Summary: Topology of $\mathcal{H}(U)$

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For $U \in \mathbb{C}$, $\mathcal{H}(U)$ is a *complex algebra* under pairwise addition and multiplication.

We would like to define a topology on $\mathcal{H}(U)$ which we shall call the topology of *uniform convergence* (Also known as the compact-open topology). To do this we will construct open sets using seminorms.

Definition 0.1. A *seminorm* on an $\mathbb{C}$-vector space X is a map $\rho : X \to \mathbb{R}^+$ such that:

$$\begin{aligned} &\rho(x + y \leq \rho(x) + \rho(y) && \forall x, y\ X \\ &\rho(\gamma x) = |\gamma|\rho(x) && \forall x \in X, \gamma \in \mathbb{C} \end{aligned}$$

We note that a norm has the additional property that $\rho(x) = 0$ if and only if $x = 0$.

For each compact subset, $K \in U$, we define $||\cdot||_K$ as:

$$||f||_K = \sup_{z\in K} |f(z)|.$$

Note that $||\cdot||_K$ is a seminorm on $\mathcal{H}(U)$.

Definition 0.2. A *neighbourhood basis* at a point $g \in \mathcal{H}(U)$ is a set $\{f \in \mathcal{H}(U) : ||f - g||_K < \epsilon\}$ for some $\epsilon \in \mathbb{R}$.

To define a topology on $\mathcal{H}(U)$ we shall look at the following theorem that defines a topology on a more general space.

Theorem 0.3. *Let V be a vector space over $(\mathbb{C} \setminus \mathbb{R})$ and $(\rho_i)_{i\in I}$ be a family of seminorms.*

For $F \subset I$ finite define:

$$U_{F,\epsilon} = \cap_{i\in F}\{x \in V : \rho_i(x) < \epsilon\}.$$

Define

$$\mathcal{U}^{curl} = \{U_{F,E} : \epsilon > 0, F \subset I \textit{ finite}\}$$

and

$$\tau = \{O \subset V : \forall x \in O \exists U \in \mathcal{U}^{curl} \textit{ s.t. } x + U \subset O\}.$$

Then τ defines the open sets of a topology on V.

Proof. To prove that τ defines a topology on V we need to show that it satisfies the three axioms of open sets.

1. $V, \emptyset \in \tau$. (trivial)

2. τ is closed under union. (take $x \in O$ and then take the ϵ and F from whichever set of the union it comes from)

3. τ is closed under finite intersection. (take $x \in O$ and then take minimum ϵ_j and union of F_j of each intersecting set)

□

We can also make $\mathcal{H}(U)$ a metric with the following measure:

$$d(x,y) = \sum_{n=0}^{\infty} \frac{2^{-n}\rho_n(x-y)}{1+\rho_n(x-y)} \ \forall x, y \in V,$$

where $\rho_n = ||\cdot||_{K_n}$ for some nested sequence of compact subsets of U, $\{K_n\}$.

Theorem 0.4. *Let $(f_j)_{j\in\mathbb{N}}$ be a sequence of functions in $\mathcal{H}(U)$. Then $(f_j)_{j\in\mathbb{N}}$ converges with respect to τ if and only if $(f_j)_{j\in\mathbb{N}}$ converges compactly on U.*

Proof. Let $K \subset U$ be compact. Then there exists j_k such that $K \subset K_{j_K}$.

If $f_j \to f$ with respect to τ then

$$\sup_{z\in K} |f_j(z) - f(z) = ||f_j - f||_K \leq ||f_j - f||_{K_{j_K}} \to 0.$$

So $(f_j)_{j\in\mathbb{N}}$ converges uniformly on K.

Conversely if

$$\lim_{j\to\infty} \sup_{z\in K} |f_j(z) - f(z)| = 0 \ \forall \text{ compact } K \subset U$$

then

$$\lim_{j\to\infty} d(f_j, f_) = \lim_{j\to\infty} \sum_{n=0}^{\infty} \frac{2^{-n}||f_j - f||_{K_n}}{1+||f_j - f||_{K_n}} = 0.$$

□

The following theorem shows that $\mathcal{H}(U)$ is complete on the topology τ.

Theorem 0.5. *If $\{f_n\}$ is a sequence of holomorphic functions on U which is uniformly convergent on each compact subset of U then the limit is also holomorphic.*

To prove this theorem we shall employ Morena's Theorem, given below.

Theorem 0.6 (Morena's Theorem)**.** *If f is continuous complex-valued function defined on a connected open subset D with*

$$\iint_\gamma f = 0$$

for all closed curves γ in D then f is holomorphic on D.

Now to prove Theorem 0.5:

Proof. We apply Morena's Theorem to each variable. This means the limit is continuous. By applying Osgood's Lemma we get that the limit is holomorphic. □

We just showed that $\mathcal{H}(U)$ is a Fréchet Space - it is a complete metric. Now we shall show it is also a Montal Space - it is a Fréchet Space with the additional property that every closed bounded subset is compact.
Theorem 0.7. *Every closed bounded subset of $\mathcal{H}(U)$ is compact.*

Proof. Since $\mathcal{H}(U)$ is a metric space it is sufficient to show that every bounded subsequence in $\mathcal{H}(U)$ has a convergent subsequence.

Let $\{f_n\}$ be a bounded sequence in $\mathcal{H}(U)$.

We would like to show that $\{f_n\}$ that $\{f_n\}$ has a uniformly convergent subsequence on each compact subset $K \subset U$. We know by the previous theorem that if the limit of the subsequence is continuous then it is holomorphic.

By the Adcoli-Arzela Theorem, a *bounded equicontinuous* sequence of functions on a compact set has a uniformly convergent subsequence.

By the Cauchy Inequality Theorem $\{f_n\}$ has a uniformly bounded sequence of complex sequence of complex partial derivatives $\{\frac{\partial f_n}{\partial z_j}\}$ for each complex variable, z_j.

By the Cauchy Riemann Theorem, the real and imaginary partial derivatives of $\{f_n\}$ have a uniformly bounded sequence of partial derivatives.

By applying the Minimum Value Theorem we obtain a continuous limit. □

References:

Scheidemann, V. Introduction to Complex Analysis in Several Variables. Basel; Boston: Birkhuser Verlag, 2005.