ASSIGNMENT 3

MATH 303, FALL 2011

If you find any errors please let me know.

Manipulation

(M1) (a) $3 = \{0, 1, 2\}$
(b) $3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$.

(M2) $\{0, 3, 4\}^+ = \{0, 3, 4\} \cup \{\{0, 3, 4\}\} = \{0, 3, 4, \{0, 3, 4\}\}$

(M3)

$$\begin{aligned} Y^X = \{&\{(a, c), (b, c)\}, \{(a, c), (b, d)\}, \{(a, c), (b, e)\}, \\ &\{(a, d), (b, c)\}, \{(a, d), (b, d)\}, \{(a, d), (b, e)\}, \\ &\{(a, e), (b, c)\}, \{(a, e), (b, d)\}, \{(a, e), (b, e)\}\} \end{aligned}$$

(M4)

$$\begin{aligned} X^Y = \{&\{(c, a), (d, a), (e, a)\}, \{(c, a), (d, a), (e, b)\} \\ &\{(c, a), (d, b), (e, a)\}, \{(c, a), (d, b), (e, b)\} \\ &\{(c, b), (d, a), (e, a)\}, \{(c, b), (d, a), (e, b)\} \\ &\{(c, b), (d, b), (e, a)\}, \{(c, b), (d, b), (e, b)\}\} \end{aligned}$$

(M5) The set $\{(a, c), (1, c), (b, 2), (2, 3), (a, 4)\}$ is not a function because it contains both (a, c) and $(a, 4)$ and thus a maps to both 4 and c, which is not possible for a function.

(M6) Define the map $f : \{a, b, c\} \to 3$ by $f(a) = 0$, $f(b) = 1$, $f(c) = 2$. Then f is onto 3 as $3 = \{0, 1, 2\}$ and f is one-to-one as there are no two distinct elements of $\{a, b, c\}$ mapping to the same element of 3.

Pure Math

(P1) **(4 points)**
(a)

$$\begin{aligned} \bigcup 4 &= 0 \cup 1 \cup 2 \cup 3 \\ &= \emptyset \cup \{0\} \cup \{0, 1\} \cup \{0, 1, 2\} \\ &= \{0, 1, 2\} \\ &= 3 \end{aligned}$$

(b) *Most of the time the assignment was up this question was posed to that you needed to start your induction at 1 rather than 0. Thus, it is ok if you did that and also ok if you reindexed to start at 0.*
Let S be the set of natural numbers n to that $\bigcup(n + 1) = n$.
First notice that $\bigcup 1 = \emptyset = 0$ so $0 \in S$.

Now suppose $n \in S$. Then $n + 1 = n^+ = n \cup \{n\}$. So

$$\begin{aligned} \bigcup(n+1) &= \left(\bigcup n\right) \cup n \\ &= (n-1) \cup n \qquad \text{since } n \in S \\ &= n \qquad \text{since } (n-1) \subseteq n \end{aligned}$$

Thus $n^+ \in S$ and so by the principle of mathematical induction $S = \omega$, and so for all $n \in \omega$ we have $\bigcup(n+1) = n$.

(P2) $\omega^+ = \omega \cup \{\omega\}$ and note that $\omega \notin \omega$, so the elements of ω^+ are all the elements of ω along with the element ω. Define the following function f from ω^+ to ω. Let $f(n) = n^+$ for $n \in \omega$ and let $f(\omega) = 0$.

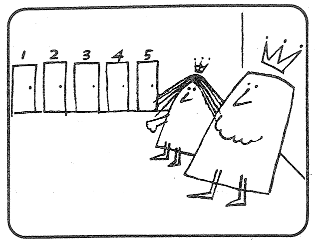
By construction f is a function between the correct sets. f is onto because if we take any $m \in \omega$ then either $m = n^+$ for some n (namely $n = m - 1$) and so $f(n) = m$, or $m = 0$ and so $f(\omega) = m$. f is one-to-one because if $f(a) = f(b)$ then either $f(a) = f(b) = 0$ so $a = b = \omega$ (since 0 is not the successor of any natural number) or $a^+ = b^+$ and so by a result from class $a = b$.

Ideas

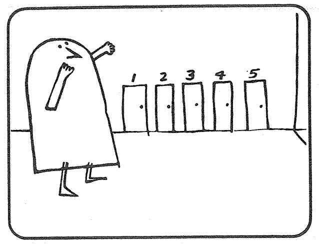
(I1) *There are lots of possible answers; either yes or no can be correct. Perhaps the more subtle answer is yes; this is Martin Gardner's answer – see the attached scan (from Aha Gotcha by Martin Gardner, WH Freeman, (1982)).*

(I2) *Answers will vary*

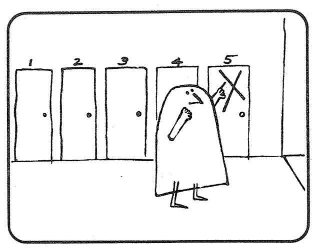
The Unexpected Tiger



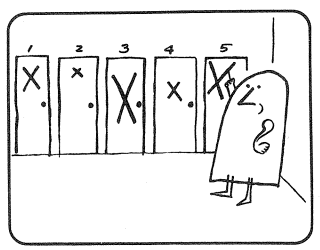
Princess: You're the king, father. May I marry Michael?
King: My dear, you may if Mike kills the tiger behind one of these five doors. Mike must open the doors in order, starting at 1. He won't know what room the tiger's in until he opens the right door. It will be an *unexpected* tiger.



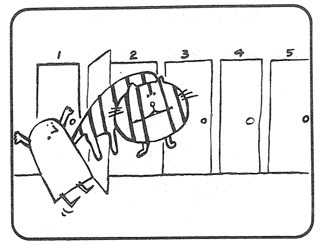
When Mike saw the doors he said to himself:
Mike: If I open four empty rooms I'll *know* the tiger's in room 5. But the king said I wouldn't know in advance. So the tiger *can't* be in room 5.



Mike: Five is out, so the tiger must be in one of the other four rooms. What happens after I open three empty rooms? The tiger will have to be in room 4. But then it won't be unexpected. So 4 is out too.



By the same reasoning, Mike proved the tiger couldn't be in room 3, or 2, or 1. Mike was overjoyed.
Mike: There's no tiger behind *any* door. If there were, it wouldn't be unexpected, as the king promised. And the king *always* keeps his word.



Having proved there was no tiger, Mike boldly started to open the doors. To his surprise, the tiger leaped from room 2. It was completely unexpected. The king had kept his word. So far logicians have been unable to agree on what is wrong with Mike's reasoning.

The paradox of the unexpected tiger has many other story forms. Of unknown origin, it first appeared in the early 1940s as a paradox about a professor who announced that an "unexpected examination" would be given on one day of the following week. He assured his students that no one could deduce the day of the examination until the day it occurred. A student "proved" it couldn't be on the last day of the week, or the next-to-last, or the day before that, and so on for all days of the week. Nevertheless, the professor was able to keep his word by giving the examination on, say, the third day.

When the Harvard University philosopher W. V. Quine wrote a paper about the paradox in 1953, it took the form of a warden who scheduled an unexpected hanging for a prisoner. For a discussion of the paradox, and a bibliography of 23 references, see the first chapter of my book, *The Unexpected Hanging and Other Mathematical Diversions.*

Most people admit that the first step in Mike's reasoning is correct, namely that the tiger cannot be in the last room. But once this is admitted as a sound deduction, the rest of Mike's reasoning seems to follow. For if the tiger cannot be in the last room, then identical reasoning rules out the next-to-last, and so on for the others.

However, even the first step of Mike's reasoning is faulty. Suppose he has opened all doors but the last. Can he deduce correctly that there is no tiger in the last room? No, because if he makes such a deduction, he might open the door and find an unexpected tiger! Indeed, the entire paradox holds even if only one room is involved.

Suppose Mr. Smith, who you believe always speaks truly, hands you a box and says, "Open it and inside you will find an unexpected egg." What can you deduce about the presence or absence of an egg in the box? If Smith is correct, the box must contain an egg, but then you will expect the egg and therefore Smith's statement is false. On the other hand, if this contradiction prompts you to deduce that the box cannot contain an egg (in which case Smith spoke falsely) and you open it to find an unexpected egg, then Smith spoke truly.

The consensus among logicians is that although the king knows he can keep his word, there is no way that Mike can know it. Therefore, there is no way he can make a valid deduction about the absence of the tiger in any room, including the last one.