} + \mathcal{R}^2 \left\{ u^3 b_{\theta\theta} \right\}_{xx} = 0 \quad ; \quad u = \frac{1}{1 - \mathcal{R}A b_{\theta}}$$

Linear Theory ($u \equiv 1$), Queney 1947

- ▷ small topographic amplitude: $A \rightarrow 0$
- ▷ dispersion relation: $m^2 = -k^2 / (1 - \mathcal{R}^2 k^2)$
 - transition at inertial wavenumber: $k_i = 1/\mathcal{R}$
 - long waves: upward decay & symmetric uplifting of streamlines
 - short waves: waves & upstream phase tilt

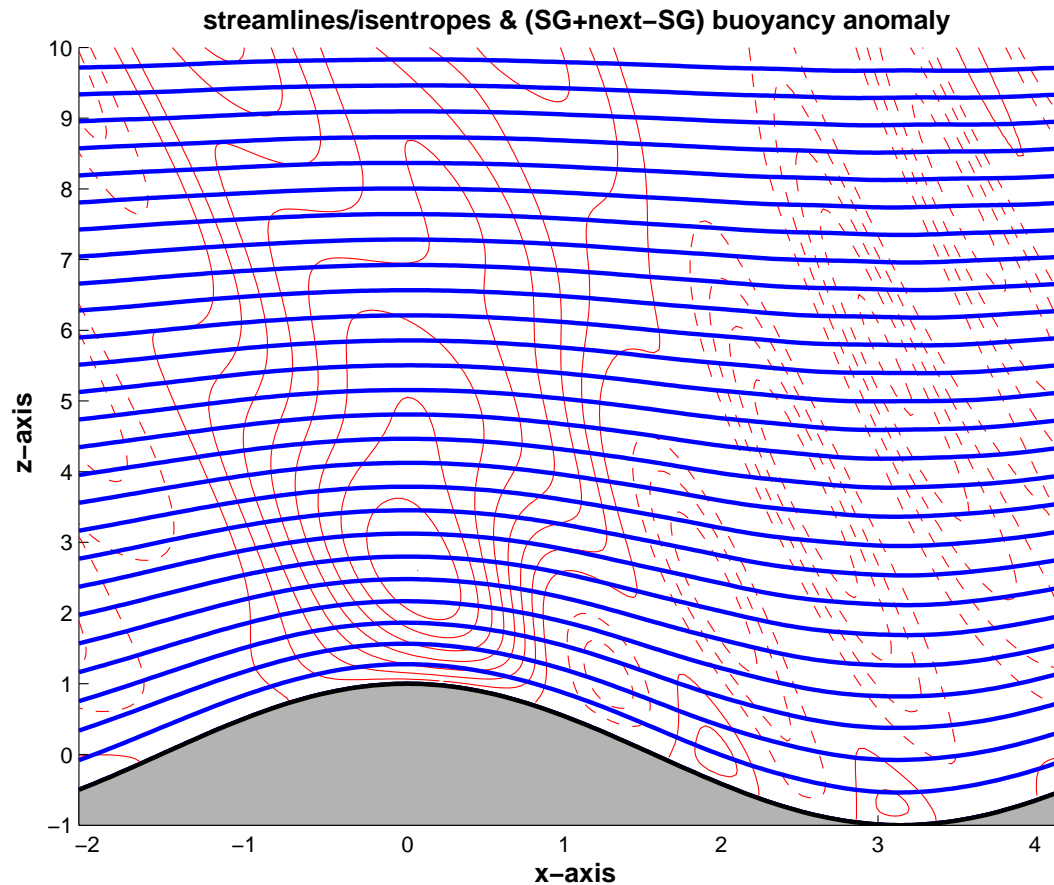


- ▷ quasigeostrophy: $\mathcal{R} \rightarrow 0$ limit banishes inertial wavenumber, $k_i = 1/\mathcal{R} \rightarrow \infty$

Nonlinear Wave Generation

First Iteration ($u = u_{linear}$)

- ▷ sinusoidal topography with $\mathcal{R} = 0.25$; $\mathcal{A} = 1.00$
- ▷ first nonlinear correction well-approximates converged solution



- ▷ establish exponentially-small wave amplitude ($e^{-1/\mathcal{R}}$) via weak nonlinear cascade

“Cascade” for a Nonlinear Eigenfunction

Linear Eigenvalue Problem

- ▷ linear boundary value problem

$$y'' + \lambda^2 y = 0 \quad ; \quad y(0) = y(\pi) = 0$$

- ▷ sinusoidal eigenfunctions, $y(x)$, for integers n

$$y(x) = A \sin nx \quad ; \quad \lambda^2 = n^2$$

Weakly Nonlinear Eigenvalue Problem

- ▷ nonlinear boundary value problem

$$y'' + \lambda^2 y = \epsilon y^3 \quad ; \quad y(0) = y(\pi) = 0$$

- ▷ exact solutions via elliptic functions

- ▷ perturbed eigenfunctions, $y(x)$, for $\epsilon A^2 \ll 1$

$$y(x) \sim A \sin nx + \frac{\epsilon A^2}{32n^2} \left(1 - \frac{3\epsilon A^2}{16n^2} \right) A \sin 3nx + \frac{3\epsilon^2 A^4}{512n^4} A \sin 5nx + \dots$$

$$\lambda^2/n^2 \sim 1 + \frac{3\epsilon A^2}{4n^2} - \frac{9\epsilon^2 A^4}{64n^4} + \dots$$

- ▷ nonlinear eigenfunction displays a cascade to full Fourier series for $\epsilon A^2 \neq 0$

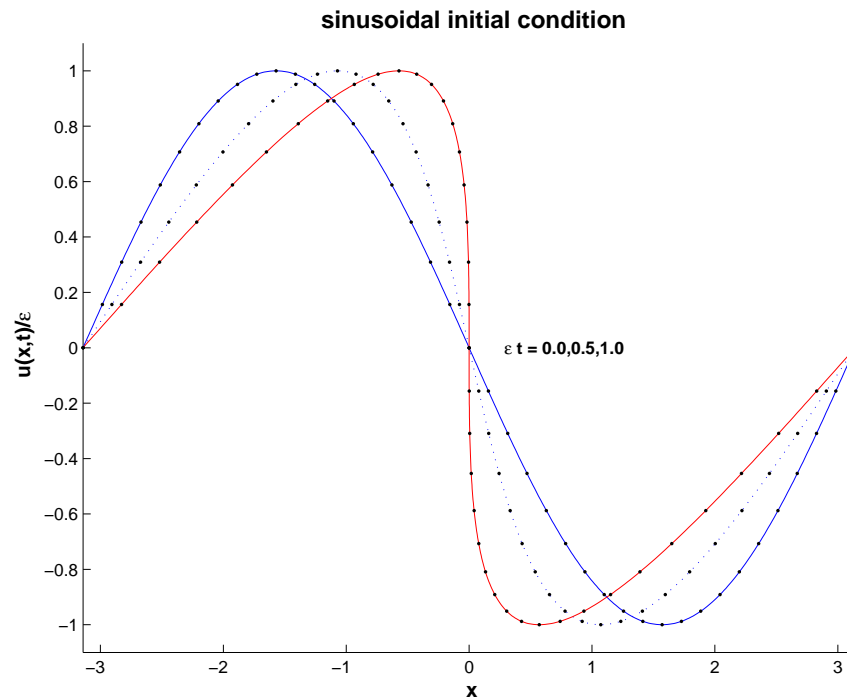
Simple Kinematic Wave Equation

Textbook Nonlinear PDE (Inviscid Burgers Equation)

- ▷ PDE of hyperbolic type, initial value problem for $u(x, t)$

$$u_t + u u_x = 0 \quad ; \quad u(x, 0) = f(x)$$

- ▷ *exact* solution by method of characteristics
- ▷ example of wave steepening & finite-time wavebreaking
- ▷ propagation of discontinuities determined by Rankine-Hugoniot conditions



- ▷ also embodies a simple one-dimensional cascade

An Exact Solution

Characteristic ODEs

- ▷ define characteristics as parametric curves $(x(s), t(s))$ originating from $(x_0, 0)$

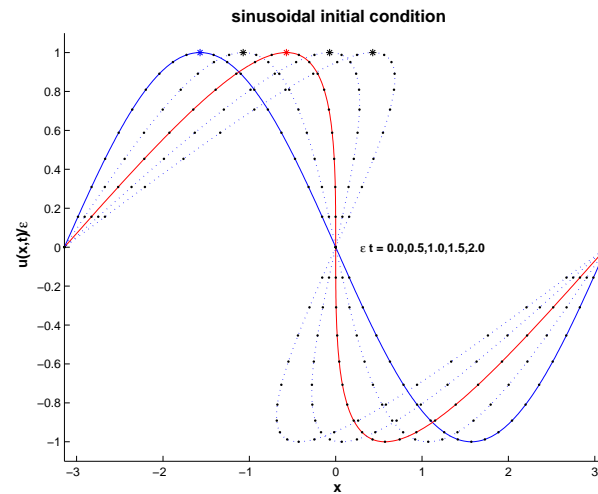
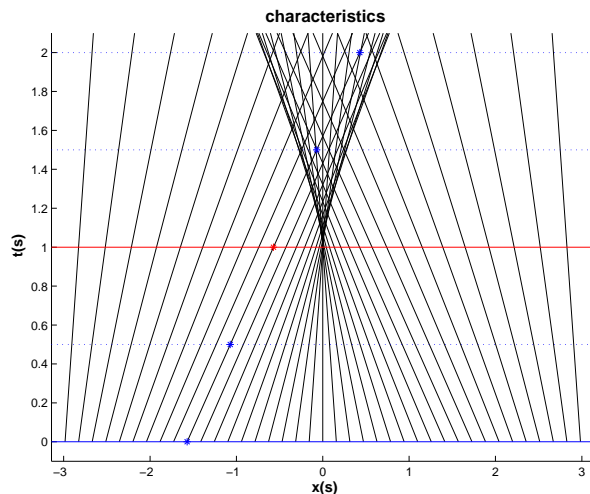
$$\begin{aligned} \frac{dx}{ds} = u & \quad ; \quad x(0) = x_0 & \rightarrow & \quad x = x_0 + s f(x_0) \\ \frac{dt}{ds} = 1 & \quad ; \quad t(0) = 0 & \rightarrow & \quad t = s \end{aligned}$$

- ▷ PDE becomes ODE for $u(s)$ along each characteristic

$$\frac{du}{ds} = 0 \quad ; \quad u(0) = f(x_0) \quad \rightarrow \quad u = f(x_0)$$

- ▷ parametric solution in terms of x_0 and t

$$u = f(x_0) \quad ; \quad x = x_0 + t f(x_0)$$



Series Representation

$$u(x, t) = a_0 + \sum_1^{\infty} \{a_n(t) \cos nx + b_n(t) \sin nx\}$$

Fourier Coefficients

- ▷ assume $u(x, t)$ continuous, prior to wavebreaking
- ▷ period integral of $u(x, t)$ is conserved \rightarrow time-independent mean

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x_0) dx_0$$

- ▷ sine coefficient & integration by parts ($n \neq 0$)

$$b_n(t) = \frac{1}{\pi} \int_{-\pi}^{+\pi} u(x, t) \sin nx dx = \frac{1}{\pi n} \int_{-\pi}^{+\pi} u_x(x, t) \cos nx dx$$

- ▷ u_x via implicit differentiation of $x = x_0 + t u$

$$b_n(t) = \frac{1}{\pi n t} \int_{-\pi}^{+\pi} \left(1 - \frac{dx_0}{dx}\right) \cos nx dx = -\frac{1}{\pi n t} \int_{-\pi}^{+\pi} \cos nx dx_0$$

- ▷ introduce IVs via parametric solution $x = x_0 + t f(x_0)$

$$b_n(t) = -\frac{1}{\pi n t} \int_{-\pi}^{+\pi} \cos [nx_0 + nt f(x_0)] dx_0$$

$$a_n(t) = \frac{1}{\pi n t} \int_{-\pi}^{+\pi} \sin [nx_0 + nt f(x_0)] dx_0$$

Platzman's 1964 Solution

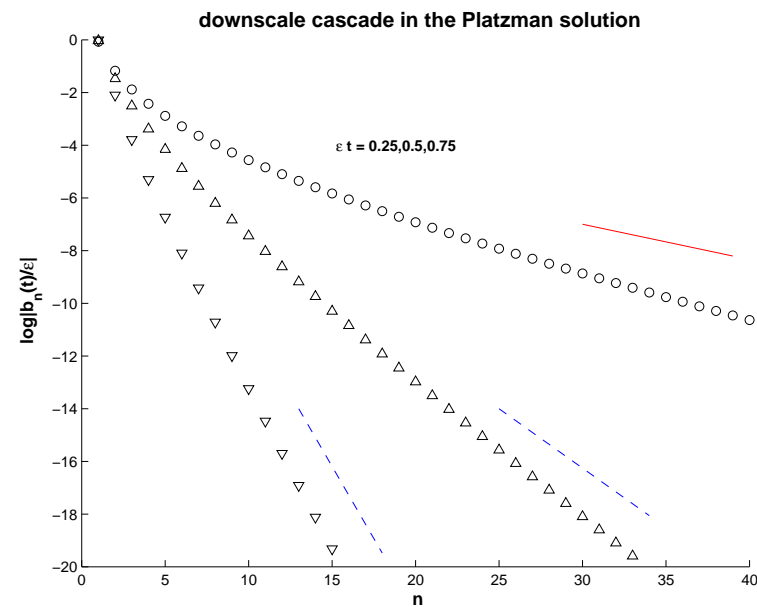
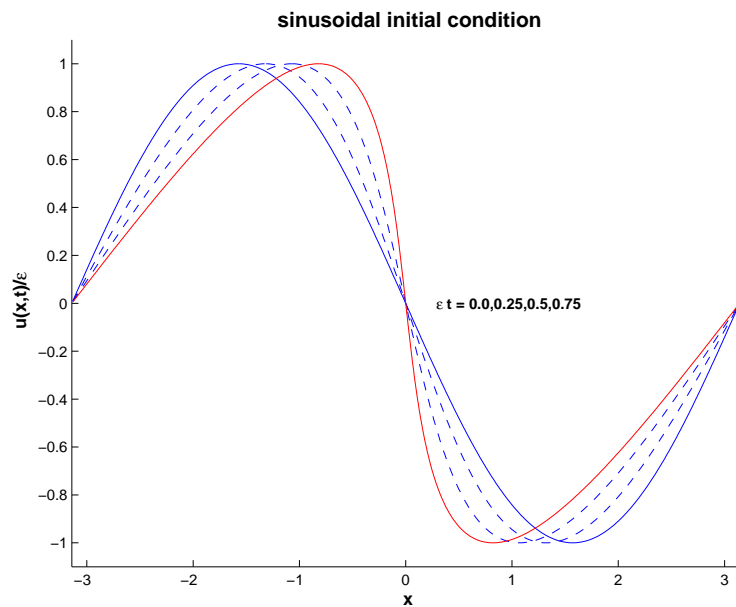
Sinusoidal Initial Condition: $f(x) = -\epsilon \sin x$

- ▷ solution remains a sine series for $t > 0$: $a_n(t) \equiv 0$
- ▷ sine coefficient is integral representation for $J_n(\cdot)$ Bessel function

$$b_n(t) = -\frac{1}{\pi n t} \int_{-\pi}^{+\pi} \cos [n x_0 - n \epsilon t \sin x_0] dx_0 = -2 \frac{J_n(n \epsilon t)}{n t}$$

- ▷ exact series solution with time-dependent fourier coefficients → cascade

$$u(x, t) = \sum_1^{\infty} -2 \frac{J_n(n \epsilon t)}{n t} \sin n x$$



Spectral Slope

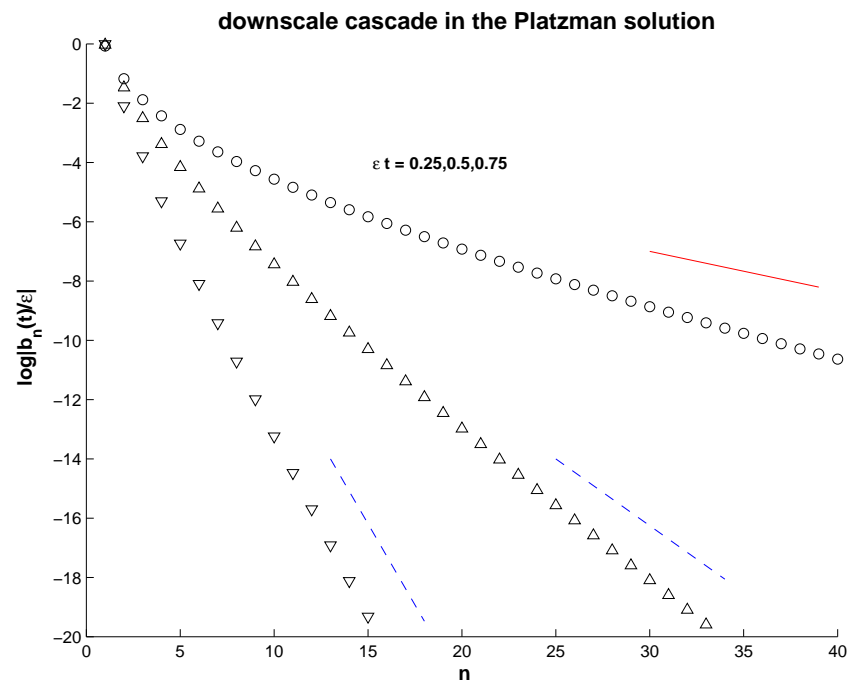
Large Wavenumber Asymptotics

- ▷ Bessel function asymptotics for large index & argument ($n \rightarrow \infty, \epsilon t < 1$)

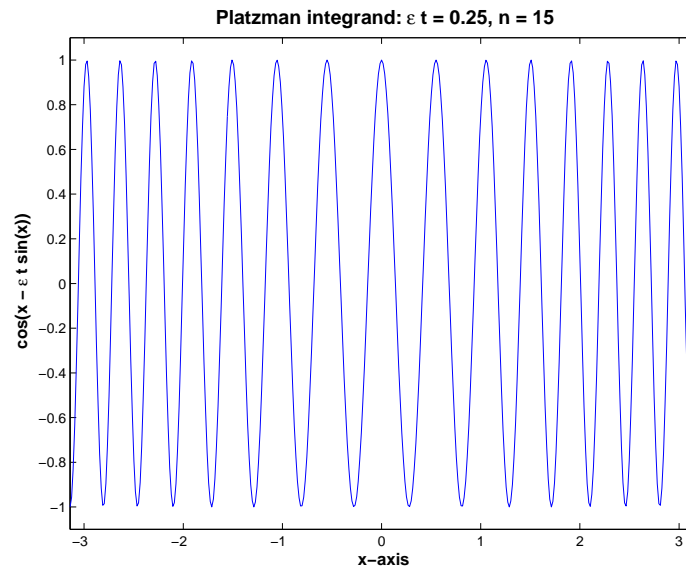
$$|b_n(t)| \sim \sqrt{\frac{2}{\pi \tanh \alpha}} t^{-1} n^{-3/2} e^{n(-\alpha + \tanh \alpha)} \quad ; \quad \cosh \alpha = \frac{1}{\epsilon t}$$

- ▷ spectral slope, $n \rightarrow \infty$

$$\frac{\ln |b_n(t)|}{n} \sim \ln \left(\frac{\epsilon t}{2} \right) + \sqrt{1 - \epsilon^2 t^2} - \ln \left(\frac{1 + \sqrt{1 - \epsilon^2 t^2}}{2} \right)$$



Integral Asymptotics



Highly-Oscillatory Integrand

- ▷ can we obtain spectral slope without Platzman's series solution?
- ▷ sine coefficient in complex exponential form

$$b_n(t) = -\frac{1}{\pi n t} \operatorname{Re} \left\{ \int_{-\pi}^{+\pi} e^{in(x_0 - \epsilon t \sin x_0)} dx_0 \right\}$$

- ▷ large n integral asymptotics
 - no stationary-phase point for $\epsilon t < 1$
 - periodicity neutralizes integration by parts
 - complex analysis & steepest descent methods

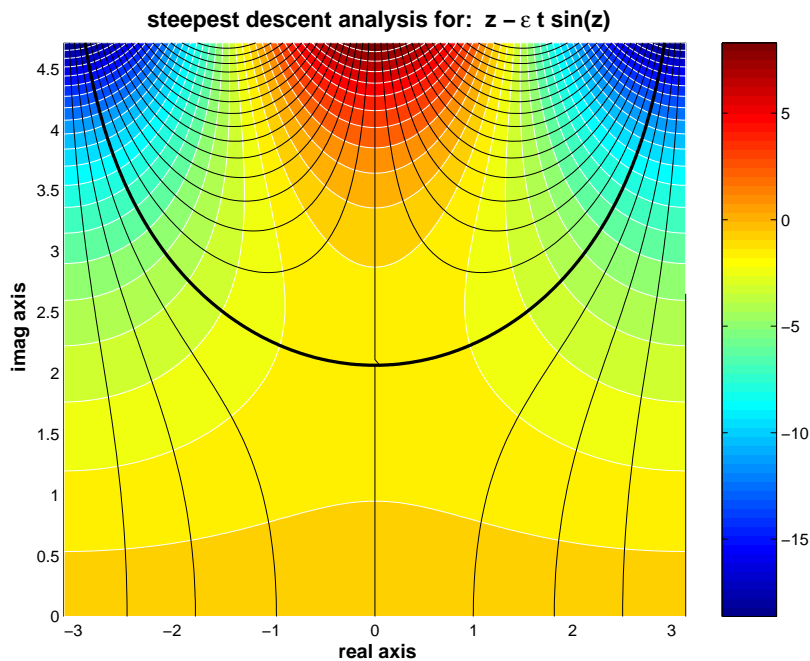
Complex Analysis

Path Deformation & Complex Phase

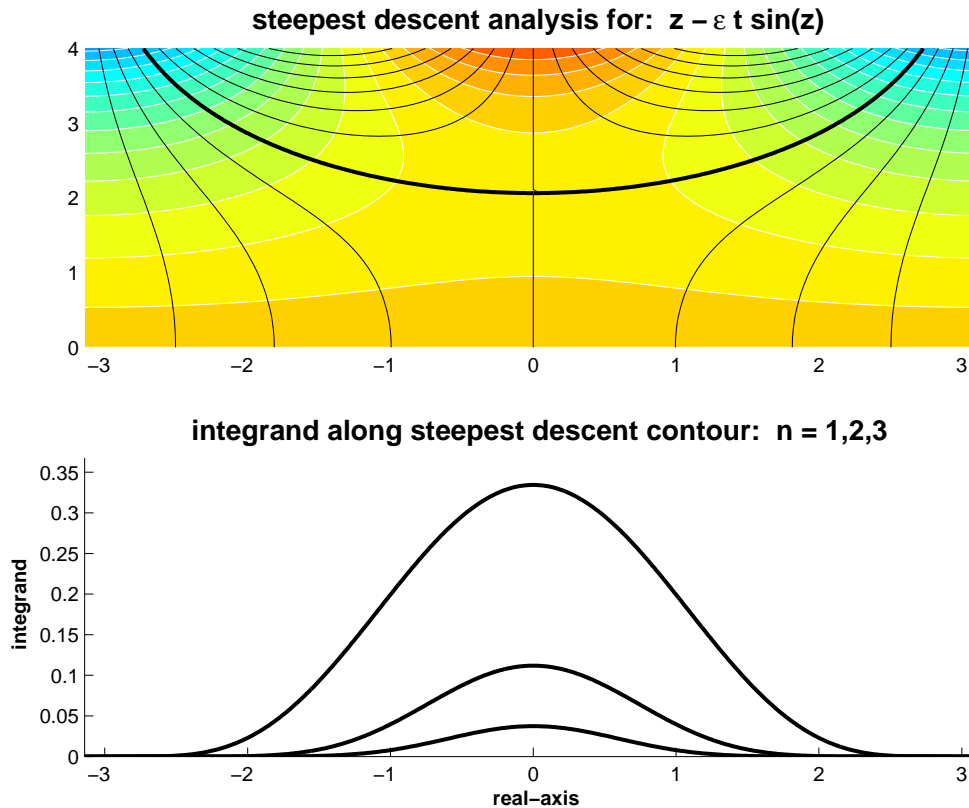
- ▷ consider integral in complex z -plane by analytic continuation

$$b_n(t) = -\frac{1}{\pi n t} \operatorname{Re} \left\{ \int_{\mathcal{C}} e^{in(z - \epsilon t \sin z)} dz \right\}$$

- ▷ integrand is 2π -periodic & has no singularities
- ▷ complex analysis of phase function, $\phi(z) = i(z - \epsilon t \sin z)$, for $\epsilon t = 0.25$
 - blue indicates regions of negative $\operatorname{Re}(\phi)$: exponentially small integrand
 - black contours are curves of constant $\operatorname{Im}(\phi)$: paths of non-oscillation



Method of Steepest Descent



Saddlepoint Contribution

- ▷ maximum of integrand occurs at $z = i\alpha$ where $\cosh \alpha = \frac{1}{\epsilon t}$
- ▷ for $n \rightarrow \infty$, integrand localizes near saddlepoint like a gaussian

$$b_n(t) \sim -\frac{1}{\pi n t} e^{n(-\alpha + \tanh \alpha)} \int_{-\infty}^{+\infty} e^{-(n \tanh \alpha)x^2/2} dx$$

An Odd Quadrature Solution

When is an Integral not an Integral?

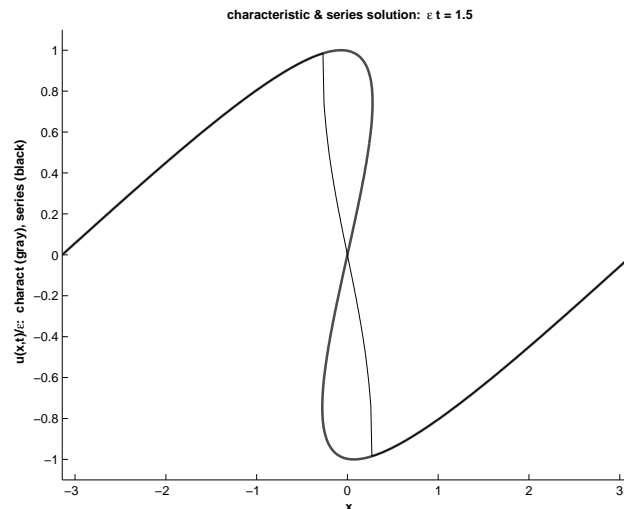
- ▷ substitute integral coefficients into series, interchange summation & integration

$$\begin{aligned}
 u(x, t) &= a_0 + \sum_{n=1}^{\infty} \frac{1}{\pi n t} \int_{-\pi}^{+\pi} \sin n[x - x_0 - t f(x_0)] dx_0 \\
 &= a_0 + \frac{1}{t} \int_{-\pi}^{+\pi} \left\{ \sum_{n=1}^{\infty} \frac{\sin n[x - x_0 - t f(x_0)]}{\pi n} \right\} dx_0
 \end{aligned}$$

- ▷ series can be summed

$$u(x, t) = a_0 + \frac{1}{t} \int_{-\pi}^{+\pi} \left\{ \left(\frac{x - x_0 - t f(x_0)}{2\pi} \bmod 1 \right) - \frac{1}{2} \right\} dx_0$$

- ▷ paradox: characteristic solution is local, solution cannot depend globally on initial function



Spectral Dynamics

$$u(x, t) = \sum_1^{\infty} b_n(t) \sin nx$$

Direct Substitution into $u_t + uu_x = 0$

- ▷ identify all terms which produce $\sin nx$

$$\begin{aligned} \dots b'_n \sin nx + \dots + \sum_1^{n-1} k b_k b_{n-k} \cos kx \sin(n-k)x + \dots \\ \dots + \sum_1^{\infty} k b_k b_{n+k} \cos kx \sin(n+k)x + \dots \\ \dots + \sum_1^{\infty} (n+k) b_{n+k} b_k \cos(n+k)x \sin kx + \dots = 0 \end{aligned}$$

- ▷ spectral dynamics ODEs: triad resonances

$$b'_n = -\frac{n}{4} \sum_1^{n-1} b_k b_{n-k} + \frac{n}{2} \sum_1^{n-1} b_k b_{n+k} + \frac{n}{2} \sum_n^{\infty} b_k b_{n+k}$$

→ 1st-sum: downscale transfer from **smaller wavenumber, long waves**

→ 2nd-sum: mixing transfer from straddling wavenumbers

→ 3rd-sum: upscale transfer from **larger wavenumber, short waves**

Spectral ODEs

$$b'_n = -\frac{n}{4} \sum_1^{n-1} b_k b_{n-k} + \frac{n}{2} \sum_1^{n-1} b_k b_{n+k} + \frac{n}{2} \sum_n^{\infty} b_k b_{n+k}$$

Solution Strategies

- ▷ Platzman solution is exact for initial conditions, $\{b_n(0)\} = \{-\epsilon, 0, 0, \dots\}$

$$\text{slope} = \ln\left(\frac{\epsilon t}{2}\right) + \sqrt{1 - \epsilon^2 t^2} - \ln\left(\frac{1 + \sqrt{1 - \epsilon^2 t^2}}{2}\right)$$

- ▷ downscale transfer only solution, asymptotically valid for $0 \leq \epsilon t \ll 1$

$$b_n(t) \sim -\epsilon \frac{n^{n-1}}{n!} \left(\frac{\epsilon t}{2}\right)^{n-1} \rightarrow \text{slope} \approx \ln\left(\frac{\epsilon t}{2}\right) + 1$$

→ first Taylor term of Platzman solution for small ϵt & Stirling approximation

- ▷ small ϵt perturbation series

$$\left. \begin{aligned} b_1(t) &\sim -\epsilon && + \epsilon \left(\frac{\epsilon t}{2}\right)^2 && + \dots \\ b_2(t) &\sim -\epsilon \left(\frac{\epsilon t}{2}\right) && + \dots \\ b_3(t) &\sim -\epsilon \frac{3}{2} \left(\frac{\epsilon t}{2}\right)^2 && + \dots \end{aligned} \right\} \rightarrow \text{slope} \approx \ln\left(\frac{\epsilon t}{2}\right)$$

Cascade Solutions

- ▷ is there a general approach for constructing approximate solutions that embody spectral cascade?

Perturbation Expansion for $f(x) = O(\epsilon)$

- ▷ PDE for disturbance about initial condition: $u(x, t) = f(x) + \hat{u}(x, t)$

$$\hat{u}_t = -ff_x - (f\hat{u})_x - \hat{u}\hat{u}_x \quad ; \quad \hat{u}(x, 0) = 0$$

- ▷ simple iterative construction $\hat{u} \sim \hat{u}_2(x, t) + \hat{u}_3(x, t) + \hat{u}_4(x, t) + \dots$

$$(\hat{u}_2)_t = -ff_x$$

$$(\hat{u}_3)_t = - (f\hat{u}_2)_x$$

$$(\hat{u}_4)_t = - (f\hat{u}_3)_x - (\hat{u}_2)(\hat{u}_2)_x$$

→ generates polynomial-in- ϵt solutions; \hat{u}_2 -error is $O(\epsilon^3)$

→ solution up to \hat{u}_k contains wavenumbers up to k — partial cascade only

- ▷ 1st newton iterate

$$(\hat{u}_2)_t + (f\hat{u}_2)_x = -ff_x$$

→ non-constant coefficient PDE; \hat{u}_2 -error is $O(\epsilon^4)$

→ solution by characteristics

1st Newton Iterate

Characteristic ODEs

- ▷ define characteristics as parametric curves $(x(s), t(s))$ originating from $(x_0, 0)$

$$\begin{aligned}\frac{dx}{ds} &= f(x) & ; & & x(0) &= x_0 \\ \frac{dt}{ds} &= 1 & ; & & t(0) &= 0\end{aligned}$$

- ▷ f times PDE becomes ODE for $f\hat{u}_2$ along each characteristic

$$\frac{d}{ds}(f\hat{u}_2) = -\frac{1}{2}f(f^2)_x = -\frac{1}{2}\frac{d}{ds}(f^2) \quad ; \quad u(0) = 0$$

- ▷ can integrate to obtain \hat{u}_2

$$\hat{u}_2 = \frac{1}{2} \frac{f^2(x_0) - f^2(x)}{f(x)}$$

→ but relation determining $x_0(x, t)$ requires solution to nonlinear ODE

Platzman to the Rescue (again)

- ▷ exact characteristics for $f(x) = -\epsilon \sin x$

$$\frac{dx}{dt} = -\epsilon \sin x \quad ; \quad x(0) = x_0 \quad \rightarrow \quad \sin x_0 = \frac{\operatorname{sech} \epsilon t}{1 - \tanh \epsilon t \cos x} \sin x$$

Approximate Cascade Solution

Two Small ϵt Asymptotic Miracles

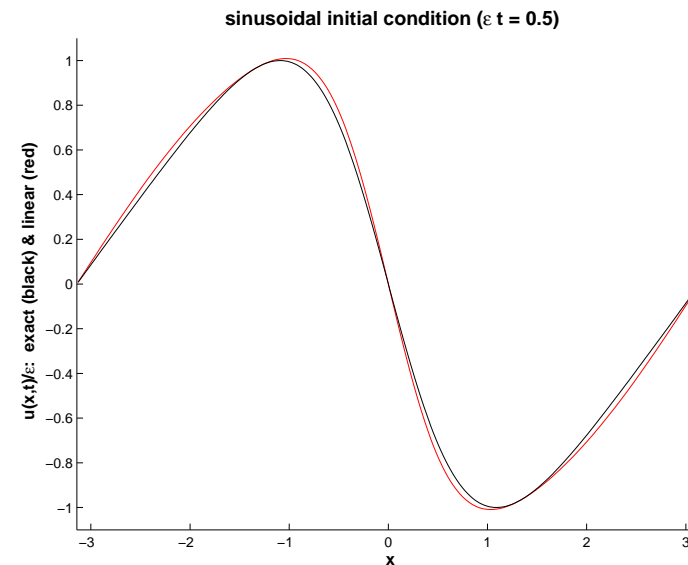
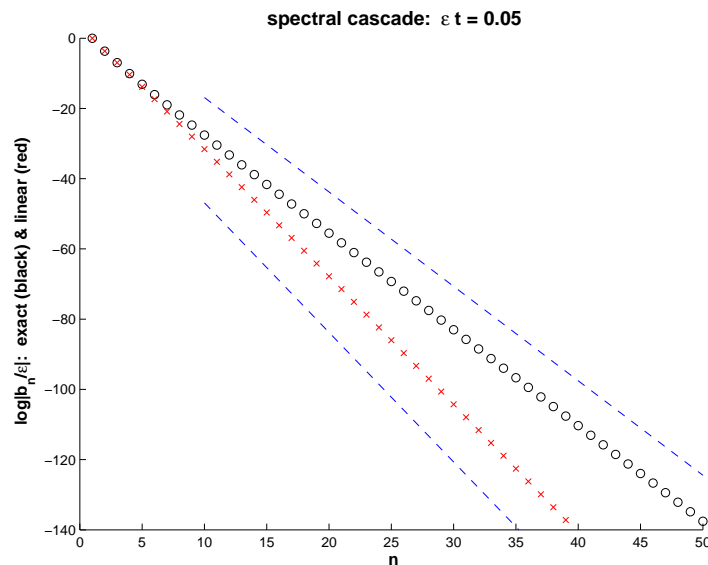
- ▷ formally $O(\epsilon^3)$ -accurate solution from linearizing truncation

$$u(x, t) \sim -\epsilon \sin x + \frac{\epsilon}{2} \left(1 - \frac{\operatorname{sech}^2 \epsilon t}{(1 - \tanh \epsilon t \cos x)^2} \right) \sin x$$

- ▷ need Fourier series representation

$$u(x, t) \sim -\frac{\epsilon}{2} \left\{ \sin x + \operatorname{sech}^2 \left(\frac{\epsilon t}{2} \right) \sum_1^{\infty} n \tanh^{n-1} \left(\frac{\epsilon t}{2} \right) \sin nx \right\}$$

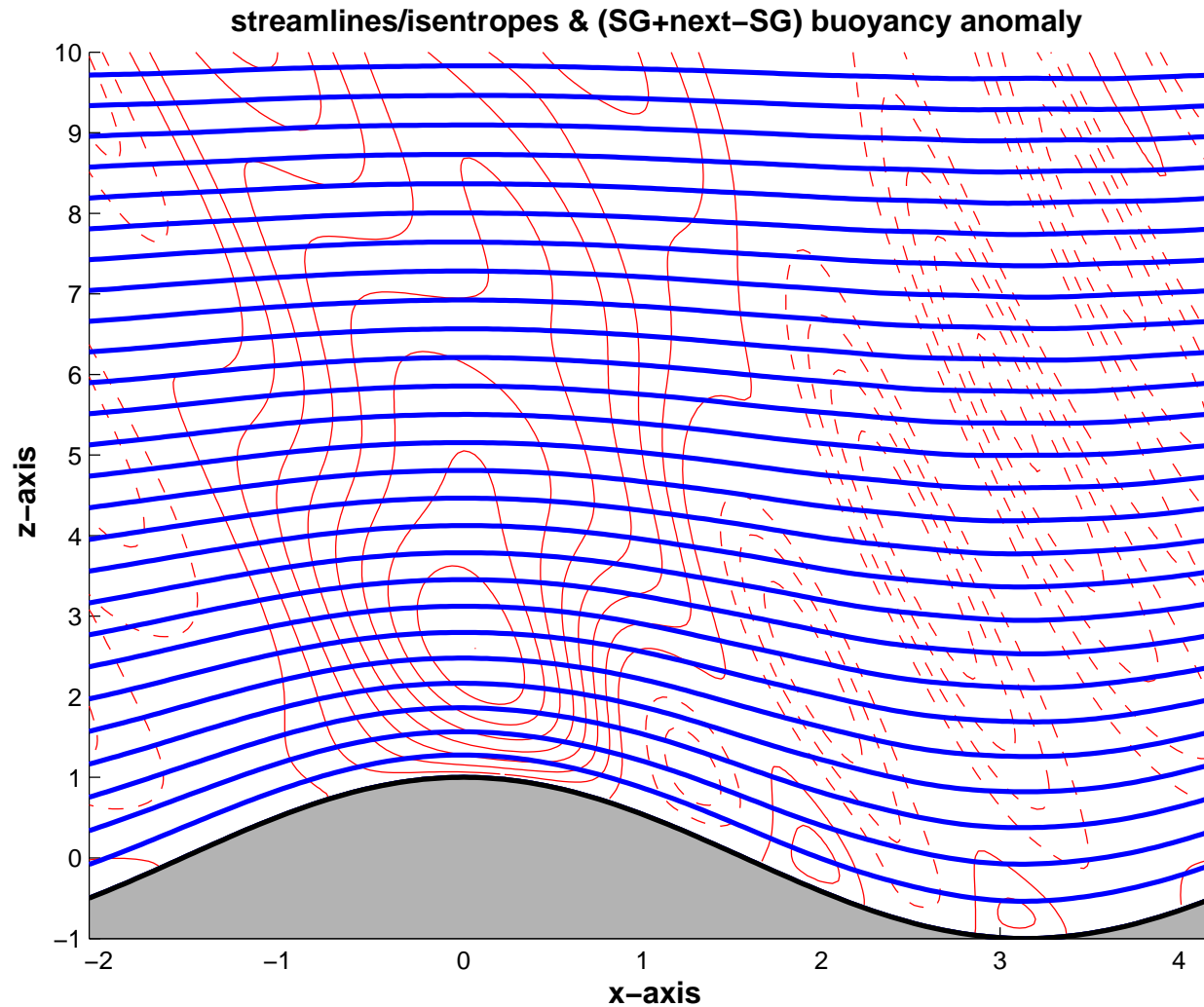
→ contains a cascade with spectral slope = $\ln \left(\tanh \frac{\epsilon t}{2} \right) \sim \ln \left(\frac{\epsilon t}{2} \right)$ for $0 < \epsilon t \ll 1$



2D Flow Over Sinusoidal Topography

Small Rossby Number Limit ($\mathcal{R} = 1/4$) & Large Amplitude Topography

- ▷ no visible waves in streamlines, but $n = 4$ short waves are present via weak nonlinear cascade!



Quantifying the Weak Cascade

Simple Quantitative Illustration of a Spectral Cascade

- ▷ perturbative construction of a solution containing approximate cascade
- ▷ use Platzman solution for kinematic wave equation as benchmark
- ▷ 1st newton iterate uses non-constant coefficient in PDE to generate cascade
 - robust methodology may be adapted to more difficult problems
 - formal accuracy is not improved
 - yet, full spectral content with leading behaviour of spectral slope
- ▷ likely new general results on an old problem: $u_t + uu_x = 0$
 - fourier series solution for continuous evolutions
 - exact characteristic solution of linearizing truncation about initial conditions

Emerging Applications to Atmospheric Fluid Dynamics

- ▷ generation of short-scale inertia-gravity waves by large-scale flows