Summary: Section 9.2

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We shall restrict ourself to two variables, since the extension is by induction it takes little to move to more variables. We will also assume $F = \frac{G}{H} \in \mathbb{C}(x, y)$.

Last time we found when we choose a direction $\delta_* = (r_*, s_*)$ then we have an associated height function $h_* : \text{Relog } \mathbb{V} \to \mathbb{R}$, $\mathbb{V} = \mathbb{V}(h)$.

Theorem 0.1 (Theorem 9.2.1). *If $F \in \mathbb{C}(x, y)$ with $\delta_* = (r_*, s_*)$, h_* the associated height function and h_* has a strictly minimal smooth point, z_*, then*

$$a_{r,s} \sim_{\substack{r+s\to\infty \\ (r,s)\|\delta_*}} \frac{1}{2\pi i} \int_{\mathcal{N}} x^{-r} \phi(x)^{-s} \Psi(x) \frac{dx}{x},$$

where

$$\Psi(x) = Res \left\{ \frac{F(x, y)}{y}; y = \phi(x) \right\}.$$

Proof. See above Theorem 9.2.1 in Pemantle and Wilson.

The general idea of the proof is to calculate the residue of $\left(\int_{C_-} - \int_{C_+} \right) \omega$. □

Example 1. *Let $H = 1 - x - y$ and $\delta_* = (r, s)$, $(x_*, y_*) = (\frac{r}{r+s}, \frac{s}{r+s})$.*

Then $\phi(x) = 1 - x$ which is an entire function.

Take $\mathcal{N}$ small around $\frac{r}{r+s}$ on $\delta_* = (r, r)$ then the approximation is good until $r \approx 75$.

If you take $\mathcal{N} = T$ then $\int_{\mathcal{N}} = a_{r,s}$.

Now we consider the case where $z_* = (x_*, y_*)$ is a smooth critical point for h_* and $\mathbb{V} \cap T(\log |x_*|, \log |y_*|) = E$ is a finite set of smooth points of $\mathbb{V}$.

We construct the following tori in Figure **??**:

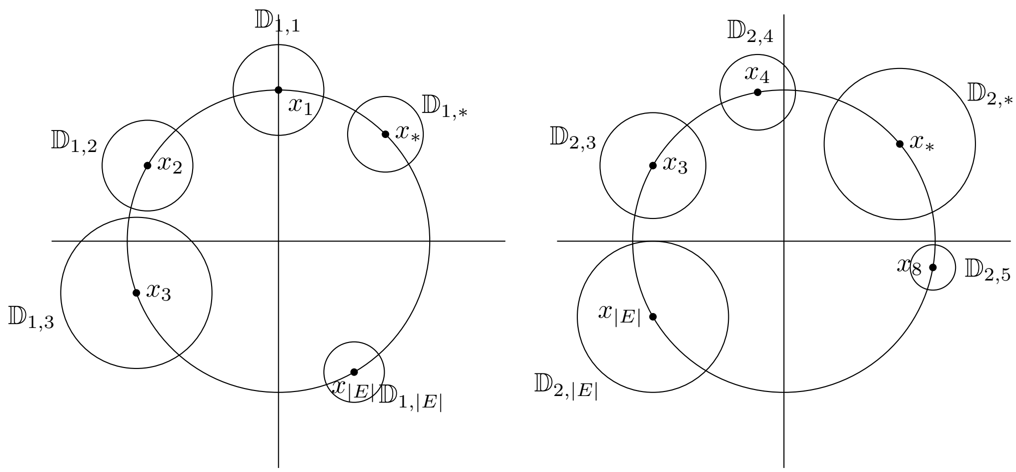
For each $p \in E$ we get $\phi_p : \mathbb{D}_{i,p} \to \mathbb{D}_{2,p}$ mapping x-coordinates of poles to y-coordinates of poles and a neighbourhood $\mathcal{N}_p = T(\log |x_*|) \cap \mathbb{D}_{1,p}$. With these we get:

Corollary 0.2 (Corollary 9.2.3). *Let $F \in \mathbb{C}(x, y)$ with δ_* a fixed direction and h_* is the associated height function. If z_* is a smooth critical point and E, $\mathbb{D}_{i,p}$, ϕ_p and $\mathcal{N}_p$ are as above then*

$$a_{r,s} \sim_{\substack{r+s\to\infty \\ (r,s)\|\delta_*}} \frac{1}{2\pi i} \sum_{p\in E} \int_{\mathcal{N}_p} x^{-r} \phi_p(x)^{-s} \Psi_p(x) \frac{dx}{x},$$

where

$$\Psi_p(x) = Res \left\{ \frac{F(x, y)}{y}; y = \phi_p(x) \right\}.$$



(a) x-plane (b) y-plane

Figure 1: The tori constructed for the case of multiple smooth points

Next we will consider the case for torally minimal points.

Definition 0.3. Let $T(u,v)$ be a torus. If H is an analytic function and $\mathbb{V}(H)$ its variety then we say H satisfies the *torality hypothesis* (TH) on $T(u,v)$ if

$$z = (x,y) \in \mathbb{V}(H) \implies (\text{Relog } x = u \implies \text{Relog } y = v)\,.$$

This condition allows us to use the same annulus in Theorem 9.2.1 to extend the theorem.

Example 2. *$H(x,y) = (1-x-y)$ does not satisfy TH on $T(\log 1, \log 1)$ since $|x| = 1$ implies that $|1-y| = 1$.*

Consider $H(x,y) = 1-x-y+xy$. When $H(x,y) = 0$. $y = \frac{x-1}{x-1} = 1$ and $x = \frac{y-1}{y-1} = 1$ so H satisfies TH on $T(\log 1, \log 1)$.

So we get the corollary:

Corollary 0.4 (Corollary 9.2.4)**.** *Suppose that $F = \mathbb{C}(x,y)$, δ_* is a fixed direction, h_* is the associated height function, z_*, a smooth critical point for h_* and H satisfies TH on $T = T(\log|x_*|, \log|y_*|)$ then*

$$a_{r,s} \sim_{\substack{r+s\to\infty \\ (r,s)\|\delta_*}} \frac{1}{2\pi i}\int_T x^{-r}\phi(x)^{-s}\Psi(x)\frac{dx}{x},$$

for an appropriate Ψ.

We are primarily interested in how this relates to saddle point integrals which are of the form:

$$\int_C e^{-\lambda f(\theta)}A(\theta)d\theta.$$

We will apply a change of variables to get our integrals in the above form.

For $\delta_* = (r_*, s_*)$ a fixed direction and x_*, y_* a strictly minimal point of H_* the associated height function we make the change of variables $x = x_* e^{i\theta}$ giving $dx = ixd\theta$. Let $\mathcal{N}'$ be the diffeomorphic image of $\mathcal{N}$ resulting from our change of variables.

$x = x_* e^{i\theta}$ is merely a rotation of x_* by θ so $\mathcal{N}' = [-\epsilon, \epsilon]$ for some $\epsilon > 0$.

We define $f := \log\left({}^{\phi}/_{\phi_*}\right)$ centered by $\frac{ir\theta}{s}$, where $\phi_* = \phi(x_*)$ and $A := \Psi$. In other words:

$$f(\theta) = \log(\frac{\phi(x_* e^{i\theta})}{\phi(x_*)}) + \frac{ir\theta}{s}$$

and

$$A(\theta) = \Psi(x_* e^{i\theta}).$$

Proposition 0.5 (Proposition 9.2.5 (reduction to Fourier-Laplace Integral))**.** *Let $F \in \mathbb{C}(x, y)$ and (x_*, y_*) be a strictly minimal point for a height function of a fixed direction, δ_*. Let f, A and $\mathcal{N}'$ be as above then*

$$a_{r,s} \sim \frac{1}{2\pi} x_*^{-r} y_*^{-s} \int_{\mathcal{N}'} e^{sf(\theta)} A(\theta) d\theta.$$

Furthermore when $(r, s) \parallel \delta_*$ then f vanishes to at least order 2 at $\underline{0}$.

Proof. See proof of Proposition 9.2.5 in Pemantle and Wilson.

To prove the first part just apply the change of variables to the result of Theorem 9.2.1.

To prove the second part we just need to show that $f(0) = 0$ and $\left.\frac{df}{d\theta}\right|_0 = 0$. ∎