HOMEWORK 1 SOLUTIONS

MATH 818, FALL 2010

Sh, I.2.5: If they were isomorphic then their coordinate rings would be isomorphic, but $k[\mathbb{A}^1] = k[t]$ while $k[V(xy = 1)] = k[x,y]/\langle xy = 1\rangle = k[x, x^{-1}]$. The only units in $k[t]$ are the nonzero constants, while in $k[x, x^{-1}]$ we also have all the monomials as units. Thus the rings are not isomorphic.

Sh, I.2.11: With notation as in the question suppose $J(f) \in k[x_1, \ldots, x_n] \smallsetminus k$. Then there is some $x \in \mathbb{A}^n$ such that $J(f)(x) = 0$ but then f has no inverse at x so f is not an automorphism. For the same reason $J(f)$ is not identically 0, so $f \mapsto J(f)$ maps into k^*. Finally then just note that, as in calculus, $J(f \circ g) = J(f)J(g)$.

Sh, I.2.15: Let g be the map which takes X to the graph of f. (This was a better question before we discussed graphs.)

F, 1-10: Let L_n be the line through the origin with slope $1/n$. This is an algebraic subset of $\mathbb{A}^2$ but $\bigcup_{n=1}^{\infty}$ is the region in the first quadrant below the line $y = x$ and the region above the line $y = x$ in the third quadrant. We know this is not an algebraic subset of $\mathbb{A}^2$ because we characterized all such subsets.

F, 1-26: Suppose $F = Y^2 + X^2(X-1)^2$ factored. If one factor is quadratic in Y then the other factor contains no Y and hence, since the coefficient of Y^2 in F is 1, the other factor must also be a constant. So it must be the case that $F = (Y + A)(Y + B)$ for some polynomials $A, B \in k[X]$. But F has no linear term in Y so $A = -B$, and $-A^2 = X^2(X-1)^2$ which is not possible over $\mathbb{R}$. Thus F does not factor over $\mathbb{R}$.

Now consider $V(F)$. $P = (a, b) \in V(F)$ implies $b^2 = -a^2(a-1)^2$, so over $\mathbb{R}$, $b = 0$ and $a = 0$ or $a = 1$. Thus

$$V(F) = V(Y, X) \cup V(Y, X-1).$$

Sh, I.3.4: First note that $z = 0 \Leftrightarrow y = 0$ in X and in that case x is free so $V(y, z) \subset X$. $V(y, z)$ is irreducible since $\langle x, y\rangle$ is prime, and it is not only birational, but in fact isomorphic, with $\mathbb{A}^1$ via $(x, 0, 0) \mapsto x$.

For $y \neq 0$, plugging the second equation into the first squared and cancelling gives $y = x^2$. So $V(y = x^2, z^2 = y^3) \cap \{y \neq 0\} \subset X$. Further the only point in $V(y = x^2, z^2 = y^3)$ with $y = 0$ is $(0, 0, 0)$ which is a point of X, so $V(y = x^2, z^2 = y^3) \subset X$. We can parametrize:

$$V(y = x^2, z^2 = y^3) = \{(t, t^2, t^3) : t \in k\}$$

which gives a regular map $\mathbb{A}^1 \to V(y = x^2, z^2 = y^3)$. The inverse is given by, for instance, $(x, y, z) \mapsto y/x$ and hence is rational. Also $\langle y = x^2, z^2 = y^3\rangle$ is prime because if $fg \in \langle y = x^2, z^2 = y^3\rangle$, then $fg(t, t^2, t^3) = 0$ as a polynomial in t, and hence at least one of $f(t, t^2, t^3)$ and $g(t, t^2, t^3)$ is also 0 as polynomials in t and hence is in $\langle y = x^2, z^2 = y^3\rangle$.

Finally, note that we have covered the case $y = 0$ and the case $y \neq 0$ and so

$$X = V(y, z) \cup V(y = x^2, z^2 = y^3)$$

Sh. I.3.6: The given rational function is manifestly regular if $x \neq 0$. Furthermore, on $V(x^2 + y^2 = 1)$ we have $x^2 = (1-y)(1+y)$ so as rational functions

$$\frac{1-y}{x} = \frac{x}{1+y}$$

Thus the rational function is also regular when $y \neq -1$. If $x = 0$ and $y = -1$ then in the original representation the numerator is nonzero and the denominator is zero, so no trickery will fix the rational function at this point. Therefore the rational function is regular at $P \neq (0, -1)$.

diagram: See last page.

F. 4-18: (a) Check well defined: If $V(\sum a_i X_i) = V(\sum b_i X_i)$ then for all projective points $(c_0 : \cdots : c_n)$, $\sum a_i c_i = 0 \Leftrightarrow \sum b_i c_i = 0$. In particular with the points $(0 : \cdots : 0 : 1 : 0 : \cdots : 0)$ we get $a_i = 0 \Leftrightarrow b_i = 0$ for all i so $a_i = \lambda_i b_i$ for some nonzero λ_is. Then with the points $(0 : \cdots : 1 : \cdots : x : \cdots : 0)$ we get $a_i + a_j x = 0 \Leftrightarrow \lambda_i a_i + \lambda_j a_j x = 0$ for all x and so $\lambda_i = \lambda_j$. Thus with any representative of the hyperplane we get the same projective point.

Check one-to-one: Suppose $(a_0 : \cdots : a_n) = (b_0 : \cdots : b_n)$, then there exists a λ such that $(a_0 : \cdots : a_n) = (\lambda b_0 : \cdots : \lambda b_n)$, and so $V(\sum b_i X_i) = V(\sum \lambda b_i X_i) = V(\sum a_i X_i)$.

Check onto: Take any point $(a_0 : \cdots : a_n) \in \mathbb{P}^n$, then the hyperplane $V(\sum a_i X_i)$ maps to this point.

(b)

$$V(\sum a_i X_i) \mapsto (a_0 : \cdots : a_n) \mapsto V(\sum a_i X_i)$$

so the double duals work out. More interestingly let $P = (a_0 : \cdots : a_n)$, and $H = V(\sum b_i X_i)$ then $P \in H$ iff $\sum a_i b_i = 0$. But $P^* = \sum a_i X_i$ and $H^* = (b_0 : \cdots : b_n)$ so $P^* \in H^*$ iff $\sum a_i b_i = 0$. Thus $P \in H$ iff $P^* \in H^*$.

F. 4-20: (a) Suppose we have a polynomial $f \in k[X, Y, Z]$ such that $b - a^2 = 0, c - a^3 = 0$ implies $f(a, b, c) = 0$. Then by polynomial division write

$$f = f_1(Y - X^2) + f_2(Z - X^3) + f_3$$

where $f_1 \in k[X, Y, Z]$, $f_2 \in k[X, Z]$, $f_3 \in k[X]$. Then $b - a^2 = 0, c - a^3 = 0$ implies $f_3(a) = 0$. But there are infinitely many such a so $f \in \langle Y - X^2, Z - X^3 \rangle$ and so $I(V) = \langle Y - X^2, Z - X^3 \rangle$.

(b) $Z - XY = Z - X^3 - X(Y - X^2)$ so $Z - XY \in I(V)$ so $ZW - XY \in I(V)*$. On the other hand if $ZW - XY \in \langle (Y - X^2)*, (Z - X^3)* \rangle$ then

$$ZW - XY = F(YW - X^2) + G(ZW^2 - X^3)$$

for some $F, G \in k[W, X, Y, Z]$. But by degree G must be 0 and F a constant which is impossible, thus $ZW - XY \notin \langle (Y - X^2)*, (Z - X^3)* \rangle$