Math 821, Spring 2013, Lecture 14

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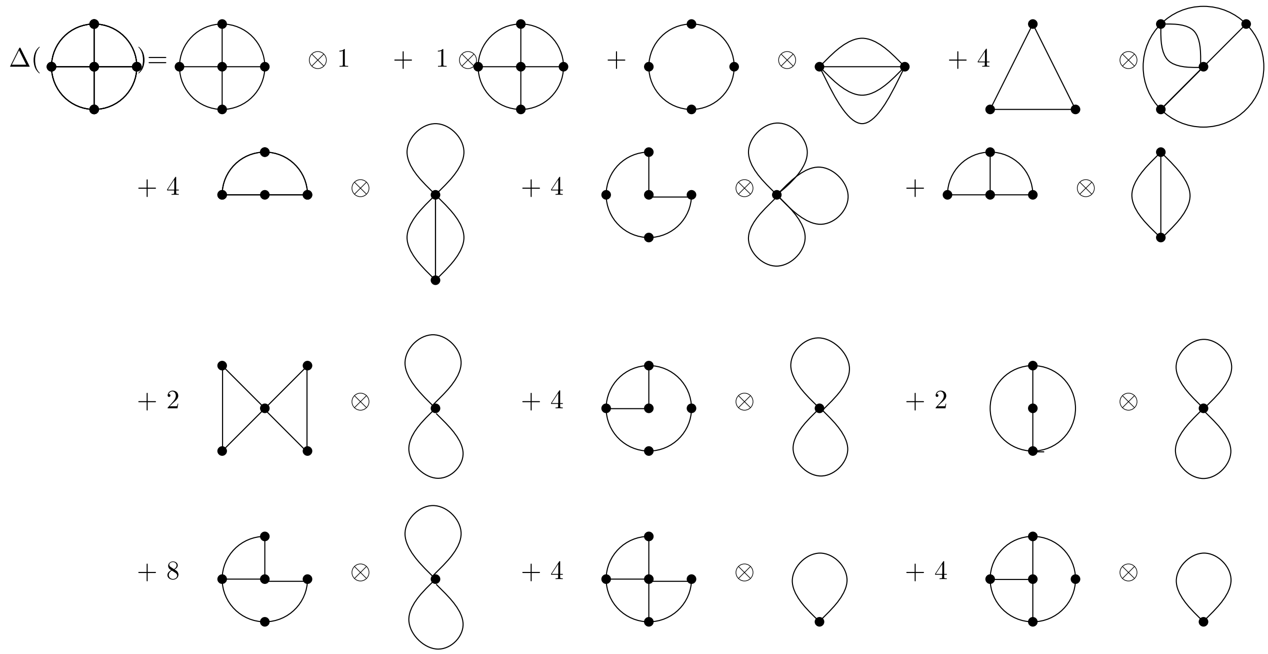
1 Cycle Hopf algebra of graph

Eg: Let G be a graph. Define $\Delta(G) = \sum C \otimes G/C$, where C is subgraph of G and all edges of C are in at least one cycle of C. G/C means each connected component of C is contracted to a vertex without contracting multiple edges.

Let $\mathcal{G}$ be the combinatorial class of equivalent classes of bridgeless graphs under the equivalence $G \sim G\cdot$ where $G\cdot$ means G disjoint union with an isolated vertex. The size of each element G is the number of edges of G, multiple edges and loops are allowed. Then $V\mathcal{G}$ is a Hopf algebra with multiplication disjoint union and Δ as above. $\mathbf{1}$ is the equivalent class of any number of isolated vertices. This is connected as a Hopf algebra, clearly graded, finite type.

Subeg:



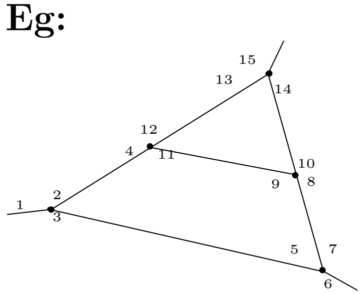
Now we could restrict this example. Take only G satisfying some condition and then also require the subgraphs in the coproduct to satisfy the condition.

An important family of such restrictions are called renormalization.

2 Hopf algebras built on Feynman graphs:

Graphs built out of half edges, i.e. a Feynman graph is a set of half edges and a set of half edge – half edge adjacency: these are the internal edges, each half edge appears in at most one internal edge and a set of tuples of half edges which gives the vertices and partition the set of half edges.

Eg:



Half edges are: $\{1, \cdots, 15\}$, internal edges: $\{\{2,4\},\{3,5\},\{7,8\},\{9,11\},\{12,13\},\{10,14\}\}$, the vertices:$\{\{1,2,3\},\{5,6,7\},\{4,11,12\},\{8,9,10\},\{13,14,15\}\}$.

But this is usually too cumbersome, but nice to know what we're talking about.

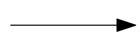
The half edges not part of an internal edge are called external edges.

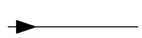
The only information we need from the physical theory (for us) is the following:

- a finite set of half edge type (i.e. for each graph we'll want function from the half edges of the graph to the set of half edge types, giving the type of each half edges),
- a set of permissible pairs of half edge types, these are the edge type, eg: for an undirected edge type we require both half edges to be the same type, for a directed edge type requiring one of the half edges to be of beginning type and the other of end type,
- a set of permissible subsets of half edge types for the vertex types,
- an integer weight for each edge type and vertex type,
- an integer dimension of space-time (typically 4).

Eg: QED (quantum electrodynamics)

half edge type: half photon 

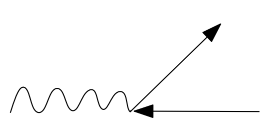
back half fermion 

front half fermion 

edge type: two half photons:  weight: 2

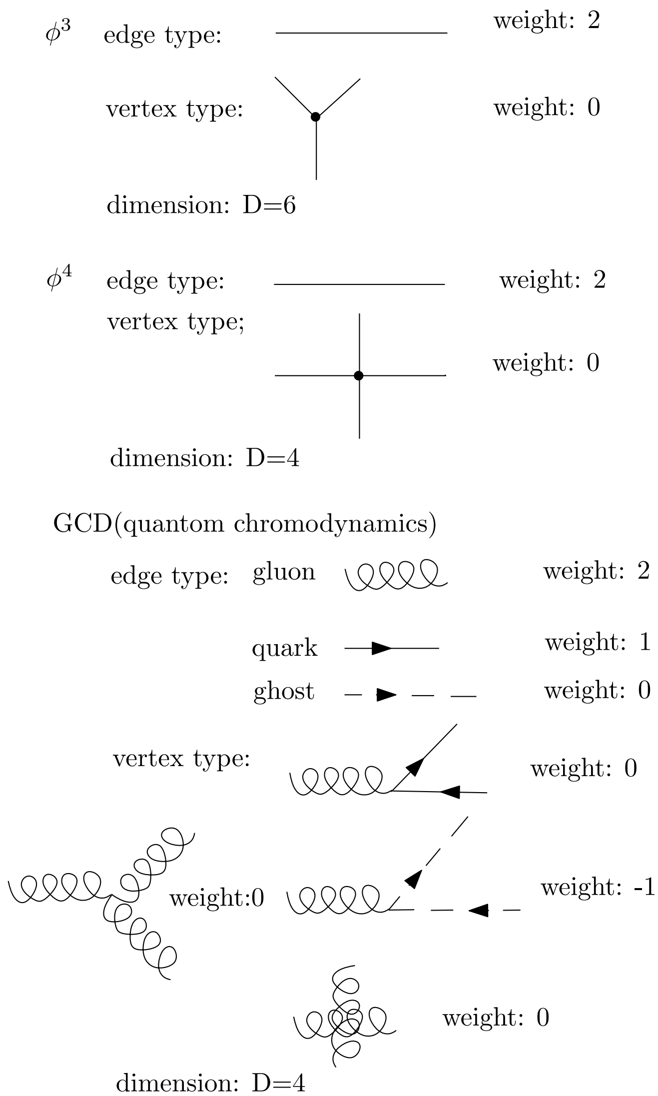
a back half fermion and a front half fermion:  weight: 1

vertex type: one of each half type:



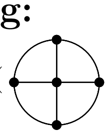
weight: 0

dimension of space time: D=4



Defination 1 *Let G be a graph as above. Let* **l(G)**=*the number of independent cycles of G = dimension of cycle space = 1^{st} Betti number = contract a spanning tree and count loops.*

Eg:

$l($  $) = 4$

Defination 2 *Let G be a Feynman graph in the giving theory. Then the superficial degree of divergence of G is*

$$w(G) = D \cdot l(G) - \sum_e w(e) - \sum_v w(v)$$

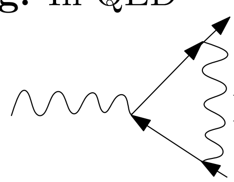
where e *is the internal edge of* G *and* v *is the vertex of* G*, and* $w(e)$ *and* $w(v)$ *are the weight given by the physical theory. If* $w(G) \geq 0$ *we say* G *is divergent.*

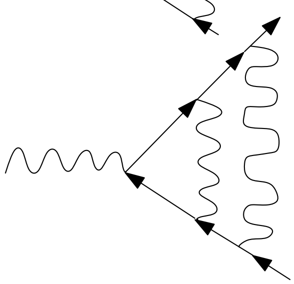
NOTE: Each internal edge and vertex contribute a factor to the Feynman integral. The weight counts how many powers of integration variables, for example:

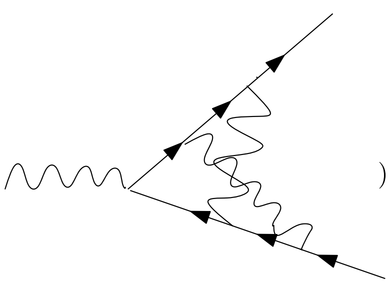
 contributes 2 powers to the denominator.

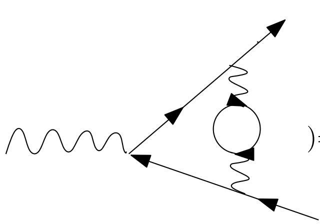
each independent cycle contributes an integration and we're integrating over D- dimensional space.

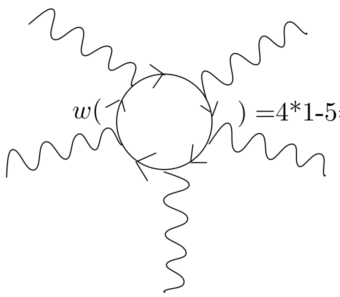
Eg: In QED

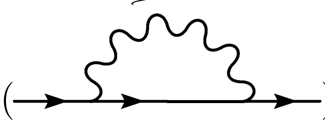
w()=4-1-1-2=0

w()=4*2-2-2-1-1-1-1=0

w()=4*2-2-2-1-1-1-1=0

w()=4*2-1-1-2-2-1-1=0

$w($ ) =4*1-5=-1 converge

$w($  $) = 4 * 1 - 3 \geq 0$

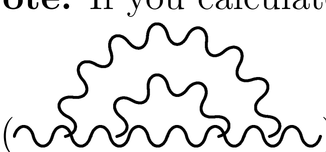
Defination 3 *Given the combinatorial information about a physical theory, we say the theory is renormalizable if the superficial degree of divergence depend only on the external edges.*

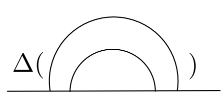
Defination 4 *Given the combinatorial representation of a physical theory, let $\mathcal{F}$ be the set of bridgeless divergent graphs graded usually by $l(G)$ (but edges also ok), then let*

$$\Delta(G) = \sum_{C \subseteq G} C \otimes G/C$$

(where C is bridgeless and all connected components of C are divergent), and otherwise as the cycle Hopf algebra.

Note: If you calculate

$\Delta($  $)$ in QCD,and



in ϕ^3

you get different answers because the divergence condition changes.

References

[1] Karen Yeats, Growth estimates for Dyson-Schwinger equations, arXiv:0810.2249v1, 13 Oct 2008.

[2] Karen Yeats, Scribe: Karel Casteels, Renormalization Hopf Algebras I: The Connes-Kreimer Algebra of Rooted Trees, April 22 2010. website:http://people.math.sfu.ca/ kyeats/seminars/RHA.pdf