MATH 821, Spring 2013, Lecture 4

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Outline: Labelled combinatorial classes, exponential generating functions, order shuffles, the usual constructions, Set and PSet, set partitions and permutations, 'fake' labelled counting, Feynman graphs and symmetry factors.

1 Labelled Counting

Let $\mathcal{C}$ be a combinatorial class formed (either iteratively or recursively) out of copies (potentially different) of $\mathcal{Z}$ (and possibly $\mathcal{E}$). Many important classes are built in this manner. Call the copies of $\mathcal{Z}$ *atoms* . Some examples are: letters in words, vertices in graphs, and boxes in partitions.

Definition. Let $\mathcal{C}$ be such a class as defined above. A *labelling* of an element $C \in \mathcal{C}$ is a total order on the atoms of C. The order is usually represented by associating an element of $\{1, 2, \ldots, |C|\}$ to each atom bijectively.

Definition. A *labelled combinatorial class* $\mathcal{D}$ is a countable set of pairs $(C, <)$ where C is a combinatorial object built of atoms, and '$<$' is a labelling of C satisfying:

- $|(C, <)| = |C|$, and
- $\{(C, <) \in \mathcal{D} : |(C, <)| = n\}$ is finite for all $n \geq 0$,

potentially allowing objects of size 0 with the empty ordering.

Example (Labelled Simple Graphs)**.** In Figure 1 we enumerate the labelled simple graphs of sizes 1, 2, and 3, where we label the vertices and take the size function to be the number of vertices.

The generating functions for labelled combinatorial classes are the *exponential generating functions.*

Definition. Let $\mathcal{C}$ be a labelled combinatorial class. Then the *exponential generating function* of $\mathcal{C}$ is

$$C(x) = \sum_{C \in \mathcal{C}} \frac{x^{|C|}}{|C|!} = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!}.$$

We will make no distinction between labelled and unlabelled classes for the notation '$C(x)$' as the meaning is usually clear from the context.

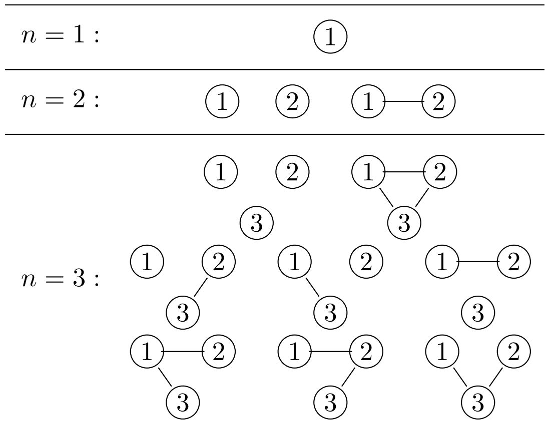


Table 1: Labelled simple graphs of size 1, 2, and 3

Lemma 1. *If* $A(x) = \sum_{n=0}^{\infty} a_n x^n/n!$ *and* $B(x) = \sum_{n=0}^{\infty} b_n x^n/n!$ *are exponential generating functions, then their product is computed as follows:*

$$\begin{aligned} A(x)B(x) &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \left(\frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!} \right) x^n \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a_k b_{n-k} \frac{x^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k} \right) \frac{x^n}{n!}. \quad \square \end{aligned}$$

Definition. Let A and B be finite sets with total orders $<_A$ and $<_B$, respectively, and such that $A \cap B = \emptyset$. Define the *shuffle* of $<_A$ and $<_B$ to be the set of all total orders on $A \cup B$ which agree with $<_A$ on A and with $<_B$ on B. Denote this by $(<_A) ⧢ (<_B)$.

Example. Viewing finite sets with total orders as words, the shuffle product of the word `abc` (i.e. the set $\{\mathtt{a}, \mathtt{b}, \mathtt{c}\}$ with total order $\mathtt{a} < \mathtt{b} < \mathtt{c}$) and the word `qr` is:

$$\begin{aligned} \mathtt{abc} ⧢ \mathtt{qr} = \{&\mathtt{abcqr}, \mathtt{abqcr}, \mathtt{aqbcr}, \mathtt{qabcr}, \mathtt{abqrc}, \\ &\mathtt{aqbrc}, \mathtt{qabrc}, \mathtt{aqrbc}, \mathtt{qarbc}, \mathtt{qrabc}\} \end{aligned}$$

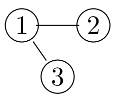
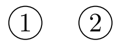
Now we are ready to define the labelled product of two classes.

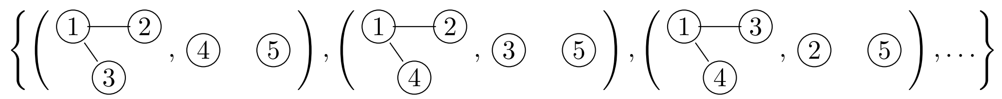
Definition. Let $\mathcal{B}$ and $\mathcal{C}$ be labelled combinatorial classes. Define the *labelled product* of $\mathcal{B}$ and $\mathcal{C}$, denoted $\mathcal{A} = \mathcal{B} \star \mathcal{C}$, to be the combinatorial class with underlying set

$$\left\{ \Big(((b, <_b), (c, <_c)), <_A \Big) \ : \ \begin{array}{c} (b, <_b) \in \mathcal{B} \\ (c, <_c) \in \mathcal{C} \\ <_A \in (<_b \sqcup <_c) \end{array} \right\},$$

and with size function $|(b, c)| = |b| + |c|$.

Example. Take the class of labelled graphs $\mathcal{G}$. The contribution of the graphs

 and  to the class $\mathcal{G} \star \mathcal{G}$ starts out as follows:



and it consists of the $\binom{5}{2}$ possibilities of labelling the two graphs with the numbers $\{1, \ldots, 5\}$ such that the original orders on the individual graphs hold.

Proposition 2. *Let $\mathcal{B}$ and $\mathcal{C}$ be labelled combinatorial classes and let $\mathcal{A} = \mathcal{B} \star \mathcal{C}$. Then $A(x) = B(x)C(x)$ (with exponential generating functions).*

Proof. If $(b, <_b) \in \mathcal{B}$ and $(c, <_c) \in \mathcal{C}$, then the size of $<_b \sqcup <_c$ is just $\binom{|b|+|c|}{|b|}$ or $\binom{|b|+|c|}{|c|}$. Since the size of $|(b, c)|$ is equal to $|b| + |c|$, we have that

$$\begin{aligned} \mathcal{A}_n &= \left| \left\{ \Big(((b, <_b), (c, <_c)), <_A \Big) \ : \ \begin{array}{c} (b, <_b) \in \mathcal{B} \\ (c, <_c) \in \mathcal{C} \\ <_A \in (<_b \sqcup <_c) \\ |b| + |c| = n \end{array} \right\} \right| \\ &= \sum_{|b|=0}^{n} \binom{|b| + |c|}{|b|} \ |\{(b, c) : \ b \in \mathcal{B}, c \in \mathcal{C}, |c| = n - |b|\}| \\ &= \sum_{k=0}^{n} \binom{n}{k} b_k c_{n-k}, \end{aligned}$$

and the result follows from Lemma 1. $\square$

Note that this also proves admissibility for the labelled product.

Some of the combinatorial constructions for unlabelled classes translate to labelled combinatorial classes directly:

- The disjoint union '+' is defined exactly as before, and we get the same relation for the exponential generating functions.
- If $\mathcal{B}$ is a labelled class, we define $\mathcal{B}^{\star k} = \mathcal{B} \star \mathcal{B} \star \cdots \star \mathcal{B}$ (k times). As before, if $\mathcal{A} = \mathcal{B}^{\star k}$, then $A(x) = B(x)^k$.
- Similarly, we define $\textsc{Seq}(\mathcal{B}) = \sum_{n=0}^{\infty} \mathcal{B}^{\star n}$ and we get the same formula: if $\mathcal{A} = \textsc{Seq}(\mathcal{B})$, then $A(x) = \frac{1}{1-B(x)}$.

On the other hand, the other constructions are much nicer in the labelled world. Since we do not need to keep track of automorphisms (everything is distinguishable!), the generating functions lose the $B(x^k)$ terms and turn out to be analytically easier to handle.

Proposition 3. *Let $\mathcal{B}$ be a labelled combinatorial class with $\mathcal{B}_0 = \emptyset$. Let $\mathcal{A} = \textsc{DCyc}_k(\mathcal{B}) = \mathcal{B}^{\star k}/C_k$ be the combinatorial class of labelled directed k-cycles of elements of $\mathcal{B}$. Then, $A(x) = \frac{1}{k} B(x)^k$.*

Proof. Since everything is distinguishable, each labelled directed k-cycle has only the k cyclic shifts as automorphisms. This implies the result. $\square$

Corollary 4. *Let $\mathcal{B}$ be a labelled combinatorial class with $\mathcal{B}_0 = \emptyset$, and let $\mathcal{A} = \textsc{DCyc}(\mathcal{B}) = \sum_{k=1}^{\infty} \textsc{DCyc}_k(\mathcal{B})$. Then, $A(x) = \sum_{k=1}^{\infty} \frac{1}{k} B(x)^k = \log\left(\frac{1}{1-B(x)}\right)$.*

2 The SET and PSet Constructions

The MSet construction does not translate directly to the labelled case because labelled objects are distinguished. Instead we have Set, which is defined in a similar manner.

Definition. Let $\mathcal{B}$ be a labelled combinatorial class with $\mathcal{B}_0 = \emptyset$. Then define $\textsc{Set}_k(\mathcal{B}) = \mathcal{B}^{\star k}/S_k$.

Proposition 5. *Let $\mathcal{B}$ be as above and let $\mathcal{A} = \textsc{Set}_k(\mathcal{B})$. Then $A(x) = B(x)^k/k!$.*

Proof. Objects are distinguishable, so all permutations on a k-sequence give a different k-sequence. $\square$

Corollary 6. *Let $\mathcal{B}$ be as above and let $\mathcal{A} = \textsc{Set}(\mathcal{B}) = \sum_{k=0}^{\infty} \textsc{Set}_k(\mathcal{B})$. Then $A(x) = \sum_{k=0}^{\infty} B(x)^k/k! = \exp(B(x))$.*

This result implies that there is an exponential connection between a combinatorial class of connected components and the full class of all components.

Now let us return to the unlabelled case and define a similar construction. We will construct a 'true set' (in the sense that it is not a multiset) by forcing elements to have no duplicates.

Definition. Let $\mathcal{B}$ be an unlabelled combinatorial class with $\mathcal{B}_0 = \emptyset$. Define $\textsc{PSet}(\mathcal{B})$ to be the combinatorial class whose underlying set is the set of finite subsets of $\mathcal{B}$, and with size function $|\{b_1, b_2, \ldots, b_k\}| = |b_1| + \cdots + |b_k|$.

Proposition 7. *Let $\mathcal{B}$ be an unlabelled combinatorial class with $\mathcal{B}_0 = \emptyset$ and let $\mathcal{A} = \textsc{PSet}(\mathcal{B})$. Then $A(x) = \exp\left(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{B(x^k)}{k}\right)$.*

Proof. Since each element of $\mathcal{B}$ is either in the subset we choose or not, each element contributes a factor $(1 + x^{|b|})$ to the generating function $A(x)$. Then we take the product of all these factors, and take the logarithm to convert it to a sum:

$$
\begin{aligned}
A(x) &= \prod_{b \in \mathcal{B}} (1 + x^{|b|}) \\
&= \exp\left(\log\left(\prod_{b \in \mathcal{B}} (1 + x^{|b|})\right)\right) \\
&= \exp\left(\sum_{b \in \mathcal{B}} \log(1 + x^{|b|})\right) \\
&= \exp\left(\sum_{n=1}^{\infty} b_n \log(1 + x^n)\right) && \text{(count elements of size } n\text{)} \\
&= \exp\left(\sum_{n=1}^{\infty} b_n \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} (x^n)^k\right) && \text{(expand log)} \\
&= \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=1}^{\infty} b_n (x^k)^n\right) && \text{(switch order of summation)} \\
&= \exp\left(\sum_{k=1}^{\infty} (-1)^{k-1} \frac{B(x^k)}{k}\right)
\end{aligned}
$$

and this is the desired result. $\square$

This generating function is in essence performing an inclusion-exclusion on the subsets so that duplication does not occur: we exclude doubles, then include triples (which were over-excluded), and so on. Also, compare this to the generating function for $\textsc{MSet}(\mathcal{B})$, which is $\exp\left(\sum_{k=1}^{\infty} B(x^k)/k\right)$. The only difference is the $(-1)^{k-1}$ term, and this will cause a bit of trouble once we get to asymptotics.

This result can also be proven using an idea similar to the $\textsc{MSet}$ proof from before.

We end this section with a few examples for the labelled case.

Example. Given a set S, a *set partition* is a disjoint collection of nonempty subsets of S such that their union is the original set S. To get a combinatorial specification for set partitions, first we introduce a different specification for the positive integers in terms of labelled sets of atoms:

$$\textsc{Set}_{\geq 1}(\mathcal{Z}) = \left\{ \textcircled{1}\,, \begin{matrix} \textcircled{1} \\ \textcircled{2} \end{matrix}\,, \begin{matrix} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \end{matrix}\,, \dots \right\}.$$

To get set partitions, we try the $\textsc{Seq}$ construction:

$$\textsc{Seq}\left(\textsc{Set}_{\geq 1}(\mathcal{Z})\right) = \left\{ \varepsilon,\ \left(\textcircled{1}\right)\ ,\ \begin{pmatrix} \textcircled{1} \\ \textcircled{2} \end{pmatrix}\ ,\ \left(\textcircled{1}\,,\textcircled{2}\right)\ ,\ \left(\textcircled{2}\,,\textcircled{1}\right)\ ,\dots \right\}$$

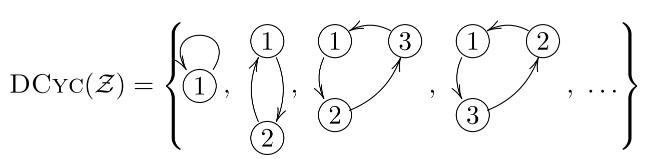
but this is incorrect as the labelled $\textsc{Seq}$ construction imposes an order on the parts. Try $\textsc{Set}$:

$$\textsc{Set}\left(\textsc{Set}_{\geq 1}(\mathcal{Z})\right) = \left\{ \begin{matrix} \varepsilon,\ \left(\textcircled{1}\right)\ ,\ \begin{pmatrix} \textcircled{1} \\ \textcircled{2} \end{pmatrix}\ ,\ \left(\textcircled{1}\,,\textcircled{2}\right)\ , \\ \begin{pmatrix} \textcircled{1} \\ \textcircled{2} \\ \textcircled{3} \end{pmatrix}\ ,\ \left(\textcircled{1}\,,\textcircled{2}\,,\textcircled{3}\right)\ , \\ \left(\textcircled{1}\,, \begin{matrix} \textcircled{2} \\ \textcircled{3} \end{matrix}\right)\ ,\ \left(\textcircled{2}\,, \begin{matrix} \textcircled{1} \\ \textcircled{3} \end{matrix}\right)\ ,\ \left(\textcircled{3}\,, \begin{matrix} \textcircled{1} \\ \textcircled{2} \end{matrix}\right)\ ,\dots \end{matrix} \right\}$$

This is now correct. Let $\mathcal{S} = \textsc{Set}\left(\textsc{Set}_{\geq 1}(\mathcal{Z})\right)$. Then the (exponential) generating function for $\mathcal{S}$ is $S(x) = e^{e^x - 1}$.

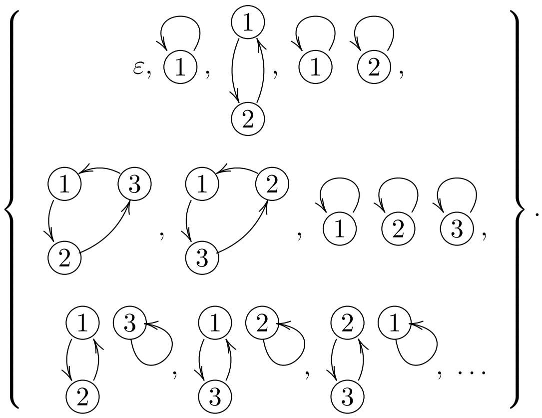
Example. We construct the combinatorial class of permutations in two ways. First, since each permutation has a unique representation as a disjoint cycle decomposition, the class of permutations $\mathcal{P}$ can be specified as $\mathcal{P} = \textsc{Set}\left(\textsc{DCyc}(\mathcal{Z})\right)$, where $\textsc{DCyc}$ is the labelled version.

This is easily seen as $\textsc{DCyc}(\mathcal{Z})$ is the set



And so $\mathcal{P}$ is the set

$$\textsc{Set}\,(\textsc{DCyc}(\mathcal{Z})) =$$



The generating function for $\mathcal{P}$ is $P(x) = \exp\left(\log\left(\frac{1}{1-x}\right)\right)$, or simply just $P(x) = \frac{1}{1-x}$. It should be noted that this is an exponential generating function and so it gives the correct number of permutations on n objects:

$$P(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} n!\left(\frac{x^n}{n!}\right).$$

A second specification for the class of permutations is obtained by looking at each permutation as a listing of the integers from 1 to n in some order, that is, $\mathcal{P} = \{\mathtt{1}, \mathtt{12}, \mathtt{21}, \mathtt{123}, \mathtt{132}, \mathtt{213}, \mathtt{231}, \mathtt{312}, \mathtt{321}, \ldots\}$. This is just a labelled sequence of atoms, and so we have $\mathcal{P} = \textsc{Seq}(\mathcal{Z})$. Hence we get the same generating function, $P(x) = \frac{1}{1-x}$.

3 Components and 'Fake' Labelled Counting

Recall some examples from the $\textsc{Set}$ construction:

Example. If $\mathcal{G}$ is the class of labelled graphs and $\mathcal{C}$ is the class of *connected* labelled graphs, then $\mathcal{G} = \textsc{Set}(\mathcal{C})$. So, $G(x) = \exp(C(x))$.

Example. If $\mathcal{T}$ is the class of labelled trees and $\mathcal{F}$ is the class of labelled forests, then $F(x) = \exp(T(x))$.

This is the exponential connection between connected objects and all objects. In the unlabelled case we have the $\textsc{MSet}$ construction, which for graphs gives us $G(x) = \exp\left(\sum_{k=1}^{\infty} C(x^k)/k\right)$. This is similar asymptotically but definitely different from the labelled case.

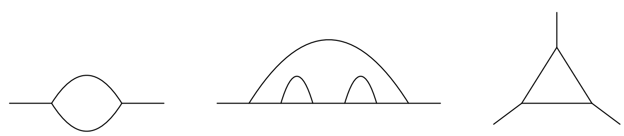
Outside of combinatorics we might see this exponential connection between connected and potentially disconnected objects, even in the unlabelled case.

In Quantum Field Theory, we see another example of labelled counting, specifically in counting Feynman diagrams. Here, each object a is counted along with a symmetry factor $1/|\textsc{Sym}(a)|$ of a graph a, where $\textsc{Sym}(a)$ is the automorphism group of a.

In fact, the sum $\sum_{a\in\mathcal{A}} x^{|a|}/|\textsc{Sym}(a)|$ is secretly just labelled counting! To see this, let $\widetilde{\mathcal{A}}$ be the same class as $\mathcal{A}$ but with all possible labellings. Then the exponential generating function for $\widetilde{\mathcal{A}}$ is:

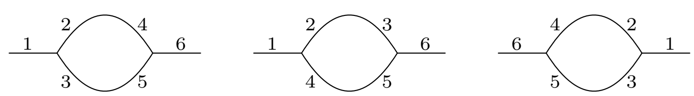
$$\widetilde{A}(x) = \sum_{a\in\widetilde{\mathcal{A}}} \frac{x^{|a|}}{|a|!} = \sum_{a\in A} \frac{x^{|a|}}{|a|!} \left(\sum_{\substack{\widetilde{a}\in\widetilde{\mathcal{A}} \\ \widetilde{a}\text{ without the} \\ \text{labelling is } a}} 1 \right) = \sum_{a\in\mathcal{A}} \frac{x^{|a|}}{|\textsc{Sym}(a)|}$$

Example. Consider the Feynman graphs in the class ϕ^3 in 6 dimensions. This is just the set of 3-regular graphs, permitting multiedges, loops, and external edges (half-edges connected to a vertex of the graph). Some examples are shown below.



To compute the symmetry factor, we use two labellings: one where all half-edges (not just external ones) are labelled, and one where only the external edges are labelled.

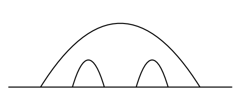
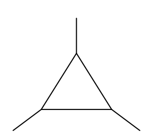
Then we count all labellings of the diagram, of the first type. For instance the labellings below are all different:



Note that the last graph is just the first one flipped, but this does not satisfy the ordering on the external vertices (1 on the left, 2 on the right).

If we continue doing this we count a total of only 2 elements in the automorphism group of this diagram: the horizontal flip. Hence the symmetry factor is $\frac{1}{2}$. If we had written out all labellings, we would obtain a total of 360 different labellings, and we get the same symmetry factor $\frac{360}{6!} = \frac{1}{2}$.

The other two graphs

 and 

have symmetry factors $\frac{1}{4}$ and 1 respectively.