MATH 821, SPRING 2012, ASSIGNMENT 2

DUE TUESDAY FEBRUARY 19, 2013 IN CLASS

(1) In class we gave very different proofs of the generating functions for PSet and MSet.
   (a) Give a PSet-type proof for MSet and a MSet-type proof for PSet.
   (b) Give an intuitive explanation for the signs in the cycle index polynomial for PSet.
(2) This question concerns power series operators $\Theta : x\mathbb{R}[[x]] \to x\mathbb{R}[[x]]$ with the property that the dependence of $[x^n]\Theta(A(x))$ on $A(x)$ involves only $[x^j]A(x)$, $1 \leq j < n$.
   (a) Let Θ be an operator as above. Prove that $T(x) = \Theta(T(x))$ has a unique solution.
   (b) Let Θ_1 and Θ_2 be operators as above. Suppose that for all $A(x) \in x\mathbb{R}_{\geq 0}[[x]]$,
   $$\Theta_1(A(x)) \trianglelefteq \Theta_2(A(x)),$$
   and that for $A_z(x), A_2(x) \in x\mathbb{R}_{\geq 0}[[x]]$ with $A_1(x) \trianglelefteq A_2(x)$,
   $$\Theta_i(A_1(x)) \trianglelefteq \Theta_i(A_2(x))$$
   for at least one of $i = 1$ or $i = 2$. Let $T_i(x) = \Theta_i(T_i(x))$ be the unique solutions. Prove
   $$T_1(x) \trianglelefteq T_2(x).$$
(3) Let $\mathcal{T}$ be the class of binary rooted trees where all children are either left children or right children (the class from the first day). Given $t \in \mathcal{T}$, let $\ell(t)$ be the number of leaves of t (i.e. the number of nodes with no nonempty children). Define
   $$T(x, y) = \sum_{t \in T} x^{|t|} y^{\ell(t)}$$
   (a) Give a polynomial equation satisfied by $T(x, y)$.
   (b) For each $n \geq 1$ determine the average number of leaves of a tree in $\mathcal{T}_n$.
(4) Consider the combinatorial class $\mathcal{D}^{(r)}$ of permutations with no cycles of length $\leq r$.
   (a) Give a specification for $\mathcal{D}^{(r)}$.
   (b) A derangement is an element of $\mathcal{D}^{(1)}$. Give a formula for the number $d_n^{(1)}$ of derangements of size n. What is $\lim_{n\to\infty} d_n^{(1)}$?
(5) (a) Give an example of a combinatorial class built of atoms with the property that for each size there are the same number of labelled objects as unlabelled objects of that size.
   (b) Give an example of a combinatorial class built of atoms with the property that for each size n, there are $n!$ times as many labelled objects as unlabelled objects of size n.
   (c) Give and example of a combinatorial class where the number of labelled objects compared to unlabelled objects is intermediate between the extremes of the previous parts. Plot the first 100 terms of the counting sequence for the labelled and unlabelled version of this class.