ASSIGNMENT 2 – SOLUTIONS

MATH 303, FALL 2011

If you find any errors please let me know.

Manipulation

(M1)

$$\begin{aligned}(A' \cup B)' &= (\{b,d,q\} \cup \{b,d\})' \\ &= (\{b,d,q\})' \\ &= \{a\} = A\end{aligned}$$

(M2)

$$\begin{aligned}A - B &= \{1,\{3,8\},4\} - \{1,2,3,4,5,6,7,8\} \\ &= \{\{3,8\}\}\end{aligned}$$

(M3) Yes it is an ordered pair. Let us unravel this set. It has two elements, one of which is a singleton containing $\{\{\emptyset\}\}$, and so the first coordinate of the pair is $\{\{\emptyset\}\}$. The other element is a set with two elements, one of which is $\{\{\emptyset\}\}$ and the other is $\{\emptyset\}$, thus the second coordinate is $\{\emptyset\}$, and the whole thing is an ordered pair.

(M4) No this set is not an ordered pair. It has a single element, so if it were an ordered pair it would be an ordered pair (c,c) for some c. Thus this set would be $\{\{c\}\}$. However the single element of the given set is not a singleton, and so it is not an ordered pair.

(M5) Lets work from the inside out:

$$\begin{aligned}A \cup B &= \{a,b\} \\ \mathcal{P}(A \cup B) &= \big\{\emptyset, \{a\}, \{b\}, \{a,b\}\big\} \\ \mathcal{P}(\mathcal{P}(A \cup B)) &= \Big\{\emptyset, \{\emptyset\}, \boxed{\{\{a\}\}}, \{\{b\}\}, \{\{a,b\}\}, \\ &\quad \{\emptyset,\{a\}\}, \{\emptyset,\{b\}\}, \{\emptyset,\{a,b\}\}, \{\{a\},\{b\}\}, \boxed{\{\{a\},\{a,b\}\}}, \{\{b\},\{a,b\}\}, \\ &\quad \{\emptyset,\{a\},\{b\}\}, \{\emptyset,\{a\},\{a,b\}\}, \{\emptyset,\{b\},\{a,b\}\}, \{\{a\},\{b\},\{a,b\}\}, \\ &\quad \{\emptyset,\{a\},\{b\},\{a,b\}\}\Big\}\end{aligned}$$

The elements in boxes are the ones in $A \times B$.

Pure Math

(P1) Take E and $A, B \subseteq E$.

- Claim $A - B = A \cap B'$
  Proof: suppose $x \in A - B$, then $x \in A$ but $x \notin B$. Since $A \subseteq E$ we also have $x \in E$, thus $x \in A$, and $x \in E$ but $x \notin B$. Thus $x \in A$ and $x \in B'$. Thus $x \in A \cap B'$.
  Now suppose $x \in A \cap B'$, then $x \in A$ and $x \in B'$. Thus $x \in A$ and $x \in E$ but $x \notin B$. In particular then $x \in A$ but $x \notin B$, and so $x \in A - B$.
- Claim $A \subseteq B$ if and only if $A - B = \emptyset$.
  Proof: suppose $A \subseteq B$. Suppose $x \in A - B$. Then $x \in A$, but $x \notin B$. However $A \subseteq B$, so $x \in A$ implies $x \in B$. This is a contradiction so there is no x in $A - B$. Thus $A - B = \emptyset$.
  Suppose $A - B = \emptyset$. Take $x \in A$. If $x \notin B$ then $x \in A - B$ but this is impossible, thus $x \in B$. So $A \subseteq B$.
- Claim $A - (A - B) = A \cap B$.
  Proof: suppose $x \in A - (A - B)$. Then $x \in A$ but $x \notin A - B$. But the only way to be in A but not in $A - B$ is to be in B, so $x \in B$. Thus $x \in A \cap B$.
  Suppose $x \in A \cap B$. Then $x \in A$ and $x \in B$. Thus $x \notin A - B$, but again $x \in A$, so $x \in A - (A - B)$.
- Claim $A \cap (B - C) = (A \cap B) - (A \cap C)$
  Proof: suppose $x \in A \cap (B - C)$. Then $x \in A$ and $x \in B - C$. Thus $x \in A$ and $x \in B$ but $x \notin C$. $x \in A$ and $x \in B$ gives $x \in A \cap B$, while $x \in A$ but $x \notin C$ gives $x \notin A \cap C$. Together we get $x \in (A \cap B) - (A \cap C)$.
  Suppose $x \in (A \cap B) - (A \cap C)$. Then $x \in A \cap B$ but $x \notin A \cap C$. From $x \in A \cap B$ we get $x \in A$ and $x \in B$. Since $x \in A$, the only way to have $x \notin A \cap C$ is $x \notin C$. Thus $x \in A$ and $x \in B$ but $x \notin C$, thus $x \in A$ and $x \in B - C$, thus $x \in A \cap (B - C)$.
- Claim $A \cap B \subseteq (A \cap C) \cup (B \cap C')$.
  Proof: suppose $x \in A \cap B$. Then $x \in A$ and $x \in B$. There are two cases, either $x \in C$ or $x \in C'$. First suppose $x \in C$. Then $x \in A$ and $x \in C$ so $x \in A \cap C$ and so $x \in (A \cap C) \cup (B \cap C')$.
  On the other hand suppose $x \in C'$, then $x \in B$ and $x \in C'$, so $x \in B \cap C'$ and so $x \in (A \cap C) \cup (B \cap C')$. In both cases $x \in (A \cap C) \cup (B \cap C')$ and so $A \cap B \subseteq (A \cap C) \cup (B \cap C')$.
- Claim $(A \cup C) \cap (B \cup C') \subseteq A \cup B$.
  Proof: suppose $x \in (A \cup C) \cap (B \cup C')$. Then $x \in A \cup C$ and $x \in B \cup C'$. This gives that

$$x \in A \text{ or } x \in C' \tag{1}$$

  and

$$x \in B \text{ or } x \in C' \tag{2}$$

  Again lets break this up depending on whether or not x is in C.
  If $x \in C$, then $x \notin C'$, so from (1) $x \in A$, and thus $x \in A \cup B$. If $x \in C'$ then $x \notin C$, so from (2) $x \in B$, and thus $x \in A \cup B$. Therefore in all cases $x \in A \cup B$.

(P2) First I claim that $\bigcup \mathcal{P}(E) = E$. To see this first note that $E \in \mathcal{P}(E)$, so $E \subseteq \bigcup \mathcal{P}(E)$. Furthermore, every element of $\mathcal{P}(E)$ is a subset of E, and so their union is also a subset of E. Thus $\bigcup \mathcal{P}(E) \subseteq E$. Together these give that $\bigcup \mathcal{P}(E) = E$.

Now consider $\mathcal{P}(\bigcup E)$. Take any $e \in E$. Then $e \subseteq \bigcup E$, and so $e \in \mathcal{P}(\bigcup E)$. So $E \subseteq \mathcal{P}(\bigcup E)$. However, typically E is a proper subset.

Specifically, if there is a $T \subseteq \bigcup E$ with $T \notin E$ then T is in $\mathcal{P}(\bigcup E)$ but not in E. If every subset of $\bigcup E$ is in E, then $\bigcup E$ is in E and so are all its subsets, and so $E = \mathcal{P}(\bigcup E)$. Thus $\bigcup \mathcal{P}(E) = E = \mathcal{P}(\bigcup E)$ if and only if E contains as an element every subset of $\bigcup E$.

(P3) (a) There are many possibilities, for example let $A = \{1\}$ and let $B = \{2\}$, then $A \times B = \{(1, 2)\}$ and $B \times A = \{(2, 1)\}$.

(b) If $A = B$ then $A \times B = A \times A = B \times A$. Also if $A =$ then $A \times B == B \times A$ and similarly if $B =$.

In fact these are the only conditions where $A \times B = B \times A$. To see that suppose A and B are non empty and not equal, then either there is some $a \in A$ with $a \notin B$ or some $b \in B$ with $b \notin A$. In the first case then take any other $c \in B$ and we get $(a, c) \in A \times B$, but not in $B \times A$ since $a \notin B$. Argue similarly in the other case.

Ideas

(I1) *Many things are possible, all you need is that* $(a, b) = (c, d)$ *implies* $a = c$ *and* $b = d$ *and* $(a, b) \neq (b, a)$ *for* $a \neq b$.

(I2) Let $E = \omega$. Let $A_p = \{2p, 3p, 4p, \ldots\}$ be the multiples of p other than p itself. Let $\mathcal{C} = \{A'_2, A'_3, A'_5, A'_7, \ldots\}$. Then the sieve of Eratosthenes gives the primes as $\bigcup \mathcal{C} - \{0, 1\}$. Note that the sieve of Eratosthenes is actually more than this as it tells us that when we do this iteratively, then the next prime is the first element still there (and so it generates the primes, without knowing them in advance), but none-the-less it is an intersection-type process.

(I3) **(3 points)** De Morgan's laws for logic have a very similar form to those for sets, negation takes the place of complementation and "and" and "or" take the place of intersection and union. Furthermore we use the one to prove the other, see the attached pages for where we used De Morgan's laws for logic in the proof of De Morgan's laws for sets.

7 De Morgan's Laws

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

Suppose $x \in (A \cup B)'$

So $x \in \mathcal{E}$ and $x \notin A \cup B$

So $x \in \mathcal{E}$ and $(x \notin A$ and $x \notin B)$

so $\boxed{x \in \mathcal{E} \text{ and } x \notin A}$ and $x \notin B$

so $x \in A'$ and $x \in B'$

so $x \in A' \cap B'$

Conclusion: $\boxed{(A \cup B)' \subseteq A' \cap B'}$

Now suppose $x \in A' \cap B'$.

so $x \in A'$ and $x \in B'$

so $(x \in E$ and $x \notin A)$ and $(x \in E$ and $x \notin B)$

so $x \in E$ and $\boxed{x \notin A \text{ and } x \notin B}$

so $\boxed{x \in E \text{ and } x \notin A \cup B}$

so $x \in (A \cup B)'$

Conclude $\boxed{A' \cap B' \subseteq (A \cup B)'}$

Similarly for the other one.

gives $(A \cup B)' = A' \cap B'$