COMMUTATIVE ALGEBRA, FALL 2013

ASSIGNMENT 1 SOLUTIONS

(1) We need to assume $\mathcal{L}$ has a 0, but we shouldn't have the 0 condition in the definition of filter; instead the question should ask that every proper filter is contained in an ultrafilter.

Let F be a proper filter of $\mathcal{L}$. Let $\mathcal{A}$ be the set of all proper filters containing F. $\mathcal{A}$ is nonempty since $F \in \mathcal{A}$. Let $\mathcal{S}$ be a chain in $\mathcal{A}$.

I claim that $\bigcup \mathcal{S} \in \mathcal{A}$.

First note that if $a \in \bigcup \mathcal{S}$ then $a \in G$ for some filter G in $\mathcal{S}$, so if we take any $b \geq a$, $b \in \mathcal{L}$ then $b \in G$ and hence $b \in \bigcup \mathcal{S}$.

Next take $a, b \in \bigcup \mathcal{S}$, say with $a \in G$, $b \in H$, $G, H \in \mathcal{S}$. Since $\mathcal{S}$ is a chain wlog $G \subseteq H$ and so $a, b \in H$ and hence $a \wedge b \in H$ so $a \wedge b \in \bigcup \mathcal{S}$.

Next note that each element of $\mathcal{S}$ is proper so none contain 0. Thus their union does not contain 0 and so their union is also proper. Furthermore each element of $\mathcal{S}$ contains F and so their union also contains F.

Thus $\bigcup \mathcal{S} \in \mathcal{A}$.

So by Zorn's lemma $\mathcal{A}$ has a maximal element, which by definition is an ultrafilter containing F.

(2) Take A cofinite. Take B with $A \subseteq B$. Then $S \setminus B \subseteq S \setminus A$ which is finite, so $S \setminus B$ is finite, and so B is cofinite.

Take A and B cofinite. Then $S \setminus (A \cap B) = (S \setminus A) \cup (S \setminus B)$ which is finite since the union of two finite sets is finite. So $A \cap B$ is cofinite.

Thus the cofinite sets form a filter.

(3) Say U is a principal ultrafilter on $\mathcal{P}(S)$. So there is some $A \subseteq S$ so that $U = \{B \subseteq S : A \subseteq B\}$. Take a finite set C with $C \cap A \neq \varnothing$. This is always possible as we can just take C to be any singleton in A. Then $S \setminus C$ is in the cofinite filter but A is not a subset of $S \setminus C$ because $C \cap A \neq \varnothing$. So $S \setminus C$ is not in U.

Now say U is an ultrafilter on $\mathcal{P}(S)$ and say U does not contain the cofinite filter. Take B cofinite, with $B \notin U$ then by maximality of U there is an $A \in U$ such that $A \cap B = \varnothing$. Consequently $A \in S \setminus B$ which is a finite set. So U contains at least one finite set. Furthermore, given any two distinct finite sets of the same size in U their intersection contains fewer elements than either of them, so there is a unique finite set C of minimal size in U. $C \neq \varnothing$ since U is proper. I claim that U is principal generated by C. By definition of filter every superset of C is in U, and if any other set is in U, then its intersection with C is smaller than C and is in U which is a contraction proving the claim.

(4) Assume $N_1 \subseteq N_2$ and K are submodules of a module M.

Let $f : (K \cap N_2)/(K \cap N_1) \to N_2/N_1$ and $g : N_2/N_1 \to (K + N_2)/(K + N_1)$ be the natural maps.

Specifically, $f(n + K \cap N_1) = n + N_1$ for $n \in K \cap N_2$ and $g(n + N_1) = m + K + N_1$ for $m \in N_2$. Therefore $\text{im} f = (K \cap N_2)/N_1$. Also if $n + N_1 \in \ker g$ then $n \in K + N_1$ so $\ker g = ((N + N_1) \cap N_2)/N_1$.

So

$$\begin{aligned} \text{im} f = \ker g \;&\Leftrightarrow\; ((K + N_1) \cap N_2)/N_1 = (K \cap N_2)/N_1 \\ &\Leftrightarrow\; (K + N_1) \cap N_2 = (K \cap N_2) + N_1 \end{aligned}$$

So the sequence in the problem being exact is equivalent to the modularity property.

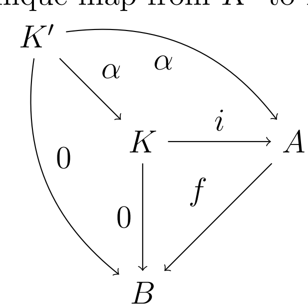
(5) I'll do it as an element chase. Did anyone do it element-free?

Take $a \in B_3$. I want to show that $a \in \text{im} g_3$. Take $b \in A_4$ with $g_4(b) = h_3(a)$ which is possible as g_4 is an isomorphism. Then $0 = h_4(h_3(a)) = h_4(g_4(b)) = g_5(f_4(b))$ and so $f_4(b) = 0$ since g_5 is an isomorphism. So b is in the image of f_3, say $f_3(c) = b$. Consider $g_3(c) - a$. $h_3(g_3(c)) = h_3(f_3(c)) = h_3(a)$ so $h_3(g_3(c) - a) = 0$. Therefore $g_3(c) - a$ is in the image of h_2, say $h_2(d) = g_3(c) - a$. Take $e \in A_2$ with $g_2(e) = d$ which is possible as g_2 is an isomorphism. Then $g_3(c) - a = h_2(g_2(e)) = g_3(f_2(e))$ so $a = g_3(c - g_2(e))$ and so $a \in \text{im} g_3$ as desired.

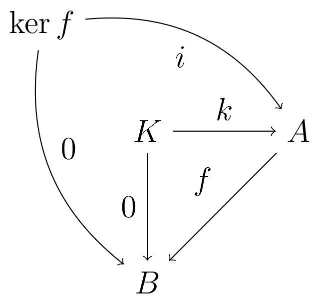
Now take $a \in \ker g_3$. Then $0 = h_3(g_3(a)) = g_4(f_3(a))$ and so $f_3(a) = 0$ since g_4 is an isomorphism. Thus there is a b with $f_2(b) = a$. Then $0 = g_3(f_2(b)) = h_2(g_2(b))$ so $g_2(b) \in \ker h_2$. Thus there is a c with $h_1(c) = g_2(b)$. Since g_1 is an isomorphism there is a d with $g_1(d) = c$, so $g_2(b) = h_1(g_1(d)) = g_2(f_1(d))$. But g_2 is an isomorphism so $b = f_1(d)$. Thus $a = f_2(b) = 0$ as desired.

(6) This was harder than I thought it would be (at least finding one not making one up). Maybe some of you came up with better ones. I found one at `http://mathoverflow.net/questions/1083/do-good-math-jokes-exist`: "How are Goethe's Faust novels like isomorphisms of sets? Dey're de monic epics."

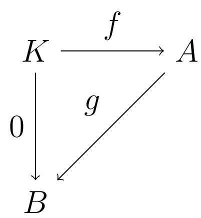
(7) Say $K = \ker f$ in the module sense. Let $i : K \to A$ be the natural injection. Then $fi = 0$, and if $\alpha : K' \to A$ monic with $f\alpha = 0$ then $\alpha(K') \subseteq K$ so $\alpha K' \to K$ is the unique map from K' to K which satisfies



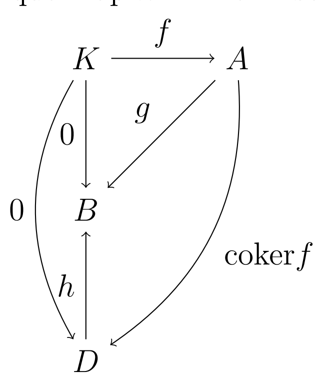
Say $k : K \to A$ is a categorical kernel of f. Then $f(k(K)) = 0$ so $k(K)$ is a subset of the module-theoretic kernel, $\ker f$ of f. But if $k(K)$ is strictly smaller than $\ker f$ then there is no map from $\ker f$ to K which satisfies



(8) f is a kernel, say of g, so



Say coker$f$: $A \to D$, then by the universal property of cokernels there exists a unique map $h : D \to B$ so that the following commutes



Now suppose we have $j : K' \to A$ monic with $(\text{coker}f)j = 0$. Then $hj = h(\text{coker}f)j = 0$ so by the universal property of kernels applied to f we have a unique map $K' \to K$ making the following diagram commute: