Math 303, Fall 2011, Lecture 19

① Review of well orders

Recall that if X with $\leq$ is a partially ordered set and

④ for all $a, b \in X$ $\quad a \leq b$ or $b \leq a$

⑤ for any nonempty $Y \subseteq X$, Y has a least element.
i.e. there is an $a \in Y$ such that $a \leq x$ for all $x \in Y$.

Then we say $\leq$ is a **well order** and we say X is **well ordered** by $\leq$

Note ④ isn't needed because ⑤ implies ④

take $a, b \in X$ $\quad$ let $Y = \{a, b\}$

Then by ⑤ Y has a least element

if the least element is a then $a \leq b$

if the least element is b then $b \leq a$

So you get either $a \leq b$ or $b \leq a$.

Note if X with $\leq$ is a partial order and satisfies ④
then $\leq$ is a **total order** on X

We saw that ω is well ordered by the usual $\leq$

eg Consider $\omega \times \omega$

What ordering shall we use?

(a) say $(a,b) \leq (c,d)$ if $a \leq c$ and $b \leq d$

eg $(1,256) \leq (3,8000000)$

$(1,256) \not\leq (3,4)$

Is $\omega \times \omega$ well ordered?

No. eg $\{(1,256),(3,4)\}$
has no least element
since $(1,256) \not\leq (3,4)$
and $(3,4) \not\leq (1,256)$

(b) Say $(a,b) \leq (c,d)$ if $a \leq c$
or $(a=c$ and $b \leq d)$

This is called lexicographic order because it is how we put things in alphabetical order.

Is $\omega \times \omega$ well ordered?

Yes! Let $Y \subseteq \omega \times \omega$

Let $Z \subseteq Y$ be the set of $(a,b) \in Y$ where a is smallest among all first coordinates of elements of Y

so $Z = \{(a,b), (a,b_1), (a,b_2), \ldots\}$

since ω is well orderd the first coordinates of elements of Y are well ordered so such an a exists.

Furthermore Z has a least element because all elements of Z have the same first coordinate and the second coordinates are well ordered.

Let (a,b) be the least element of Z. All elements of Y are at least as big so (a,b) is the least element of Y.

② Transfinite induction

Well ordered sets have the nice property that we have a notion of induction. First we need the following definition

Definition Let X with $\leq$ be a partially ordered set. Take $a \in X$. Then set

$$s(a) = \{ x \in X : x \boxed{<} a \}$$

is called the *initial segment* of a

eg $X = \omega$ $a = 12$, $\underbrace{s(a) = \{0, 1, 2, \dots, 11\}}$ $\overset{\text{as it turns out}}{=} 12$

eg $E = \{a,b,c\}$, $X = P(E)$ ordered by $\subseteq$,

$$s(\{a,b\}) = \{ \emptyset, \{a\}, \{b\} \}$$

Let X be a well ordered set and let $S \subseteq X$
if for any $x \in X$ it is the case that
$s(x) \subseteq S$ implies $x \in S$
Then $S = X$

This is called the principle of transfinite induction

First lets see that this is true and then see what we can do with it

To check the fact suppose S has the property
but $S \neq X$.

then $X \setminus S$ is nonempty so since X is well ordered $X \setminus S$ has a least element, call it a.

Consider $s(a)$ every $x \in s(a)$ has $x < a$
so by minimality of a, $x \in S$ for all $x \in s(a)$
Thus $s(a) \subseteq S$, so $a \in S$, contradicts $a \in X \setminus S$

so $X - S$ can't be nonempty

so $X = S$

How does this relate to the principle of mathematical induction which we have already seen?

let $X = \omega$

First

Take $0 \in \omega$

$s(0) = \emptyset$

so transfinite induction says

$s(0) \subseteq S$ implies $0 \in S$

but $\emptyset \subseteq S$ for any S

so if the transfinite induction property holds for $S \subseteq \omega$ then automatically $0 \in S$

Next note for $X = \omega$ transfinite induction becomes strong induction, that is to conclude $x \in S$ you need the entire initial segment inside S. I.e. your property needs to hold for all $a < x$.

On the other hand the principle of mathematical induction is weak induction, that is, to conclude

$x \in S$ you only need $(x-1) \in S$

For ω strong and weak induction are equally powerful, they can prove the same statements.

But for other well ordered sets transfinite induction is necessary

eg let $X = \omega^+ = \omega \cup \{\omega\} = \{0, 1, 2, \ldots, \omega\}$

use $\leq$ given by, for $m, n \in \omega$

let $m \leq n$ in ω^+ if $m \leq n$ in ω and vice versa

for $m \in \omega$, let $m \leq \omega$.

This is a well order.

Suppose we try to use the old principle of mathematical induction on X. What goes wrong?

For the break

$\boxed{\text{Can you find an } S \subsetneq \omega^+ \text{ with } 0 \in S \text{ and for all } n \in S,\ n^+ \in S\text{?}}$

answer ω The usual principle of mathematical induction is not strong enough to distinguish ω from ω^+

Transfinite induction fixes this problem

$$s(x) \subseteq S \Rightarrow x \in S$$

for ω: what is $s(\omega)$ in ω^+

$$s(\omega) = \{0, 1, 2, 3, \ldots\}$$

so $s(\omega) = \omega$ so certainly $s(\omega) \subseteq \omega$

but $\omega \notin \omega$

and so transfinite induction distinguishes ω from ω^+

One more note: transfinite induction also allows us to take base cases other than 0 without reindexing

How? just use the well ordered set $\{1, 2, \ldots\}$ (or $\{5, 6, \ldots\}$ or wherever you want).

③ Ordinals

We had

$$0,\quad 0^+ = 1,\quad 1^+ = 2,\quad 2^+ = 3,\quad 4,\quad 5,\quad \ldots$$

all together we have

$$\{0, 1, 2, 3, \ldots\} = \omega$$

Now consider

$$\omega, \quad \omega^+, \quad (\omega^+)^+, \quad ((\omega^+)^+)^+, \quad (((\omega^+)^+)^+)^+, \quad \ldots$$

what's next? Idea what is the next number after this, in the same way ω was after all natural numbers.

It should be $\{0, 1, 2, \ldots, \omega, \omega^+, (\omega^+)^+, \ldots\}$ but is this a set?

Suppose f is a function with domain $n \in \omega$

Say f is an ω-successor function if $f(0) = \omega$ & $f(m^+) = (f(m))^+$

eg $n = 3 = \{0, 1, 2\}$.

$$f(0) = \omega$$
$$f(1) = \omega^+ \qquad f(1) = f(0^+) = (f(0))^+ = \omega^+$$
$$f(2) = (\omega^+)^+$$

then f is an ω-successor function.

In fact for each n there is a unique ω-successor function. Intuition we never had a chance to make a choice in defining f, so it must have been unique.

Suppose f and g were both ω-successor functions with domain n

- $f(0) = \omega = g(0)$ so they agree on 0
- Let $i \in n$ be the $\boxed{\text{smallest number}}$ for which $f(i) \neq g(i)$.

  using the well ordering of ω also need it to start at 0

- $i \neq 0$ so $i = j^+$ for some $j \in \omega$ $(j = i-1)$

  but i was the smallest number where they disagreed so $f(j) = g(j)$

  so $f(i) = f(j^+) = f(j)^+ = g(j)^+ = g(j^+) = g(i)$

  contradicting $f(i) \neq g(i)$.

Thus f is unique.

What we want is to join all those things together

ie show $\{0, 1, \ldots, \omega, \omega^+, (\omega^+)^+, \ldots\}$ is a set

Let $S(n, x)$ be the property

"$n \in \omega$ and x is in the range of an ω-successor function with domain n"

in logic it can be done

but lets not bother

7 vs 2

The set we are looking for is

$$\{x : \exists n\,(n \in \omega \wedge S(n,x))\}$$

We only know this is a set by the axiom of replacement

Intuitively S is acting like a function
it takes n to the range of the unique ω-successor function defined on n
Call this function $F(n) = \{x : S(n,x)\}$
We want to know
either (a) F is a function in the set theoretic sense (i.e. can write the set of ordered pairs defining F)
or (b) The image of any $X \subseteq \omega$ under F is a set.

These are equivalent. If F is a set theoretic function then its range is a set and can pull out ranges of subsets by the axiom of subset selection

If the images are sets, then the range of F, Y is a set and so have $\omega \times Y$ and can pull out F using the axiom of subset selection

(b) would come from Cohen's version of the axiom of replacement

(a) is Halmos' version which he calls the axiom of substitution

Axiom of Substitution If $S(a,b)$ is a sentence such that for each $a \in A$ the set $\{b : S(a,b)\}$ can be formed, then there exists a function F with domain A such that $F(a) = \{b : S(a,b)\}$ for each $a \in A$

Our use of the axiom of substitution will be, as above to extend our ability to count beyond ω, ω^+, ...

So we have

$$0, 1, 2, 3, \ldots\ldots$$

$$\omega, \omega^+, (\omega^+)^+, ((\omega^+)^+)^+, \ldots \qquad (*)$$

and by the above we can define a set theoretic function F with domain ω such that

$$F(0) = \omega, \quad F(n^+) = (F(n))^+$$

let X be the range of F then $X = \{\omega, \omega^+, (\omega^+)^+, \ldots\}$

Then the next number after the ones in $(*)$ is

$$X \cup \omega = \{0, 1, 2, \ldots, \omega, \omega^+, (\omega^+)^+, \ldots\}$$

So this really is a set

Note we went to all that work just to show $X \cup \omega$ is a set

What are these new bigger counting "numbers"

ordinals

Definition An ordinal is a well ordered set S such that for all $x \in S$ $s(x) = x$

eg lets check 3 is an ordinal

$3 = \{0, 1, 2\}$ ordered in the usual way $0 \leq 1 \leq 2$

$s(0) = \emptyset = 0$

$s(1) = \{0\} = 1$

$s(2) = \{0, 1\} = 2$

Likewise every natural number is an ordinal.

eg check ω is an ordinal.

Two useful facts

① If X is an ordinal then X^+ is an ordinal

proof Use the order on X^+ given by

This is a well order as

Finally we can check the ordinal property.

② Let X be a set. There is at most one well order which makes X into an ordinal

proof Suppose there is a well order which makes X into an ordinal. Take any other well order of X

④ Next time

More on ordinals

please read Halmos section 20