Chapter 4: Fourier-Laplace Integrals in One Variable, Part 1 Summary

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Recall the classical saddle point integral result:

$$\int_\gamma A(z)e^{-\lambda\phi(z)}dz \sim A(z_0)\sqrt{\frac{2\pi}{\phi''(z_0)\lambda}}e^{-\lambda\phi(z_0)},$$

where the asymptotic estimate is as $\lambda \to \infty$ and z_0 is chosen such $\phi'(z_0) = 0$ and the modulus of the integrand is maximised at that point.

In the above integral, the function $A(z)$ is called the *amplitude* of the integrand and $\phi(z)$ is called the *phase* (when $\phi = i\rho$ is purely imaginary, then ρ is the called the phase).

We follow the exposition of Sections 4.1 and 4.2 of [1] to show the following result in the case of real amplitude and phase.

Theorem 1. *Let A and ϕ be analytic on a neighbourhood $N \subset \mathbb{C}$ containing the origin. let*

$$\begin{aligned} A(z) &= \sum_{j=l}^{\infty} b_j z^j \quad (b_l \neq 0), \\ \phi(z) &= \sum_{j=k}^{\infty} c_j z^j \quad (c_k \neq 0) \end{aligned}$$

be the power series for A and ϕ at 0. Let $\gamma : [-\epsilon, \epsilon] \to \mathbb{C}$ be a smooth curve with $\gamma(0) = 0$, $\gamma'(0) \neq 0$ and $\Re(\phi(\gamma(t))) \geq 0$ with equality only for $t = 0$.

Define

$$\begin{aligned} I_+(\lambda) &:= \int_{\gamma|_{[0,\epsilon]}} A(z)e^{-\lambda\phi(z)}dz, \\ I(\lambda) &:= \int_\gamma A(z)e^{-\lambda\phi(z)}dz, \end{aligned}$$

$$C(k,m) := \frac{\Gamma\left(\frac{1+m}{k}\right)}{k}.$$

Then

$$\begin{aligned} I_+(\lambda) &\sim \sum_{j=l}^{\infty} a_j C(k,j)(c_k\lambda)^{-(1+j)/k}, \\ I(\lambda) &\sim \sum_{j=l}^{\infty} \alpha_j C(k,j)(c_k\lambda)^{-(1+j)/k}, \end{aligned}$$

where:

1. *a_j and α_j are explicitly constructed polynomials in $b_l, ..., b_j, c_k^{-1}, c_k, ..., c_{k+j-1}$ ($a_l = b_l$ and $\alpha_j = 2a_j$ if j, k even, see [1] for more detail);*

2. the k^{th} root is determined by the principal root of $x^{-1}(c_k \lambda x^k)^{1/k}$, where $x = \gamma'(0)$.

Before moving on to a proof for real amplitude and phase, we show that this is equivalent to the classical first order approximation we gave at the beginning when $l = 0$, $k = 2$ and $z_0 = 0$ and give an example showing an application of this theorem to a generating function.

$$
\begin{aligned}
I(\lambda) &= \int_\gamma A(z) e^{-\lambda\phi(z)} dz \\
&\sim \alpha_l C(k,l)(c_k\lambda)^{-(1+l)/k} \\
&\sim \alpha_l C(2,0)(c_2\lambda)^{-(1)/2} \\
&\sim 2b_0 \left(\frac{Gamma(1/2)}{2}\right)\sqrt{\frac{1}{c_2\lambda}} \\
&\sim b_0\sqrt{\pi}\sqrt{\frac{1}{\frac{\phi''(0)\lambda}{2}}} \\
&\sim A(0)\sqrt{\frac{2\pi}{\phi''(0)\lambda}} e^{-\lambda\phi(0)}.
\end{aligned}
$$

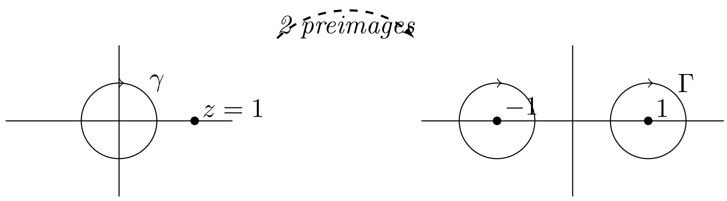
Example 2. *Let $f(z) = (1-z)^{-1/2}$. The binomial theorem tells us*

$$[z^n]f(z) = (-1)^n \binom{-1/2}{n} \sim \sqrt{\frac{1}{\pi n}}.$$

We prove this using our theorem. First, make the change of variables $z = 1 - y^2$. Then $dz = -2ydy$, and the Cauchy theorem states

$$
\begin{aligned}
f_n &= \frac{1}{2\pi i}\int_\gamma z^{-(n+1)}(1-z)^{-1/2} dz, \\
&= \frac{1}{2\pi i}\int_\Gamma (1-y^2)^{-(n+1)} y^{-1}(-2y) dy, \\
&= \frac{i}{\pi}\int_\Gamma (1-y^2)^{-(n+1)} dy.
\end{aligned}
$$

Under the change of variables, we have two choices for the preimage of γ. We choose Γ to be the circle centred at $z = 1$.



Since $(1-y^2)^{-n-1} = e^{-(n+1)\log(1-y^2)}$, we get

$$
\begin{aligned}
\phi(y) &= \log(1-y^2), \\
\phi'(y) &= \frac{-2y}{1-y^2},
\end{aligned}
$$

giving $y = 0$ as the critical point. We then deform Γ to Γ' passing through 0. A change of variables $y = it$ gives

$$f_n = \frac{1}{\pi}\int (1+t^2)^{-(n+1)} dt.$$

Since $\phi(y) = \phi(it) = t^2 - 1/2t^4 + O(t^6)$, we get $\phi(it) \sim t^2$ at the origin, so applying our theorem gives

$$\begin{aligned} f_n &\sim \frac{1}{\pi}\left[\sqrt{\frac{2\pi}{\phi''(0)n}}\right] \\ &= \frac{1}{\pi}\sqrt{\frac{2\pi}{2n}} \\ &= \sqrt{\frac{1}{\pi n}}. \end{aligned}$$

Real Integrands

When A and ϕ are real functions, we show that

$$\int_0^\epsilon A(x)e^{-\lambda\phi(x)}dx \sim \sum_{j=l}^{\infty} a_j \lambda^{-(1+j)/k}.$$

We prove this in three steps:

1. A and ϕ monomials;
2. ϕ monomial, A unrestricted;
3. A, ϕ unrestricted.

A, ϕ monomials

Let α, β be real, non-negative powers.

Substitute $y = \lambda x^\alpha$ to get

$$\begin{aligned} \int_0^\infty x^\beta e^{-\lambda x^\alpha} dx &= \int_0^\infty \left(\frac{y}{\lambda}\right)^{\beta/\alpha} e^{-y}\left(\frac{1}{\alpha}\frac{y^{1/\alpha-1}}{\lambda^{1/\alpha}}\right) dy, \\ &= \frac{1}{\alpha}\lambda^{-(1+\beta)/\alpha}\int_0^\infty y^{\frac{1+\beta}{\alpha}-1}e^{-y}dy, \\ &= \lambda^{-(1+\beta)/\alpha}C(\alpha,\beta). \end{aligned}$$

The last equality above is given by the definition of the Gamma function.

Now, the major contribution comes from a neighbourhood of zero: for any $\epsilon > 0$, the contribution from $x \in (\epsilon, \infty)$ is exponentially small in λ, so

$$\left|\int_0^\infty x^\beta e^{-\lambda x^\alpha} dx - \lambda^{-(1+\beta)/\alpha}C(\alpha,\beta)\right|$$

decays exponentially.

ϕ monomial, A unrestricted

Lemma 3. *Let $k, l > 0$ with $k \in \mathbb{Z}$. If A and ϕ are real valued piecewise smooth functions with $A(x) = O(x^l)$ at $x = 0$ and $\phi(x) \sim x^k$ at $x = 0$ and non-vanishing on $(0, \epsilon]$, then*

$$\int_0^\epsilon A(x)e^{-\lambda\phi(x)}dx = O(\lambda^{-(1+l)/k})$$

as $\lambda \to \infty$.

Proof. Pick r and s such that $|A(x)| \leq r|x|^l$ and $|e^{-\lambda\phi(x)}| \leq e^{(s-\lambda)}|x|^k$ on $[0, \epsilon]$. The result follows from bounding the absolute value of the integral using these bounds and the results for monomials above. □

This allows us to prove the following.

Lemma 4. *Suppose A is a real function with*

$$A(x) = \sum_{j=l}^{M-1} b_j x^j + O(x^M)$$

as $x \to 0$. Then

$$\int_0^\epsilon A(x) e^{-\lambda x^k} dx = \sum_{j=l}^{M-1} b_j C(k,j) \lambda^{-(1+j)/k} + O(\lambda^{-(1+M)/k}).$$

Proof. First,

$$A(x) - \sum_{j=l}^{M-1} b_j x^j = O(x^M).$$

Multiply by $e^{-\lambda x^k}$ and integrate to get the result, giving the result for real amplitude and monomial phase. □

No restrictions

This requires a little more work than the previous two cases. A change of variables will reduce this case to the previous lemma, but we require the following result (the analytic inversion lemma) to ensure that we understand the asymptotic series of the change of variables.

Lemma 5. *Let $M \geq 2$ be an integer, and let*

$$y(x) = c_1 x + c_2 x^2 + ... + x_{M-1} x^{M-1} + O(x^M)$$

in a neighbourhood of the origin, $c_1 \neq 0$ ($y(0) = 0$ and $y'(0) \neq 0$). Then there is a neighbourhood of the origin on which y is invertible. The inverse function has the expansion

$$x(y) = a_1 y + ... + a_{M-1} y^{M-1} + O(y^M)$$

where a_j are polynomials in $c_1, c_2, ..., c_j, c_1^{-1}$.

Proof. Suppose $c_1 = 1$. From $y = x + O(x^2)$ we see that $y \sim x$ at zero, hence $x = y + O(x^2) = y + O(y^2)$. Now let $2 \leq n < M$ and suppose inductively that $x = y + a_2 y^2 + \ + a_{n1} y^{n1} + O(y_n)$, where $a_2, ..., a_{j1}$ are polynomials in $c_2, ..., c_{j1}$. Let a be an indeterminate, and plug in the value of y the statement of the result to the quantity

$$x(y + a_2 y^2 + \ + a_{n1} y^{n1} + a y^n)$$

to get a polynomial in x ($\mod x^M$) whose coefficients are zero up to the x^n term. These coefficients may be written as

$$a - P(a_2, ..., a_{n-1}, c_2, ..., c_{n-1})$$

where P is a polynomial in the arguments given (and thus by induction in $c_2, ..., c_n$). Setting a_n equal to this polynomial gives

$$x - (y + \sum_{j=2}^{\infty} a_j y^j) = O(x^{n+1}),$$

completing the induction.

When $n = M - 1$, observing $O(x^m) = O(y^m)$ completes the proof for $c_1 = 1$. For $c_1 \neq 1$, repeat the above, but represent x as a function of y/c_1. □

We now prove a modification of Theorem 1.

Theorem 6. *Let A and ϕ be real functions, $\phi \in C^M$, with series (for $k, l \leq M$)*

$$\begin{aligned} A(x) &= \sum_{j=1}^{M-1} b_j x^j + O(x^M) \\ \phi(x) &= sum_{j=1}^{M-1} c_j x^j + O(x^M) \end{aligned}$$

as $x \to 0$, with $b_l, c_k \neq 0$. Then as $\lambda \to \infty$ we get

$$I(\lambda) = \int_0^\epsilon A(x)e^{-\lambda\phi(x)}dx \sim sum_{j=1}^{M-1} a_j C(k,j)(c_k\lambda)^{-(1+j)/k} + O(\lambda^{-(1+j)/k}),$$

where the a_j are polynomials in $b_l, ..., b_j, c_k^{-1}, c_{k+1}, ..., c_{k+j-1}$. The leading terms are given explicitly in [1].

Proof. Let $y = \phi(x)^{1/k}$. Then from the power series expansion of ϕ above, we get

$$y = c_k^{1/k} x \left(1 + \frac{c_{k+1}}{c_k}x + ... + \frac{c_m}{c_k}x^{M-k} + O(x^{M+1-k})\right)^{1/k}.$$

Using the Taylor series expansion $(1+u)^{1/k} = 1 + u + O(u^2)$, we find (note the departure from [1] here, due to an error in their calculation)

$$\begin{aligned} y &= c_k^{1/k} x^k \left(\sum_{j=0}^{M-k} d_j x^j + O(x^{M+1-k})\right) \\ &= c_k^{1/k} \left(\sum_{j=1}^{M-k+1} d_j x^j + O(x^{M+2-k}).\right) \end{aligned}$$

We use the previous lemma to get

$$x(y) = \sum_{j=1}^{M-k+1} e_j \left(\frac{y}{c_k^{1/k}}\right)^j + O(y^{M-k+2}),$$

where the e_j are polynomials in the d_j, which are in turn polynomials in $c_{k+1}, ..., c_j$. Since ϕ is in C^M, so is it's inverse and we may differentiate term by term to get

$$x'(y) = c_k^{-1/k} \sum_{j=1}^{M-k+1} j e_{j-1} \left(\frac{y}{c_k^{1/k}}\right)^{j-1} + O(y^{M-k+1}).$$

Then

$$\begin{aligned} I(\lambda) &= \int_0^\epsilon A(x)e^{-\lambda\phi(x)}dx \\ &= \int_o^{y(\epsilon)} \tilde{A}(y)e^{-\lambda y^k}dy, \end{aligned}$$

where $\tilde{A}(y) = A((y))x'(y)$, or explicitly

$$\tilde{A}(y) = c_k^{-1/k} \sum_{j=1}^{M-k+1} \tilde{b_j} \left(\frac{y}{c_k^{1/k}}\right)^j + O(y^{M-k+1}),$$

where $\tilde{b_j}$ is a polynomial in $b_l, ..., b_j, c_k^{-1}, c_{k+1}, ..., c_j$, evaluated in [1].

The result then follows from the previous case, where A is unrestricted and ϕ a monomial. □

References

[1] R. Pemantle and M. C. Wilson. *Analytic Combinatorics in Several Variables (draft).* Cambridge University Press, 2012.