Chapter 5: Fourier-Laplace Integrals in more than 1 variable, Part 2 Summary

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We continue Chapter 5 from the previous talk by Lily Yen, February 18. Recall, we wish to approximate the integral

$$I(\lambda) = \int_{\mathcal{C}} A(\mathbf{z}) e^{-\lambda\phi(\mathbf{z})} d\mathbf{z},$$

where A, ϕ are analytic functions of the vector $\mathbf{z}$ along the countour $\mathcal{C}$, a d-chain in $\mathbb{C}^d$.

Recall when showing this for the single variable case, we made the assumption that the quadratic term in the Taylor expansion of ϕ was non-zero. This is replaced by the assumption that the Hessian matrix

$$\mathcal{H} := \left(\frac{\partial^2 \phi}{\partial z_j \partial z_k} \right)$$

is non-singular for the multi-variate case.

Last time, we saw the proof of the Fourier-Laplace theorem in the case of:

(i) Standard phase, $\phi = z_1^2 + z_2^2 + ... + z_d^2$;

(ii) ϕ has finitely many critical points;

(iii) $Re\{\phi\}$ has a strict minimum.

We move on to completing the proof in full generality, following [1, Section 5.4].

Localisation

We wish to integrate $w = A(\mathbf{z}) e^{-\lambda\phi(\mathbf{z})} d\mathbf{z}$ over a compact chain $\mathcal{C}$. We do so by localising the integral around the critical points of the integrand in order to approximate the value of the integral.

In order to do this, we extend the definition of a critical point to manifolds, and we'll see that we can deform the curve $\mathcal{C}$ so that $Re\{\phi\}$ is strictly positive, allowing us to apply the results from last weeks session.

Theorem 1 (Theorem 5.4.8, [1], critical point decomposition)**.** *Let $\mathcal{M}$ be a compact stratified space of dimension real d, embedded in $\mathbb{C}^d$. Let A, ϕ be analytic functions defined on a neighbourhood containing $\mathcal{M}$. Suppose that ϕ has finitely many critical points on $\mathcal{M}$, all quadratically non-degenerate and in a strata of dimension d. Let G be the subset of these at which the real part of ϕ is minimized and assume, without loss of generality, that this minimal value is zero. Let $\mathcal{C}$ be a chain representing $\mathcal{M}$. Then then the integral*

$$I(\lambda) = \int_{\mathcal{C}} A(\mathbf{z}) e^{-\lambda\phi(\mathbf{z})} d\mathbf{z}$$

has asymptotic expansion

$$I(\lambda) \sim \sum_{l=0}^{\infty} c_l \lambda^{d/2-l}.$$

If A is nonzero at some point of G, then the leading term is given by

$$c_0 = (2\pi)^{-d/2} \sum_{\mathbf{x} \in G} A(\mathbf{x}) e^{\lambda \phi(\mathbf{x})} (\det \mathcal{H}(\mathbf{x}))^{-1/2}.$$

In order to prove Theorem 5.4.8, we're going to need to define a suitable class of chains and develop some geometric properties of them. This begins with stratified spaces.

Whitney Stratification overview

Let I be a partially ordered set. Then an I-decomposition of a topological space Z is a partition of Z into a disjoint union of sets $\{S_\alpha\}_{\alpha \in I}$, having the property

Definition 1. *Let Z be a closed subset of the manifold $\mathcal{M} \subset \mathbb{R}^d$. A* Whitney stratification *of Z is an I-decomposition such that*

(i) Each S_α is a manifold in RR^d

(ii) If

(a) $\alpha < \beta$,

(b) the sequences $\{x_i \in S_\beta\}$, $\{y_i \in S_\alpha\}$ converge to a point $y \in S_\alpha$,

(c) the lines $\bar{x_i y_i} = l_i$ converge to a line l, and

(d) the tangent planes $T_{x_i}(S_\beta)$ converge to a plane T,

then l and $T_y(S_\alpha)$ are contained in T.

Some (non) examples of Whitney Stratifications

1. Let $Z = \bar{\mathbb{D}} = \{z \in \mathbb{C} \mid 0 \leq |z| \leq 1\}$, with stratification given by

- $I = \{0, 1\}$ with order $<$,
- $S_0 = \mathbb{S}_1 = \partial \bar{\mathbb{D}}$,
- $S_1 = \bar{\mathbb{D}} \setminus \partial \bar{\mathbb{D}} = \mathbb{D}$.

We show the sequences x_i and y_i in Figure 1. The red sequence is y_i, and the blue sequence is x_i. Both converge on the black point $z = 1$, and all lines $\bar{x_i y_i}$ are contained in the only tangent plane, $\mathbb{C}$.

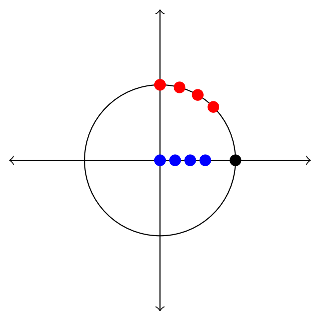

Figure 1: The above stratification of $\bar{\mathbb{D}}$ is Whitney.

2. Let $X = x^2 + y(y - z^2)$, and consider the variety of X. We impose the following stratification on the variety of X:

- $I = \{0, 1\}$ with order $<$,

- $S_0 = z$-axis,
- $S_1 = X \setminus S_0$.

In Figure 2, we show the variety and the sequences y_i in red, x_i in blue, their limit at the origin as a black point and the lines (represented by a segment) l_i in violet. This stratification is not Whitney, as the lines remain perpendicular to the tangent plane $T(x_i)$, the $z-x$-plane.

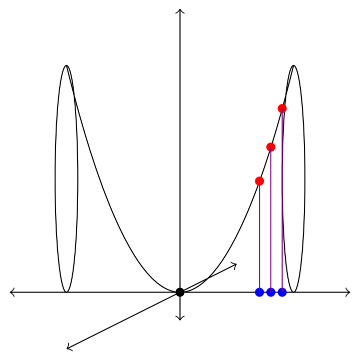

Figure 2: The given stratification for X is not Whitney

0.0.1 Stratified spaces

An important fact about stratified spaces is their local product structure: if p is contained in some strata S of $\mathcal{M}$, then p has a neighbourhood in $\mathcal{M}$ in which $\mathcal{M}$ is homeomorphic to $S \times X$, for some manifold X.

Definition 2. *Let $f : \mathcal{M} \to \mathbb{C}$, and $\mathcal{M}$ be a stratified space. The function f is smooth if it is smooth when restricted to each strata S.*

Definition 3. *A point $p \in \mathcal{M}$ is critical for a smooth function ϕ if and only if the restriction $d\phi|_S$ vanishes at p, where S is the strata containing p.*

Proposition 2 ([1], Prop. 5.4.3)**.** *Every algebraic variety in $\mathbb{R}^d$ and $\mathbb{C}^d$ admits a Whitney stratification.*

Example 3.
- *If $\mathcal{M}$ is smooth, then $(\mathcal{M})$ is a Whitney stratification (WS).*
- *If V is any space with a finite subset E such that $V \setminus E$ is a smooth manifold. The strata $(V \setminus E, E)$ form a WS.*
- *An algebraic variety $\mathcal{V}$ whose singular locus is a smooth manifold $\mathcal{V}'$ admits a Whitney stratification $(\mathcal{V} \setminus \mathcal{V}', \mathcal{V}')$.*

Tangent Vector Fields

The tangent space $T_{\mathbf{x}}(\mathcal{M})$ of a WS space $\mathcal{M}$ at a point $\mathbf{x}$ is defined to be the tangent space $T_{\mathbf{x}}(S)$ where S is the strata of $\mathcal{M}$ containing $\mathbf{x}$.

When $\mathcal{M}$ is embedded in and inherits the analytic structure of $\mathbb{C}^d$, then the tangent spaces fit together in a bundle. Each $T_{\mathbf{x}}(\mathcal{M})$ is then naturally identified with a subspace of $T_{\mathbf{x}}(\mathbb{C}^d)$, and a smooth section of a tangent bundle in $\mathcal{M}$ is a smooth vector field $f : \mathcal{M} \to \mathbb{C}^d$ such that $f(\mathbf{x}) \in T_{\mathbf{x}}(S)$

Lemma 4 ([1], Lemma 5.4.4)**.** *Let f be a smooth section of the tangent bundle to S, ie, $f(\mathbf{s}) = T_{\mathbf{s}}(S)$ for $\mathbf{s} \in S$. Then $\mathbf{s} \in S$ has a neighbourhood in $\mathcal{M}$ in which f can be extended to the smooth section of the tangent bundle of $\mathcal{M}$.*

Proof. We use the local paramterisation $S \times X$. Then given $\mathbf{s} \in S$ we can transport a vector $r \in T_{\mathbf{s}}(S)$ to a tangent space $T_{(\mathbf{s},\mathbf{x})}(\mathcal{M})$. Extend f by $f(\mathbf{s}, \mathbf{x}) = f(\mathbf{s})$. □

The following two results are used in the proof of Theorem 5.4.8, we omit their proofs and refer to [1].

Lemma 5 ([1], Lemma 5.4.5)**.** *Let $\mathbf{x} \in S$ where S is a strata of $\mathcal{M}$. Suppose that $\mathbf{x}$ is non-critical for a function ϕ. Then $\mathbf{r} \in T_{\mathbf{x}}(S \otimes U)$ such that $Re\{\phi(\mathbf{r})\}$ is strictly positive at $\mathbf{x}$ and there exists a continuous section f of the tangent bundle in a neighbourhood $\mathcal{N}$ of $\mathbf{x}$ such that $Re\{d\phi(f(\mathbf{x}))\} > 0$ for all $\mathbf{x} \in \mathcal{N}$.*

Lemma 6 ([1], Lemma 5.4.7)**.** *Let $\mathcal{M}$ be a stratified space, and ϕ be a smooth function on $\mathcal{M}$ with finitely many critical points. Then there is a global section f of the tangent bundle of $\mathcal{M}$ such that $Re\{d\phi(f(\mathbf{x}))\} > 0$ and $d\phi(f(\mathbf{x})) = 0$ only when $\mathbf{x}$ is a critical point.*

Note: The reference to Lemma 5.4.7 in the proof of Lemma 5.4.7 is an error, it should refer to Lemma 5.4.6.

Sketch of Theorem 5.4.8. We use Lemma 5.4.7 to find a tangent vector field f, which gives rise to a differentail flow.

The flow reduces the real part of ϕ everywhere except at the critical points. Consequently, it defines a homotopy H between the contour of integration $\mathcal{C}$ and a chain C on which the minima of the real part of ϕ occur precisely on the set G. The homotopy H then induces a chain homotopy $\partial\mathcal{C}_H = C - \mathcal{C} + \partial\mathcal{C} \times \sigma$ where σ is a standard 1-simplex. Let ω denote the holomorphic d form $A(\mathbf{z})\exp(-\lambda\phi(\mathbf{z}))d\mathbf{z}$. Because ω is a d form in $\mathbb{C}^d$, we have $d\omega = 0$. By Stoke's theorem,

$$\begin{aligned} 0 &= \int_{\mathcal{C}_H} d\omega \\ &= \int_{\partial\mathcal{C}_H} \omega \\ &= \int_C \omega - \int_{\mathcal{C}} \omega - \int_{\partial\mathcal{C}\times\sigma} \omega. \end{aligned}$$

The chain $\partial\mathcal{C} \times \sigma$ is supported on a finite union of spaces $S \otimes \mathcal{C}$ where S is a stratum, of dimension at most $d-1$. By Appendix A [1], ω vanishes over such a chain, therefore

$$\int_{\mathcal{C}} \omega = \int_C \omega.$$

Outside of a neighbourhood of G, the magnitude of the integrand is exponentially small, so we have shown that there are d-chains $\mathcal{C}_{\mathbf{x}}$ supported on arbitrarily small neighbourhoods $\mathcal{N}(\mathbf{x})$ of each $\mathbf{x} \in G$ such that

$$I(\lambda) - \sum_{\mathbf{x}\in G} \int_{\mathcal{C}_x} \omega$$

is exponentially small. To finish the proof, we need to show that each integral in the above summation has an asymptotic series in decreasing powers of λ whose leading term is given as in the statement of this theorem.

The d-chain $\mathcal{C}_{\mathbf{x}}$ may be parametrised by a map $\psi_{\mathbf{x}} : B \to \mathcal{N}(\mathbf{x})$, where B is the open unit ball in $\mathbb{R}^d$, and $\psi_{\mathbf{x}}$ maps the origin to $\mathbf{x}$. The real part of $\phi \circ \psi$ has a strict minimum at the origin, so we apply [1, Theorem 5.1.2]. The rest follows by calculation, the details of which are in [1]. □

References

[1] R. Pemantle and M. C. Wilson. *Analytic Combinatorics in Several Variables (draft).* Cambridge University Press, 2012.