IMPLICIT AND INVERSE MAPPING THEOREMS
5TH NOVEMBER 2012

HIMADRI GANGULI

Theorem. ***(Implicit Function Theorem)*** *Let $U \subset \mathbb{C}^n$ be an open set, $f \in H(U), f(\lambda) = 0, \frac{\partial f}{\partial z_n}(\lambda) \neq 0$. Then there exist $\Delta(\lambda, r) \subset U$ and holomorphic $g : \Delta(\lambda^{'}, r') \to \Delta(\lambda_n, r_n)$ such that for $z \in \Delta(\lambda, r)$ $f(z) = 0 \iff g(z') = z_n$.*

Proof. See [3] □

Definition. *Let $U \subset \mathbb{C}^n$ be an open domain. Let $F : U \to \mathbb{C}^n$ be a holomorphic mapping, $f_1, f_2, \cdots, f_n$ be the coordinate functions. Then we define the jacobian as*

$$J_F(z) = \left(\frac{\partial f_i}{\partial f_j}(z) \right)_{1 \leq i \leq m, 1 \leq j \leq n}$$

Theorem. ***(Implicit mapping theorem)*** *Let F be a holomorphic mapping and suppose $\lambda \in U$ and $F(\lambda) = 0$. Suppose also that the last m columns of $J_F(\lambda)$ form a non-singular $m \times m$ matrix. Then there is a polydisc $\Delta(\lambda; r) = \Delta(\lambda'; r') \times \Delta(\lambda''; r'') \subset (C)^{n-m} \times \mathbb{C}^m$ and a holomorphic map $G : \Delta(\lambda'; r') \to \Delta(\lambda''; r'')$ such that $G(\lambda') = \lambda''$ and $F(z) = 0$ for $z = (z', z'') \in \Delta(\lambda; r)$ if and only if $G(z') = z''$.*

Proof. When $m = 1$ this is the implicit function theorem which is a simple corollary of the Weierstrass preparation theorem in the case where the function is regular of degree one in its last variable. We prove the general case by induction on m. Thus, we assume that the result is true for $m - 1$ and proceed to prove it for m.
Let $J_F(\lambda) = (J^{'}_F(\lambda), J^{''}_F(\lambda))$ be the separation of $J_F(\lambda)$ into its first $n - m$ columns and its last m columns. We leave the reader to complete the rest of the proof by following the steps in [3]. □

Theorem. ***(Inverse mapping theorem)*** *If F is a holomorphic mapping from a neighborhood U of $\lambda \in \mathbb{C}^n$ into $\mathbb{C}^n$ and if $J_F(\lambda)$ is non-singular, then, on some possibly smaller neighborhood U' of λ, F is a biholomorphic mapping to some neighborhood of $F(\lambda)$.*

Proof. This follows immediately from the implicit mapping theorem applied to the mapping $H : \mathbb{C}^n \times U \to \mathbb{C}^n$ defined by $H(z', z'') = F(z'') - z'$. □

We now mention results concerning multivariate Lagrange inversion.

Theorem. *Let x' be a d dimensional vector, $g(x'), f_i(x)$ FPS, $f_i(0) \neq 0$ then the equation $w_i = x_i f_i(w_i)$ uniquely determine the w_i FPS in x'. One can also set an equality with the coefficients of the Jacobian.*

Proof. See [1]. □

There are some more specific discussions relating to directed graphs and Lagrange inversion. We request the reader to refer to [1] for further interested reading.

References

[1] Bender and Richmond *A Multivariate Lagrange Inversion for Asymptotic Calculation* 1998.

[2] Scheidemann V.*Introduction to Complex Analysis in Several Variables* Basel; Boston: Birkhuser Verlag, 2005.

[3] Taylor Joseph L., *Several Complex Variables with connections to Algebraic Geometry and Lie Groups* Volume 46 AMS, Providence, Rhode Island.