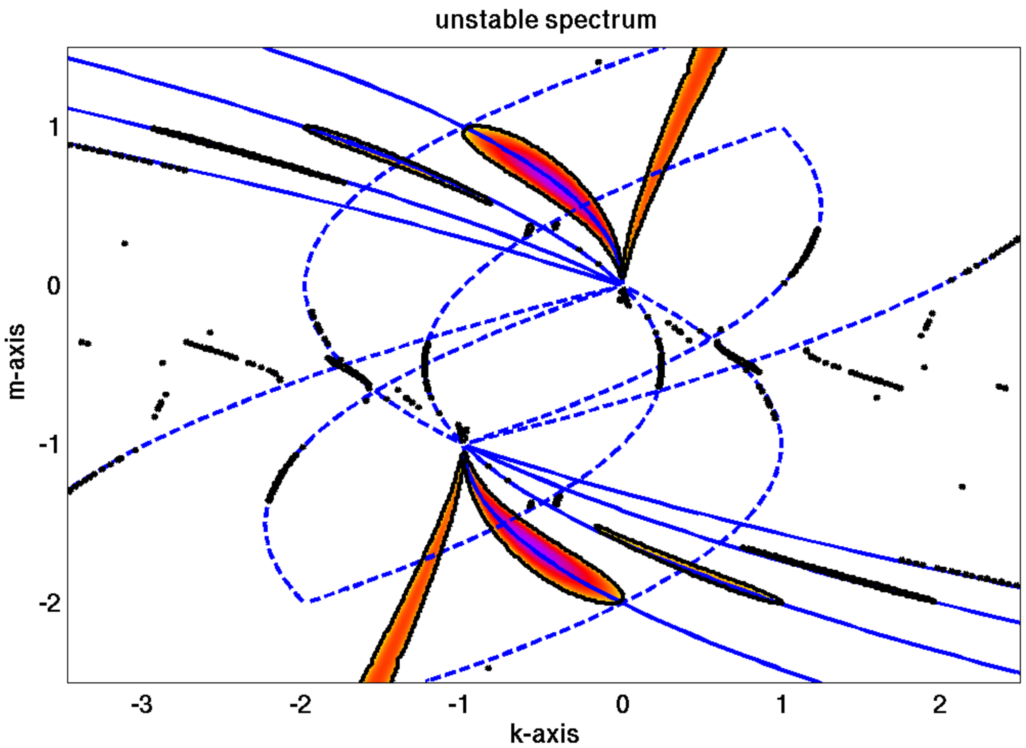
Floquet Instability & Triad Resonance in a Stratified Flow

- linear instabilities of a single-mode gravity wave
- analysis from Floquet & resonant wave perspectives



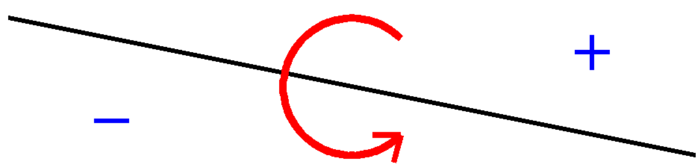
- Dave Muraki & Youngsuk Lee, Simon Fraser University

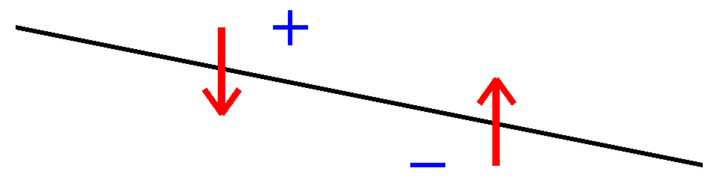
Equations for a Stratified Fluid

Vorticity/Buoyancy Dynamics

- ▷ 2D Euler fluid with Boussinesq buoyancy & constant stratification (stable) $\longrightarrow$ oscillations

$$\frac{D\eta}{Dt} = b_x \qquad\qquad \frac{Db}{Dt} = -w$$





Streamfunction Formulation

- ▷ $\psi(x,z)$, incompressible streamfunction: $u = \psi_z \;\; ; \;\; w = -\psi_x \;\; ; \;\; \eta = -\nabla^2\psi$
- ▷ uniform wind & hydrostatic scaling: $\eta \longrightarrow -\psi_{zz}$

$$\begin{aligned} \psi_{zzt} &+ \psi_{zzx} + b_x + J(\psi_{zz},\psi) = 0 \\ b_t &+ b_x - \psi_x + J(b,\psi) = 0 \end{aligned}$$

- ▷ nonlinearity via 2D streamfunction advection: Jacobian determinant

$$J(f,\psi) = \begin{vmatrix} f_x & \psi_x \\ f_z & \psi_z \end{vmatrix} = \begin{vmatrix} f_x & -w \\ f_z & u \end{vmatrix} = u f_x + w f_z$$

An Exact Wave Solution

$$\begin{array}{llllll} \psi_{zzt} & +\,\psi_{zzx} & +\,b_x & +\;J(\psi_{zz},\psi) & = & 0 \\ b_t & +\,b_x & -\,\psi_x & +\;J(b,\psi) & = & 0 \end{array}$$

Fourier Modes: $e^{i(kx+mz-\omega t)}$

- linear dispersion relation (slow/fast) for buoyancy-gravity waves

$$\omega(k,m) = k \mp \frac{k}{|m|} \qquad ; \qquad \vec{c}_g(k,m) = \left(1 \mp \frac{1}{|m|} \;\;,\;\; \pm \frac{km}{|m|^3}\right)$$

- steady wave: $k = m = 1$, $\omega = 0$ (slow wave with upward group velocity)

$$\begin{pmatrix} \psi \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} 2\epsilon\ \sin(x+z)$$

  - Jacobians are zero $\Rightarrow$ exact nonlinear solution!

Goal: to characterize the linear stability of this simple nonlinear wave

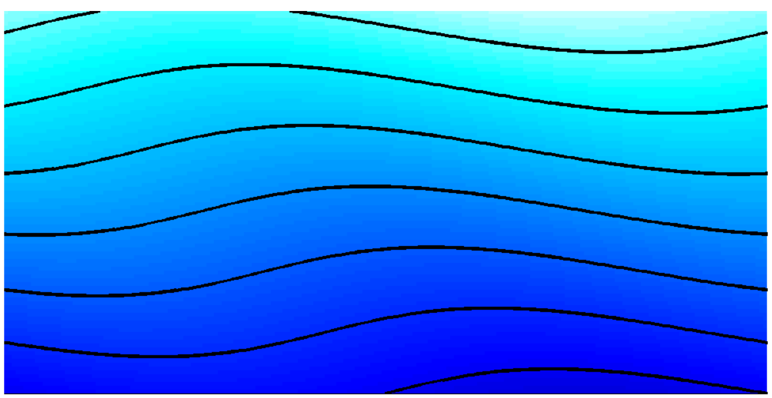
- to understand context for mountain flow instability (physical/mathematical mechanism?)
- instability: Mied 1976, Drazin 1977, Klostermeyer 1982, Sonmor & Klaassen 1997

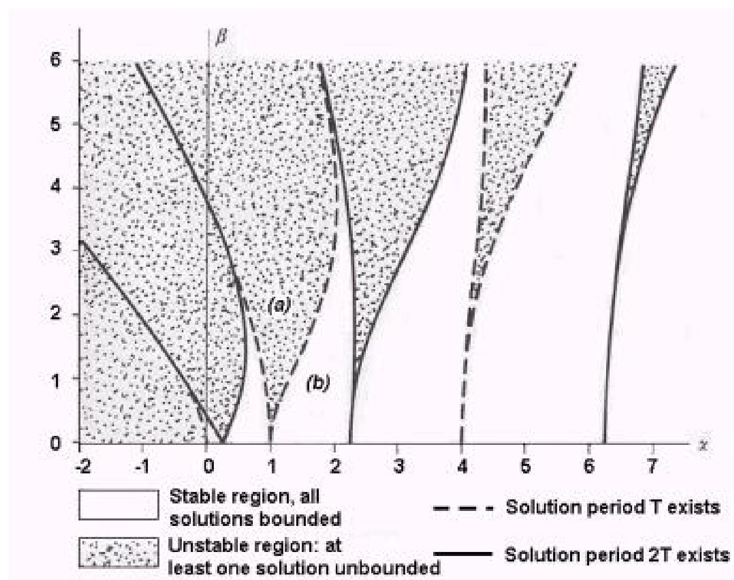
Linearized Stability

$$
\begin{array}{lllll}
\tilde{\psi}_{zzt} & + \tilde{\psi}_{zzx} & + \tilde{b}_x & + \ \epsilon J\left(\tilde{\psi}_{zz} + \tilde{\psi} \ , \ 2\sin(x+z)\right) & = \ 0 \\
\tilde{b}_t & + \tilde{b}_x & - \tilde{\psi}_x & + \ \epsilon J\left(\tilde{b} - \tilde{\psi} \ , \ 2\sin(x+z)\right) & = \ 0
\end{array}
$$

A Problem for Floquet Theory

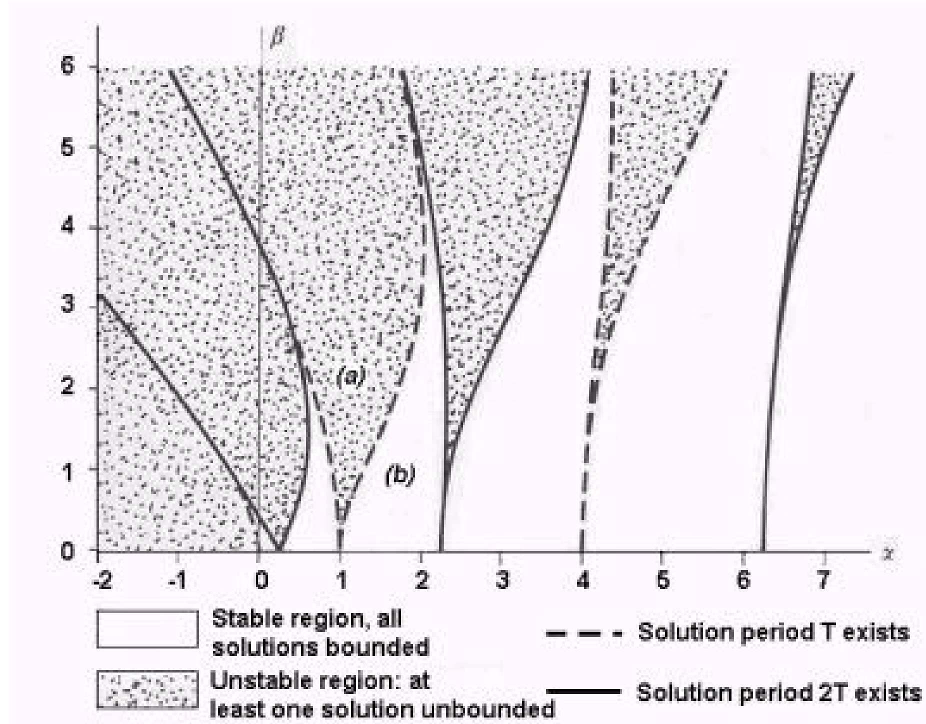
- linear PDE with periodic, non-constant coefficients
- instability via parametric resonance (as for the Mathieu ODE)





Mathieu Equation⋆

$$\psi_{tt} + (\alpha + 2\beta\ \sin t)\psi = 0$$



Floquet Stability

- ▷ analog of Fourier solution for DEs with periodic coefficients:

$$\psi(t) = e^{i\mu t} \left\{ \sum_{-\infty}^{+\infty} \vec{v}_n\ e^{int} \right\}$$

- ▷ Hill's method: μ as eigenvalue; Im $\mu > 0 \rightarrow$ instability
- ▷ instabilities organized into resonance *bands*

Floquet Approach

$$\begin{array}{llllll} \tilde\psi_{zzt} & +\,\tilde\psi_{zzx} & +\,\tilde b_x & -\;i\epsilon\, J\left(\tilde\psi_{zz}+\tilde\psi\;,\;e^{i(x+z)}-e^{-i(x+z)}\right) & = & 0 \\ \tilde b_t & +\,\tilde b_x & -\,\tilde\psi_x & -\;i\epsilon\, J\left(\tilde b-\tilde\psi\;,\;e^{i(x+z)}-e^{-i(x+z)}\right) & = & 0 \end{array}$$

Floquet, Fourier & Hill

- product of Floquet exponential & co-periodic Fourier series

$$\begin{pmatrix} \tilde\psi \\ \tilde b \end{pmatrix} = e^{i(kx+mz-\Omega t)}\left\{\sum_{-\infty}^{+\infty} \vec v_n\, e^{in(x+z)}\right\}$$

  - perturbation wavevector, $\vec K = (k,m)$ & Floquet eigenvalue, $\mathrm{Im}(\Omega) > 0 \Rightarrow$ instability

- Hill's infinite matrix

$$\begin{bmatrix} \ddots & \ddots & & \\ \ddots & \mathbf{S}_0 & \epsilon\mathbf{M}_1 & \\ & \epsilon\mathbf{M}_0 & \mathbf{S}_1 & \ddots \\ & & \ddots & \ddots \end{bmatrix} - \Omega \begin{bmatrix} \ddots & & & \\ & \mathbf{\Lambda}_0 & & \\ & & \mathbf{\Lambda}_1 & \\ & & & \ddots \end{bmatrix}$$

  - 2×2 real blocks: $\mathbf{S}_n(k,m)$, symmetric; $\mathbf{\Lambda}_n(m)$, diagonal; $\mathbf{M}_n(k,m)$

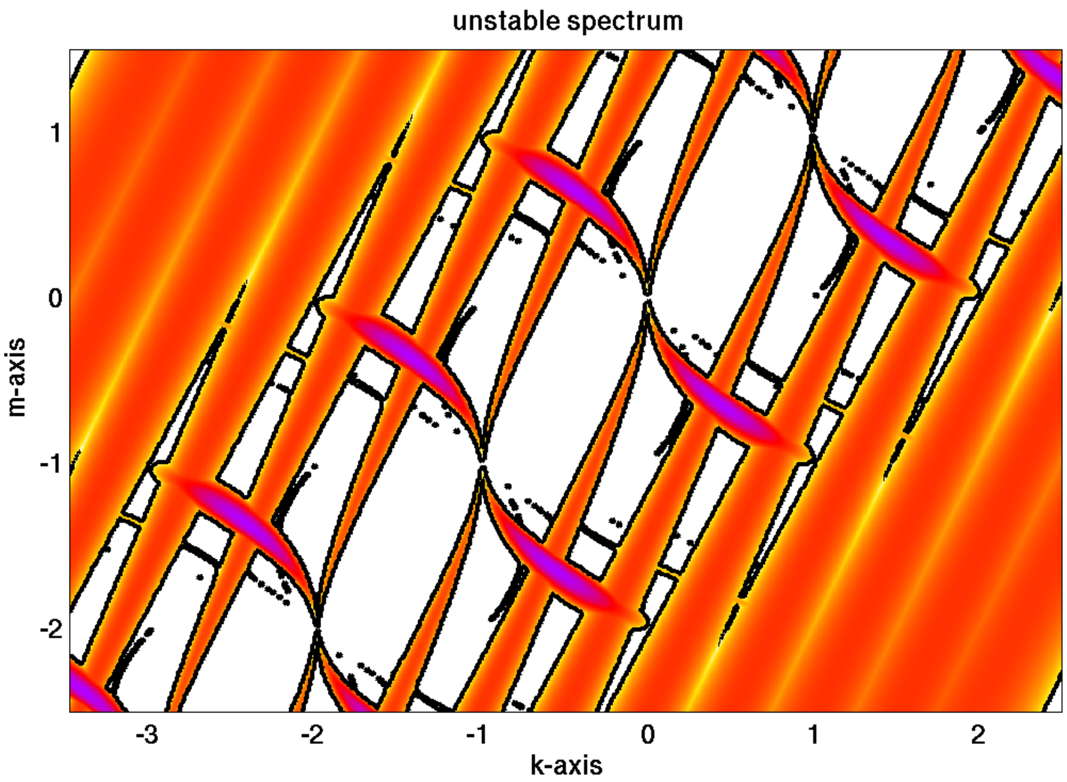
- truncate to $-N \le n \le N+1$ & compute eigenvalues: $\{\Omega(k,m)\}$

Unstable Floquet Spectrum

Maximum Growth Rate vs $\vec{K}$, $\epsilon = 0.1$

- natural periodicity due to non-uniqueness of series indexing

$$\begin{pmatrix} \tilde{\psi} \\ \tilde{b} \end{pmatrix} = e^{i((k+q)x+(m+q)z-\Omega t)} \left\{ \sum_{-\infty}^{+\infty} \vec{v}_{n+q} \, e^{in(x+z)} \right\}$$

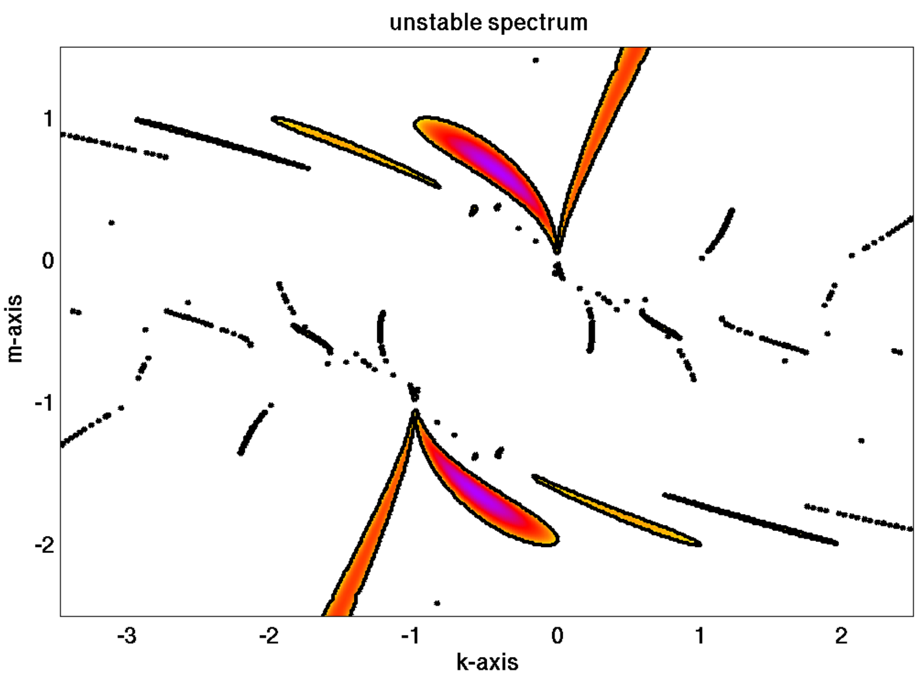


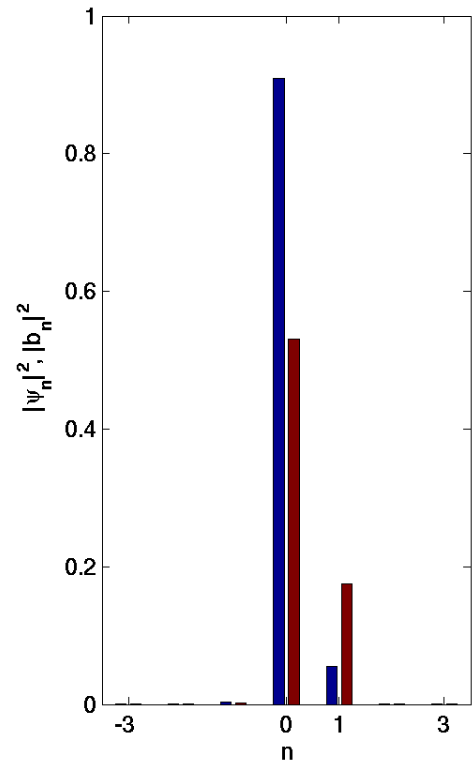
Floquet Spectrum Unwrapped

Maximum Growth Rate vs $\vec{K}$, $\epsilon = 0.1$

- *center-of-mass* criterion; preserves notion of central wavevector in (k, m)-space

$$0 \leq \frac{\sum_n n|\tilde{\psi}_n|^2}{\sum_n |\tilde{\psi}_n|^2} < 1$$





- where do the complex eigenvalues come from?

Eigenvalue Degeneracy

$$\begin{bmatrix} \ddots & \ddots & & \\ \ddots & \mathbf{S}_0 & \epsilon\mathbf{M}_1 & \\ & \epsilon\mathbf{M}_0 & \mathbf{S}_1 & \ddots \\ & & \ddots & \ddots \end{bmatrix} - \Omega \begin{bmatrix} \ddots & & & \\ & \mathbf{\Lambda}_0 & & \\ & & \mathbf{\Lambda}_1 & \\ & & & \ddots \end{bmatrix}$$

Instability of Small Amplitude Waves ($\epsilon \ll 1$)

- $\epsilon = 0$, linear dispersion relation $\Rightarrow$ real eigenvalues, $\Omega = \omega(k, m)$
- $\epsilon \neq 0$, characteristic polynomial is real
- for $0 < \epsilon \ll 1$, complex conjugate Ω's appear from multiple eigenvalues at $\epsilon = 0$

Double Root in a 2-Mode Truncation

- adjacent ($n = 0, 1$) Fourier modes $\Rightarrow$ $k_0 + 1 = k_1 \quad ; \quad m_0 + 1 = m_1$

$$\begin{pmatrix} \tilde{\psi} \\ \tilde{b} \end{pmatrix} = \vec{v}_0 \, e^{i(k_0 x + m_0 z - \Omega t)} + \vec{v}_1 \, e^{i(k_1 x + m_1 z - \Omega t)}$$

- at $\epsilon = 0$, if double root $\Longrightarrow$ $\omega_0 + 0 = \omega_1$ $\rightarrow$ triad resonance

Nonlinearity & Resonance*

$$\begin{array}{llllll} \tilde{\psi}_{zzt} & + \tilde{\psi}_{zzx} & + \tilde{b}_x & - i\epsilon J\left(\tilde{\psi}_{zz} + \tilde{\psi} \,,\ e^{i(x+z)} - e^{-i(x+z)}\right) & = & 0 \\ \tilde{b}_t & + \tilde{b}_x & - \tilde{\psi}_x & - i\epsilon J\left(\tilde{b} - \tilde{\psi} \,,\ e^{i(x+z)} - e^{-i(x+z)}\right) & = & 0 \end{array}$$

Multiplication of Fourier Modes

- $e^{i(kx+mz-\omega t)}$ Fourier mode dispersion relation $(\omega(k,m))$ from linear terms
- nonlinear interactions between $n = 0, 1$ modes:

$$e^{i(k_0 x+m_0 z-\omega_0 t)} \quad e^{+i(x+z)} = e^{i((k_0+1)x+(m_0+1)z-\omega_0 t)} \longrightarrow e^{i(k_1 x+m_1 z-\Omega t)}$$

$$e^{i(k_1 x+m_1 z-\omega_1 t)} \quad e^{-i(x+z)} = e^{i((k_1-1)x+(m_1-1)z-\omega_1 t)} \longrightarrow e^{i(k_0 x+m_0 z-\Omega t)}$$

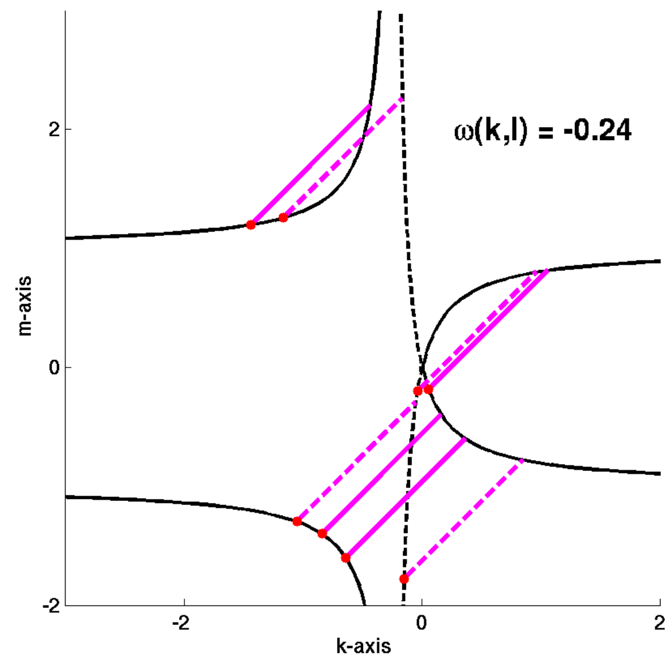
- resonance: $k_0 + 1 = k_1$, $m_0 + 1 = m_1$, $\omega_0 + 0 = \omega_1 = \Omega$

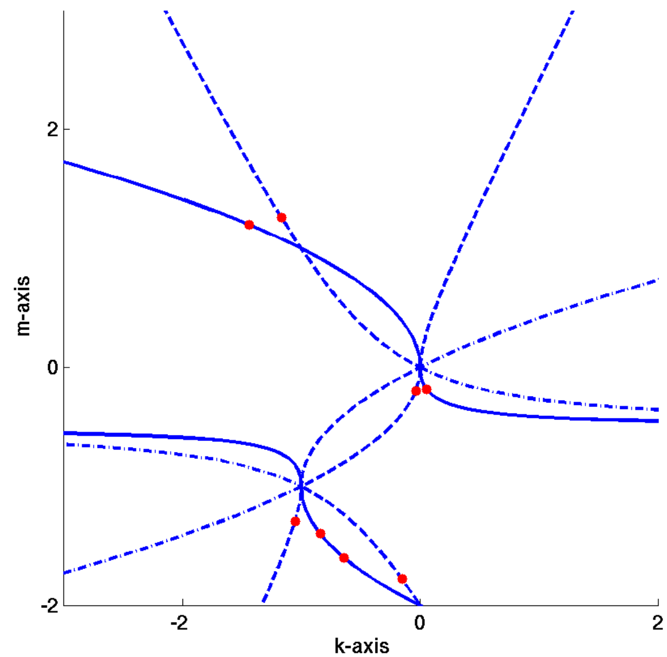
- extra resonant terms "affect" the linear dispersion calculation! (weakly nonlinear analysis)

$$\vec{k}_0 + \vec{k}_s = \vec{k}_1 \quad ; \quad \omega(\vec{k}_0) + \omega(\vec{k}_s) = \omega(\vec{k}_1)$$

Resonant Trace

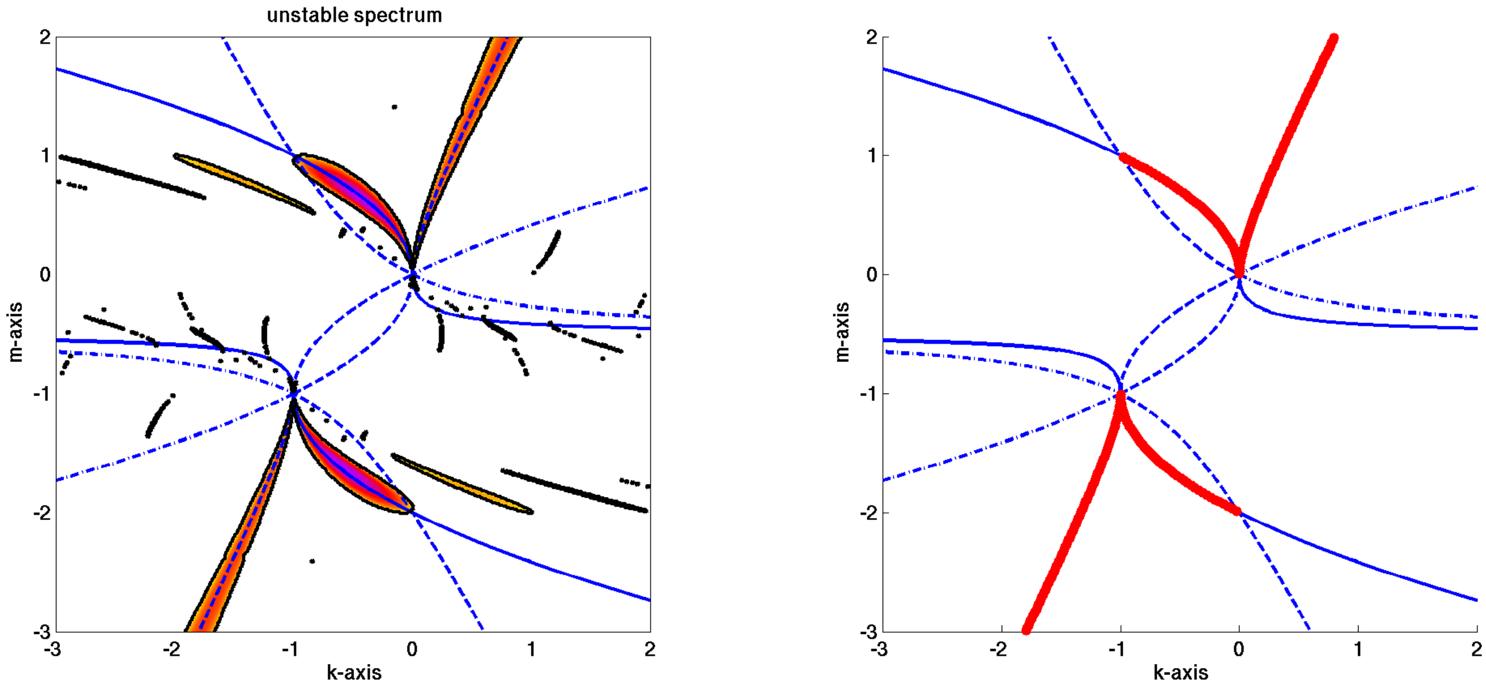
- ▷ resonances identified as $\vec{k}_s$-connections between ω_0 & ω_1 dispersion curves





- ▷ curves of all $\vec{k}_0$ generating a triad (double root) $\longrightarrow$ resonant trace

Triad Instability



Weakly Nonlinear Analysis ($\epsilon \ll 1$)

- double root only a necessary condition for small ϵ appearance of complex eigenvalues
- bifurcation analysis via eigenvalue perturbation: $\Omega(\vec{k}_0; \epsilon) = \omega_0 + \epsilon\, \Omega_1$

Mountain Wave Instability

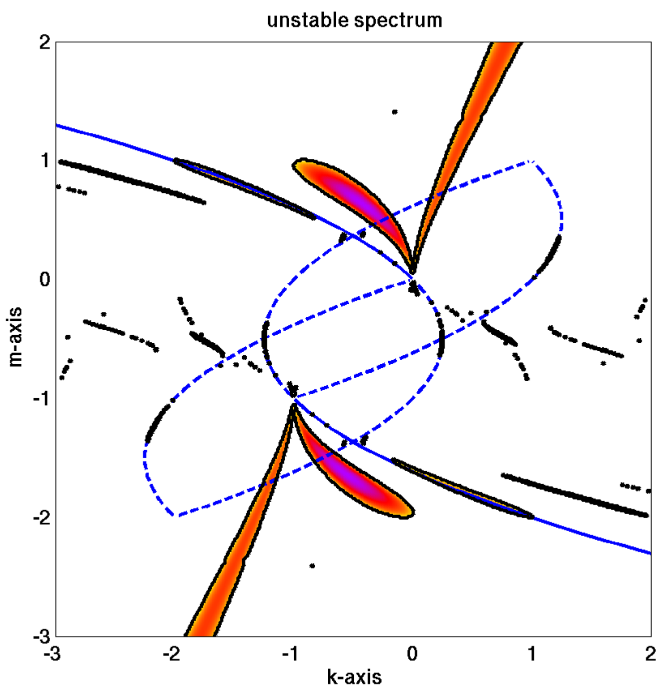
- only slow-slow resonance has counter-propagating group velocity

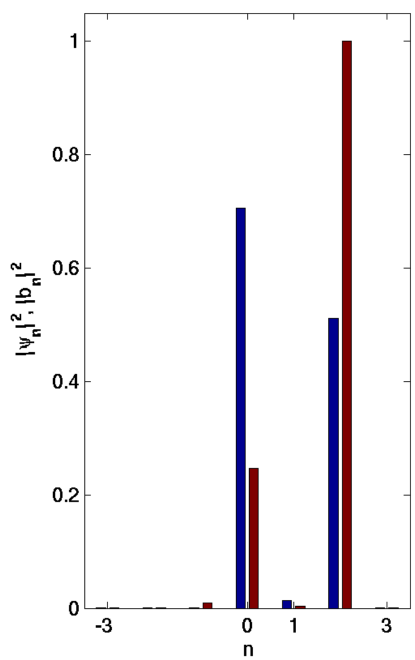
“Quartet” Instability

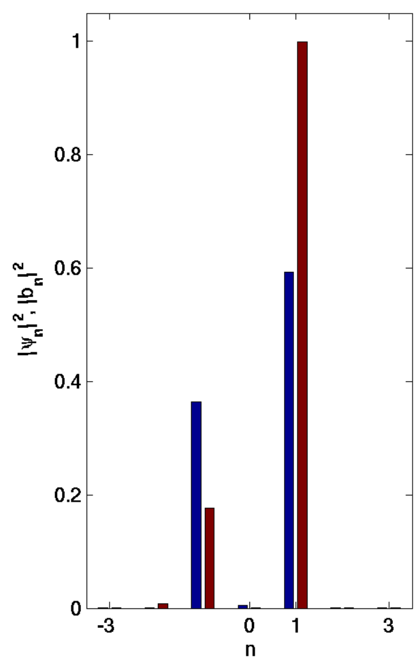
$$\vec{k}_0 + 2\,\vec{k}_s = \vec{k}_2 \quad ; \quad \omega(\vec{k}_0) + 2\,\omega(\vec{k}_s) = \omega(\vec{k}_2)$$

Next-to-Adjacent $(n = 0, 2)$ Fourier Modes

- analogous to the 2nd Mathieu instability: $\Omega(\vec{k}_0; \epsilon) = \omega_0 + \epsilon^2\,\Omega_2$
- $n = 1$ mode plays a crucial role as a “catalyst” (since nearest-neighbor coupling only)







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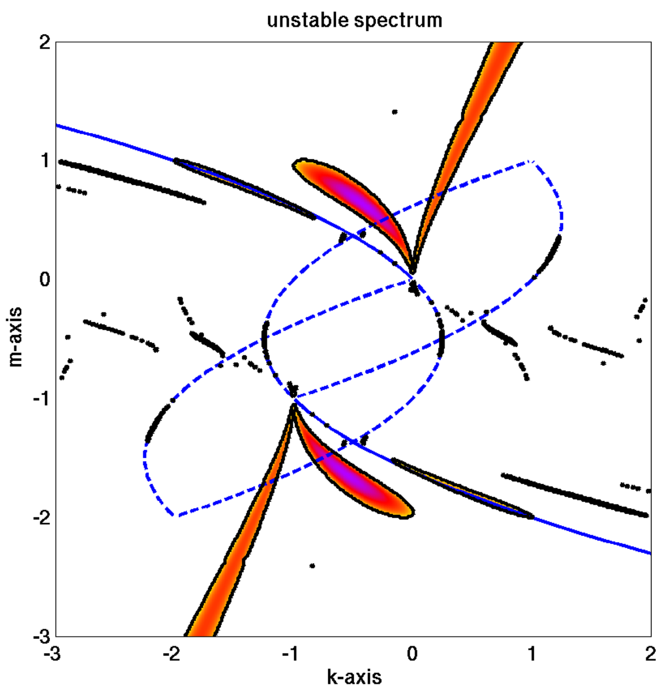
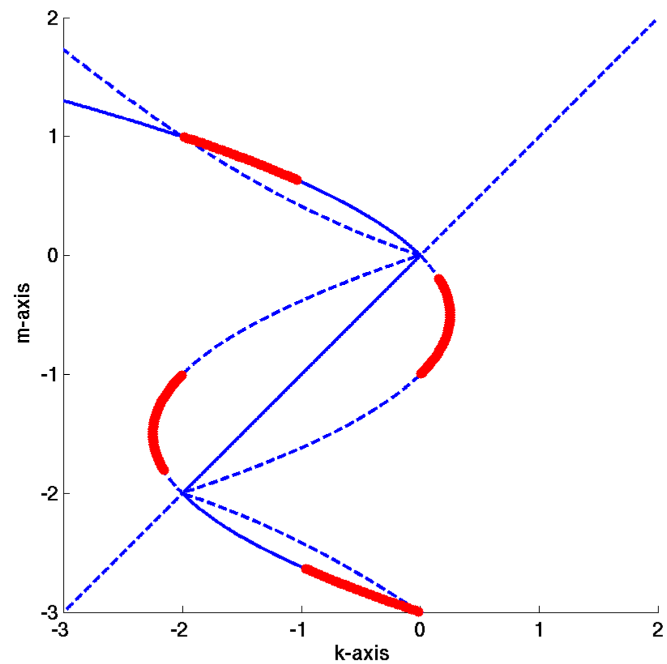


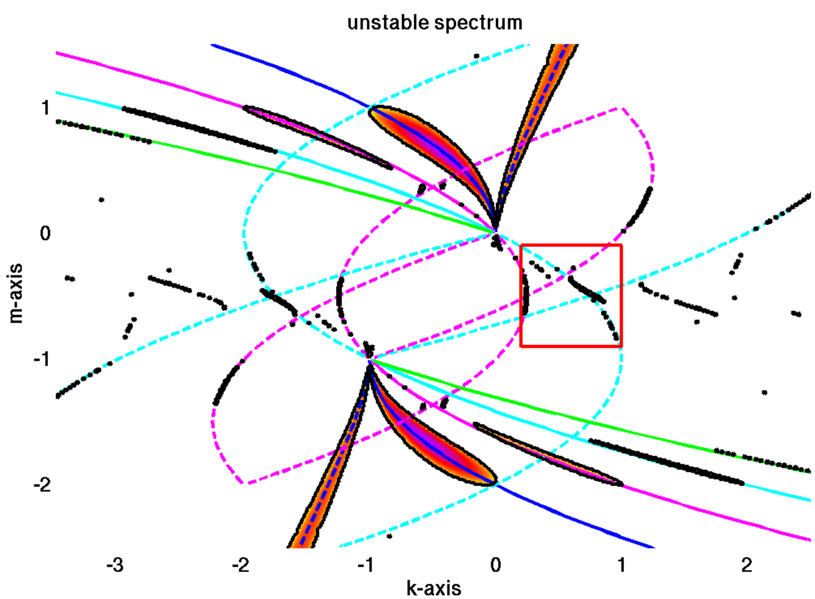


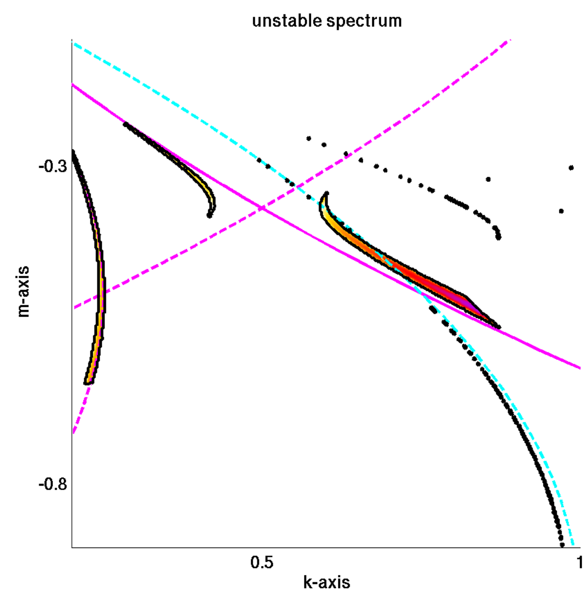
Floquet Theory & Resonant Waves

$$\vec{k}_0 + j\,\vec{k}_s = \vec{k}_j \quad ; \quad \omega(\vec{k}_0) + j\,\omega(\vec{k}_s) = \omega(\vec{k}_j)$$

2D Map of Instabilities

- Floquet theory: Fourier series $\longrightarrow$ linear eigenvalue problem
- resonant waves: Fourier resonances $\longrightarrow$ eigenvalue degeneracies
  - are all instabilities born out of degeneracy?





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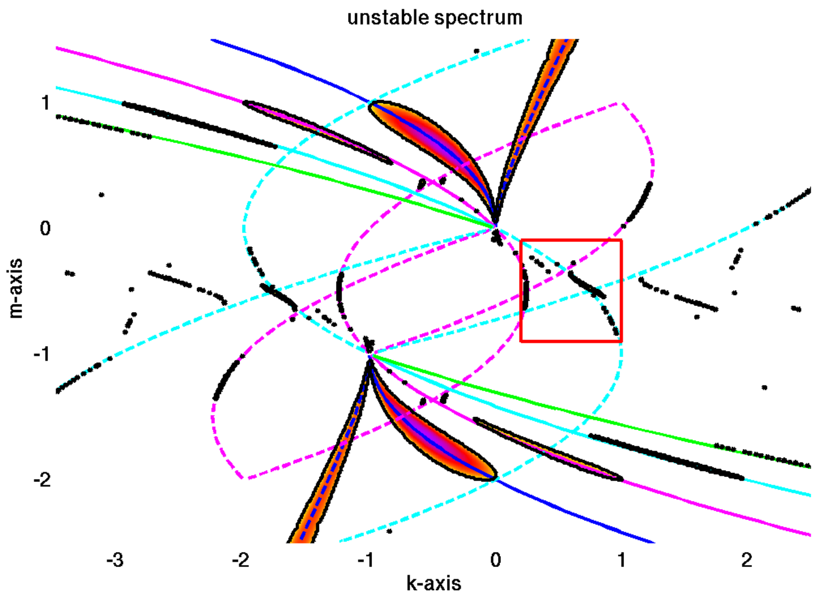
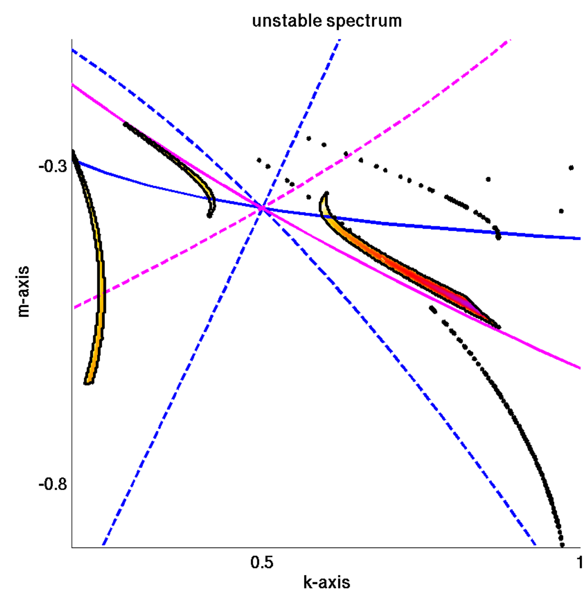
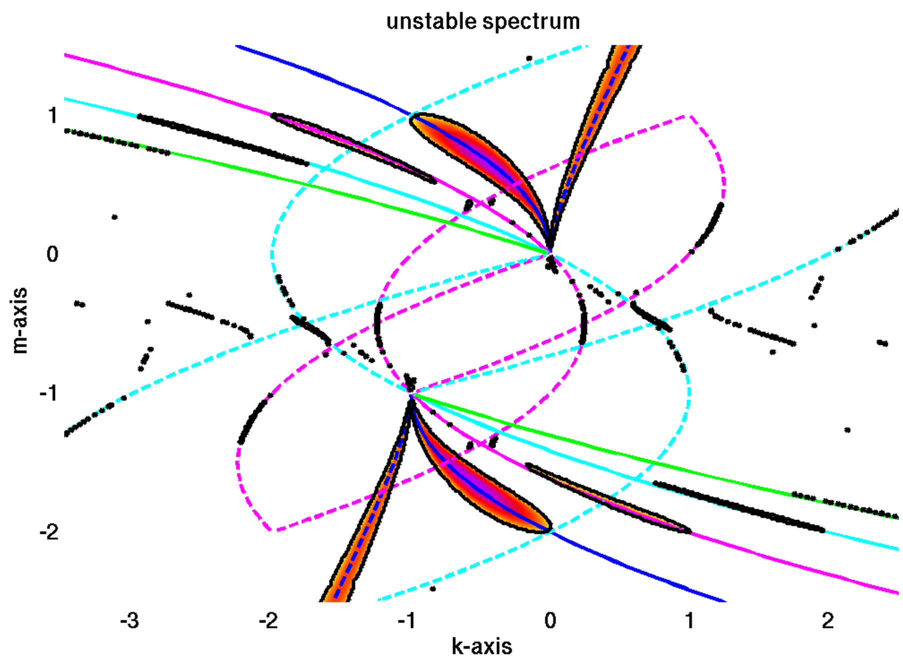




Linear Stability of a Plane Gravity Wave

- clear characterization of Floquet instabilities by wave resonances
  - neutral curve, multiple-wave stability & nonhydrostatic flow
- application of weak turbulence ideas to linear stability
  - implications for atmospheric wave turbulence?

$\epsilon = 0.1$



In Closing

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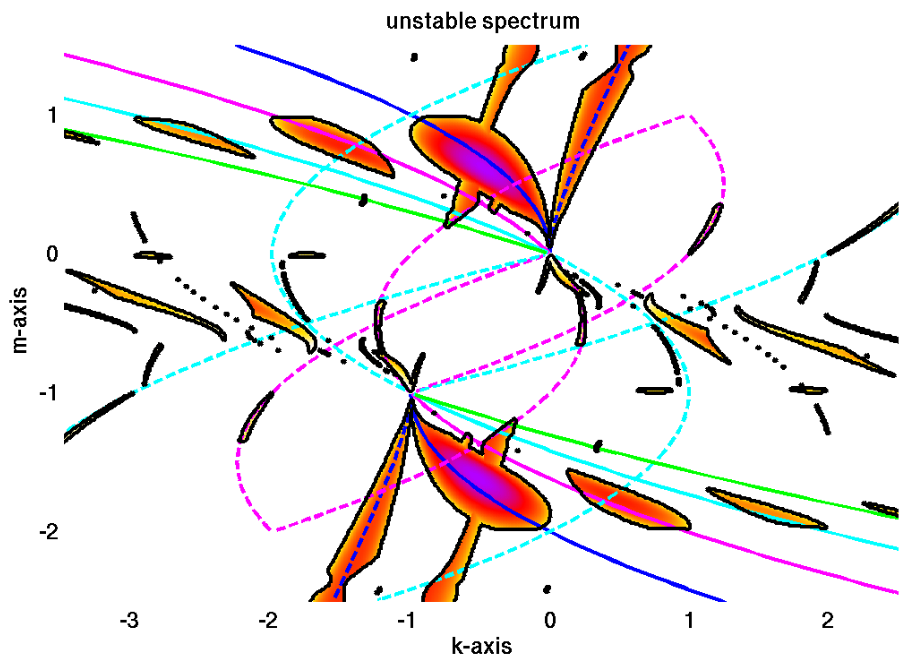
$\epsilon = 0.2$



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