Math 303, Fall 2011, Lecture 7

① Functions

Suppose X and Y are sets. We would like to encode functions

$$\underset{\text{name of function}}{f} : \underset{\text{domain}}{X} \to \underset{\text{codomain}}{Y} \qquad \text{using sets}$$

How can we do this?

use ordered pairs $(x, f(x))$

The function is determined by the set $\{(x, f(x)) \mid x \in X\}$

For our purposes this IS the function.

But is this a well defined set?

$f : \mathbb{R} \to \mathbb{R}$

To make this a precise definition we need

$$\{ (a,b) \in A \times B \mid b = f(a) \}$$

this needs to be definable with and, or, etc....

So functions are subsets of $A \times B$

Which subsets S of $X \times Y$ are functions?

Need for every $a \in A$

there is a $(a,b) \in S$ (so $f(a)$ exists)

and only one such pair (otherwise $f(a)$ would have 2 or more meanings)

$$\boxed{\text{Define} \quad Y^X = \{ f \in \mathcal{P}(X \times Y) \mid f \text{ is a function} \}}$$

eg let $X = Y = \omega \; (= \{0, 1, 2, \ldots\})$

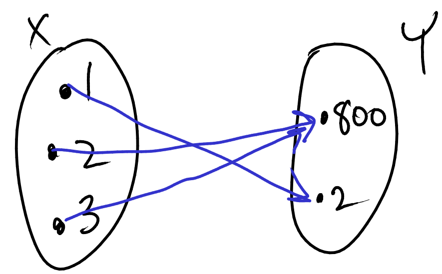
let $f(x) = x^2$

What is f as a set?

$\{(0,0), (1,1), (2,4), (3,9), \ldots\}$

eg let $X = \{1, 2, 3\}$ $\quad Y = \{800, 2\}$

let f be



as a set $\{(1,2), (2,800), (3,800)\}$

eg let $X = \emptyset$, let $Y = \omega$

What can f be?

$$f = \emptyset \subseteq X \times Y = \emptyset \times \omega = \emptyset$$

eg What is $Y^{\emptyset}$ for any set Y

Answer $Y^{\emptyset} = \{\emptyset\}$

eg let $X = \{1, 2\}$ let $Y = \{3, 4\}$

what is Y^X ?

$$\left| \begin{matrix} 1 \mapsto 3 \\ 2 \mapsto 4 \end{matrix} \right| \begin{matrix} 1 \mapsto 3 \\ 2 \mapsto 3 \end{matrix} \left| \begin{matrix} 1 \mapsto 4 \\ 2 \mapsto 3 \end{matrix} \right| \begin{matrix} 1 \mapsto 4 \\ 2 \mapsto 4 \end{matrix} \Big|$$

and that's all

so $Y^X = \{ \{(1,3), (2,4)\}, \{(1,3), (2,3)\}, \{(1,4), (2,3)\},$

$\{(1,4), (2,4)\}\}$

lets remember some function words

domain $\quad \operatorname{dom} f = \{x : \text{for some } y \ (x,y) \in f\}$

range $\quad \operatorname{ran} f = \{y : \text{for some } x \ (x,y) \in f\}$

If $\operatorname{ran} f = Y$ ← codomain then we say f is onto Y

If $X \subseteq Y$ then the function $f : X \to Y$ given by $f(x) = x$ is called the inclusion map

The inclusion map when $X = Y$ is called the identity map

The function $f : X \times Y \to Y$ given by $f(x,y) = y$ is called the projection map onto the second coordinate the function $f: X \times Y \to X$ given by $f(x,y) = x$

is called the projection map onto the first coordinate

If f maps distinct elemts to distinct elemts we say

f is one-to-one

written in function language if $f(a) = f(b)$

then $a = b$

written in set language

for each $y \in Y$

there is $\underbrace{\text{at most one}}$ x with $(x,y) \in f$

equiv. if $(a,c) \in f$ $(b,c) \in f$

then $a = b$

eg let $X = \{1, 2\}$ $\quad Y = \{1, 3\}$

what is projection from $X \times Y$ to Y?

as a set $\{((1,1), 1), ((2,3), 3), ((1,3), 3), ((2,1), 1)\}$

eg is the projection in the previous example onto Y?

Yes

is it one-to-one?

No $\quad$ 2 different things map to 1

② Finite and infinite (This is back to Halmos p52)

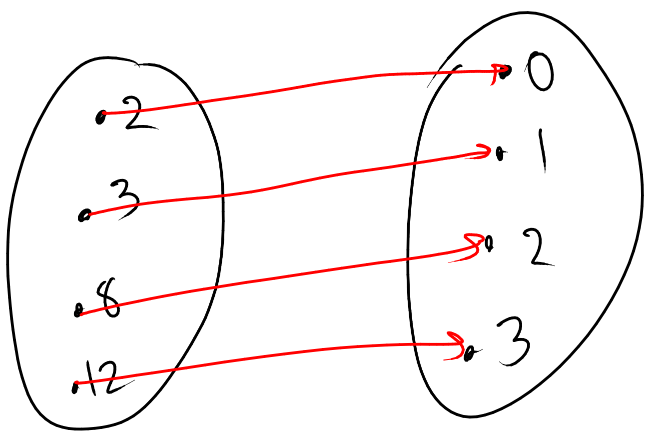
Definition Say two sets A and B are equivalent or the same size if there is a one-to-one and onto function from A to B

could mean different things in different books, contexts

eg is $\{2, 3, 8, 12\}$ equivalent to 4?

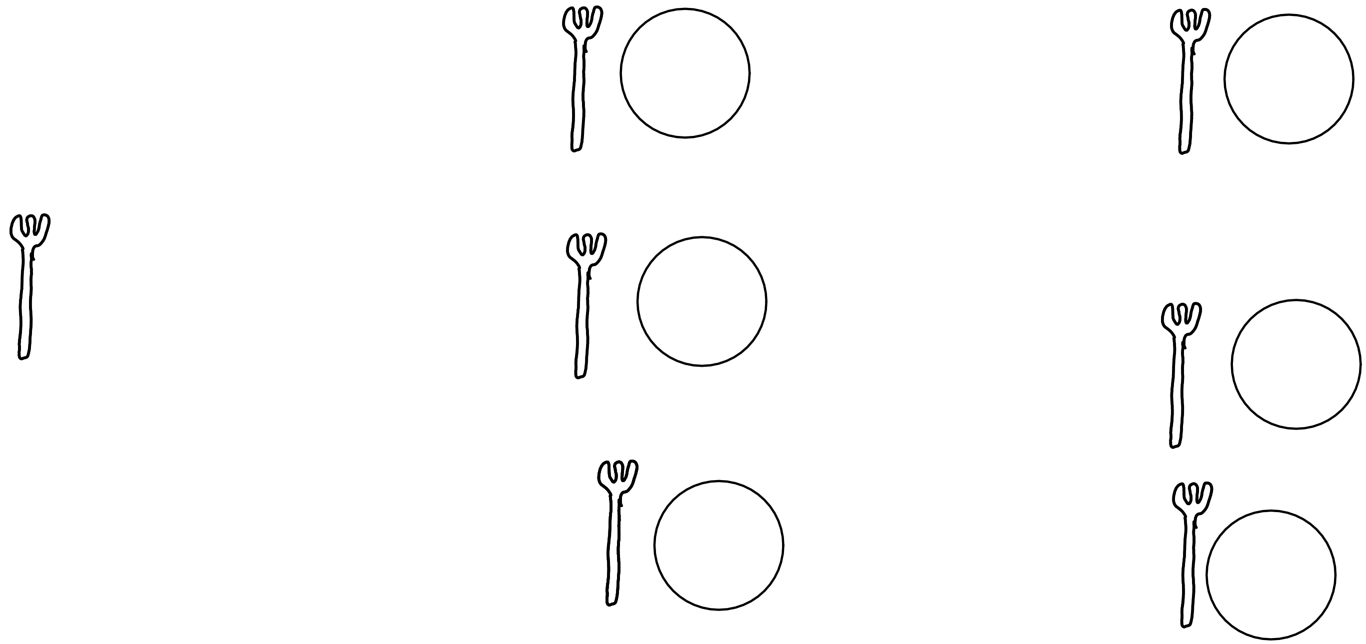
$$4 = \{0, 1, 2, 3\}$$

this is what one-to-one correspondence means



The idea here is to know if two sets have the same number of elements, I don't need to count the elements. I can just pair them up. If I don't have any left overs on either side then I must have had the same number on each side.

eg



For the break

find an example of a set which is equivalent to a proper subset of itself

eg $f: \omega \longrightarrow \{x^2 \mid x \in \omega\} \subsetneq \omega$

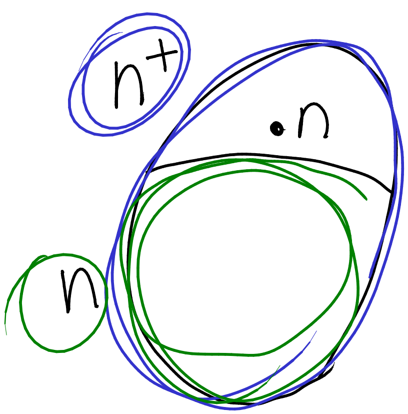
$f(x) = x^2$ is one-to-one and onto

and so ω is equivalent to a proper subset of itself.

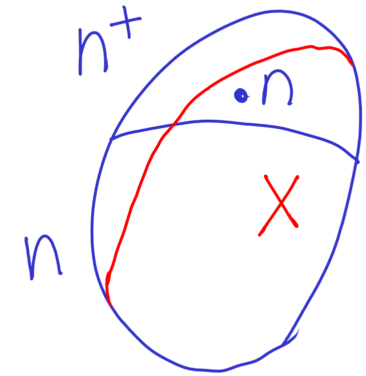
Fortunately this doesn't happen for individual natural numbers

$\boxed{\text{Claim: If } n \in \omega \text{ then } n \text{ is not equivalent to a proper subset of } n}$

proof By induction:

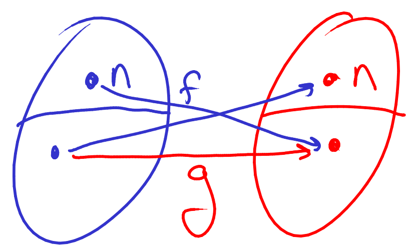


if $n \in X$

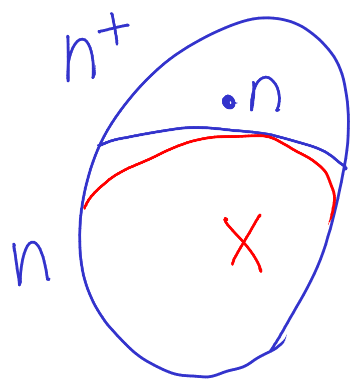


if $f(n) = n$

if $f(n) \neq n$



if $n \notin X$



Claim A set can be equivalent to at most one natural number

proof Let S be equivalent to m and n
$m, n \in \omega$.

> recall equivalent means there's a one-to-one and onto map between them.

Then m is equivalent to n

$$\begin{tikzcd}
m \arrow[rr, bend right, "g\circ f^{-1}"'] & S \arrow[l, "f"'] \arrow[r, "g"] & n
\end{tikzcd}$$

But either $m \subseteq n$ or $n \subseteq m$

and no natural number is equivalent to a proper subset of itself. So $m = n$

Now we can define a set A to be finite if it is equivalent to some natural number, and infinite otherwise

Also define the size or number of elements of a finite set A to be the unique natural number equivalent to A

use the notation $\#A$ for the size of A

This notion of size corresponds to our usual notion of size

for example if $A \subseteq B$ and A and B finite then $\#A \leq \#B$

proof Say $\#A = n$, $\#B = m$

let $f: A \to n$ be a one-to-one and onto map

let $g: B \to m$ be "

Consider g restricted to A

$g : A \to m$ not necessarily onto

but still one-to-one

so let X be the image of A under g

Then $g : A \to X$ is one-to-one and onto

and $\boxed{X} \subseteq m$

So A is equivalent to X

if $\#X > m$ Then a proper subset of m would be equivalent to m. Contradiction

So $\#X \leq m$

so $\#A \leq \#B$.

③ Next time

- Summary of our axioms so far and outlook

- families

Please read Halmos Section 9