Lectures on Renormalization Hopf algebras

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Renormalization Hopf Algebras I: The Connes-Kreimer Algebra of Rooted Trees

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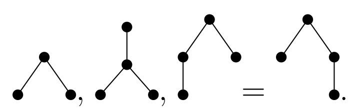
1 Comultiplication of Rooted Trees

We assume the reader is familiar with the usual notions regarding rooted trees and the basics of algebra. We denote the composition of two maps f and g as $fg := f(g(\cdot))$ (instead of the usual $f \circ g$).

We always draw rooted trees "downward" with the root at the top. The size $|T|$ of a tree T is the number of vertices in T. The *empty tree* is denoted by the symbol $\mathbb{1}$. We will usually drop the term "rooted" as all trees considered in this lecture are rooted trees. A *forest* will always refer to a disjoint union of rooted trees.

For example, the following are (rooted) trees: 

Definition 1 (Comultiplication of Rooted Trees or "The Pulling-Apart Operation")**.** Let T be a rooted tree.

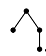
1. An *admissible cut* is a set of edges of T that contains at most one edge in any path from the root to a leaf. Note that the empty set is an admissible cut.

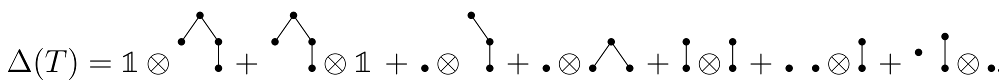
2. Given an admissible cut C and the forest $F := (V(T), E(T) \setminus C)$, we let $R_C(T)$ be the component of F that contains the root, and $P_C(T)$ the set of all other components of F.

3. The *comultiplication* $\Delta(T)$ is the formal sum

$$\Delta(T) = T \otimes \mathbb{1} + \sum_C P_C(T) \otimes R_C(T),$$

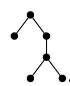
where the sum is over all admissible cuts C of T.

Example. Consider the tree $T =$ . By inspection, we see that



Note that the first term corresponds to the admissible cut $C = \emptyset$, while the second term corresponds to the admissible cut $C = E(T)$.

Exercise. Calculate $\Delta(T)$, where

$T =$ .

2 Hopf Algebras

Definition 2. Let $\mathbb{K}$ be a field and let $\mathcal{H}$ be a vector space over $\mathbb{K}$. If $\mathcal{H}$ satisfies the properties in the following list, then $\mathcal{H}$ is called a *Hopf algebra.*

1. $\mathcal{H}$ is an *associative unital algebra.* This means that there exist vector space homomorphisms $m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$ (called *multiplication*) and $\iota : \mathbb{K} \to \mathcal{H}$ (called *identity*) such that the following two diagrams commute

$$\begin{tikzcd}
 & \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H} \arrow[dl, "m \otimes \mathrm{id}"'] \arrow[dr, "\mathrm{id} \otimes m"] & \\
\mathcal{H} \otimes \mathcal{H} \arrow[dr, "m"'] & & \mathcal{H} \otimes \mathcal{H} \arrow[dl, "m"] \\
 & \mathcal{H} &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& \mathbb{K}\otimes\mathcal{H}\simeq\mathcal{H}\simeq\mathcal{H}\otimes\mathbb{K} \arrow[dl, "\iota\otimes\mathrm{id}"'] \arrow[dd, "\mathrm{id}"] \arrow[dr, "\mathrm{id}\otimes\iota"] & \\
\mathcal{H}\otimes\mathcal{H} \arrow[dr, "m"'] & & \mathcal{H}\otimes\mathcal{H} \arrow[dl, "m"] \\
& \mathcal{H} &
\end{tikzcd}.$$

We abuse notation by writing $\mathbb{1} := \iota(1)$ and $ab := m(a \otimes b)$. Note that the first diagram asserts associativity of m, i.e., $a(bc) = (ab)c$, while the second diagram simply says that $\mathbb{1}$ acts as a multiplicative identity in $\mathcal{H}$, i.e., $\mathbb{1}a = a = a\mathbb{1}$.

2. $\mathcal{H}$ is a *coassociative counital algebra*. This means that there exist vector space homomorphisms $\Delta : \mathcal{H} \to \mathcal{H}\otimes\mathcal{H}$ (called *comultiplication*) and $\varepsilon : \mathcal{H} \to \mathbb{K}$ (called *counit*) such that the following two diagrams commute:

$$\begin{tikzcd}
& \mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H} & \\
\mathcal{H}\otimes\mathcal{H} \arrow[ur, "\Delta\otimes\mathrm{id}"] & & \mathcal{H}\otimes\mathcal{H} \arrow[ul, "\mathrm{id}\otimes\Delta"'] \\
& \mathcal{H} \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

and

$$
\begin{tikzcd}
 & \mathbb{K}\otimes\mathcal{H}\simeq\mathcal{H}\simeq\mathcal{H}\otimes\mathbb{K} & \\
\mathcal{H}\otimes\mathcal{H} \arrow[ur, "\varepsilon\otimes\mathrm{id}"] & & \mathcal{H}\otimes\mathcal{H} \arrow[ul, "\mathrm{id}\otimes\varepsilon"'] \\
 & \mathcal{H} \arrow[ul, "\Delta"'] \arrow[uu, "\mathrm{id}"] \arrow[ur, "\Delta"] &
\end{tikzcd}
.
$$

In symbols, the two diagrams assert that $(\mathrm{id}\otimes\Delta)\Delta = (\Delta\otimes\mathrm{id})\Delta$ and $(\mathrm{id}\otimes\varepsilon)\Delta = (\varepsilon\otimes\mathrm{id})\Delta = \mathrm{id}$.

3. $\mathcal{H}$ is a *bialgebra*. That is, the maps Δ and ε are algebra homomorphisms (equivalently, m and ι are coalgebra homomorphisms).

4. There exists a vector space homomorphism $S : \mathcal{H} \to \mathcal{H}$ (called an *antipode*) such that the following diagram commutes:

$$
\begin{tikzcd}
 & \mathcal{H}\otimes\mathcal{H} \arrow[rr, "S\otimes\mathrm{id}"] & & \mathcal{H}\otimes\mathcal{H} \arrow[dr, "m"] & \\
\mathcal{H} \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & \mathbb{K} \arrow[rr, "\iota"] & & \mathcal{H} \\
 & \mathcal{H}\otimes\mathcal{H} \arrow[rr, "\mathrm{id}\otimes S"] & & \mathcal{H}\otimes\mathcal{H} \arrow[ur, "m"'] &
\end{tikzcd}
.
$$

In other words, $m(\mathrm{id}\otimes S)\Delta = \iota\varepsilon = m(S\otimes\mathrm{id})\Delta$.

Definition 3. Let $\mathcal{T}$ denote the collection of all forests of rooted trees. For $\mathbb{K}=\mathbb{Q}$, we make $\mathcal{T}$ into a Hopf algebra as follows.

1. Addition is formal;

2. The unit ι is defined by $\iota(1) = \mathbb{1}$ and counit is $\varepsilon(\mathbb{1}) = 1$ (note that $\varepsilon(T) = 0$ for $T \neq \mathbb{1}$);

3. Multiplication of two forests is their disjoint union and comultiplication is the map Δ from Definition 1;

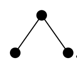
4. The antipode S is defined recursively by $S(\mathbb{1}) = \mathbb{1}$ and, for $T \neq \mathbb{1}$,

$$S(T) = -m(S \otimes \mathrm{id} - \iota\varepsilon)\Delta(T).$$

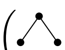
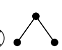
The choice of S is the only possibility given the other operations. Note that we must have $S(\mathbb{1}) = \iota\varepsilon(\mathbb{1}) = \mathbb{1}$. Moreover, $\iota\varepsilon(T) = 0$ for all $T \neq \mathbb{1}$. On the other hand, for $T \neq \mathbb{1}$ we have

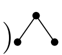
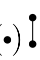
$$\begin{aligned} \iota\varepsilon(T) &= m(S \otimes \mathrm{id})\Delta(T) \\ &= m(S \otimes \mathrm{id})(T \otimes \mathbb{1} + \mathbb{1} \otimes T + \cdots) \\ &= m(S \otimes (\mathrm{id} - \iota\varepsilon))\Delta(T) + m(S \otimes \mathrm{id})(T \otimes \mathbb{1}) \\ &= m(S \otimes (\mathrm{id} - \iota\varepsilon))\Delta(T) + S(T), \end{aligned}$$

from which the recursion in Condition 4 above follows.

Example. We calculate $S(T)$ for $T =$ .

By the recursive formula for S, we have

$S(T) = -m(S \otimes (\mathrm{id} - \iota\varepsilon)) \Big($  $\otimes \mathbb{1} + \mathbb{1} \otimes$  $+ 2(\bullet \otimes \mathbb{1}) + (\bullet\bullet) \otimes \bullet \Big)$

$= S(\mathbb{1})$  $- 2S(\bullet)$  $- S(\bullet\bullet)\,\bullet$

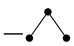
Now,

$$\begin{aligned} S(\mathbb{1}) &= \mathbb{1}, \\ S(\bullet) &= -m(S \otimes (\mathrm{id} - \iota\varepsilon))(\bullet \otimes \mathbb{1} + \mathbb{1} \otimes \bullet) \\ &= -\bullet \end{aligned}$$

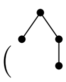
and

$$\begin{aligned} S(\bullet\bullet) &= -m(S \otimes (\mathrm{id} - \iota\varepsilon))(\Delta(\bullet)\Delta(\bullet)) \\ &= -m(S \otimes (\mathrm{id} - \iota\varepsilon))(\bullet \otimes \mathbb{1} + \mathbb{1} \otimes \bullet)^2 \\ &= -m(S \otimes (\mathrm{id} - \iota\varepsilon))(\bullet\bullet \otimes \mathbb{1} + 2(\bullet \otimes \bullet) + \mathbb{1} \otimes \bullet\bullet) \\ &= -2S(\bullet)\bullet - \bullet\bullet \\ &= -2(-\bullet)\bullet - \bullet\bullet \\ &= \bullet\bullet \end{aligned}$$

Thus

$$S(T) \quad = \quad -$$  $$+\,2($$  $$) - \bullet\bullet\bullet .$$

Exercise. 1. Calculate

$$S($$  $$).$$

2. Can you give an explicit (i.e., non-recursive) characterization of S for $\mathcal{T}$?

Properties and Vocabulary 1. For any integer $n \geq 0$, let $\mathcal{T}_n$ be the subspace of $\mathcal{T}$ generated by the forests with exactly n vertices. Notice that for all i, j and n,

$$\begin{aligned}
m : \mathcal{T}_i \otimes \mathcal{T}_j &\rightarrow \mathcal{T}_{i+j} \\
\Delta : \mathcal{T}_n &\rightarrow \bigoplus_{i+j=n} \mathcal{T}_i \otimes \mathcal{T}_j \\
S : \mathcal{T}_n &\rightarrow \mathcal{T}_n.
\end{aligned}$$

This says that, as a Hopf algebra, $\mathcal{T}$ is $\mathbb{Z}_{\geq 0}$-*graded*. We may therefore write

$$\mathcal{T} = \bigoplus_{n \geq 0} \mathcal{T}_n.$$

1. The elements of $\mathcal{T}_n$ are called *homogeneous of degree* n.

2. As $\mathcal{T}_n$ is finite dimensional for every n, we'll say $\mathcal{T}$ is of *finite type*.

3. Since $\mathcal{T}_0 \simeq \mathbb{Q}$, $\mathcal{T}$ is *connected*.

4. The multiplication m is a commutative operation since we can commute the inputs, i.e., $TR = RT$ for any T and R in $\mathcal{T}$. On the other hand, Δ is not cocommutative as we cannot commute the *outputs*, since $T \otimes R \neq R \otimes T$ in general.

5. An element F of $\mathcal{T}$ is *primitive* if $\Delta(F) = F \otimes \mathbb{1} + \mathbb{1} \otimes F$.

6. We let $\mathcal{P}(\mathcal{T})$ denote the vector space generated by the primitive elements.

For example, we have already seen above that $F = \bullet$ is primitive. Thus any scalar multiple of $\bullet$ is primitive. These are the only primitive homogeneous elements of degree 1. Now consider $F = 2\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix} - \bullet\bullet$. We have

$$\begin{aligned}\Delta(2\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix} - \bullet\bullet) &= 2\Delta(\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}) - \Delta(\bullet\bullet)\\ &= 2(\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}\otimes \mathbb{1}) + 2(\mathbb{1}\otimes \begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}) + 2(\bullet\otimes\bullet) - \bullet\bullet\otimes\mathbb{1} - \mathbb{1}\otimes\bullet\bullet - 2(\bullet\otimes\bullet).\end{aligned}$$

It follows that F is primitive. In fact, F is the only primitive homogenous element of degree 2.

Exercise. Find a basis for all primitive elements of size at most 4. What can you say in general?

3 The Restricted Dual

Recall that the *dual* of a finite-dimensional vector space V is the vector space

$$V^* := \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K}).$$

In other words, V^* is the vector space consisting of all linear maps from V to its base field. For $\mathcal{H}$ a $\mathbb{Z}_{\geq 0}$-graded Hopf algebra of finite type, the *restricted dual* is

$$\mathcal{H}^* = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} \mathcal{H}_n^*.$$

Notice that $\mathcal{H}^*$ is a Hopf algebra with the following data:

$$\begin{aligned}\varepsilon_* : \mathcal{H}^* &\to \mathbb{K}\\ f &\mapsto f(\mathbb{1})\end{aligned}$$

$$\begin{aligned}\iota_* : \mathbb{K} &\to \mathcal{H}^*\\ k &\mapsto (T \mapsto kT)\end{aligned}$$

$$\begin{array}{rcl} m_* : \mathcal{H}^* \otimes \mathcal{H}^* & \to & \mathcal{H}^* \\ f \otimes g & \mapsto & (f \otimes g)\Delta \end{array}$$

$$\begin{array}{rcl} \Delta_* : \mathcal{H}^* & \to & \mathcal{H}^* \otimes \mathcal{H}^* \\ f & \mapsto & fm \end{array}$$

$$\begin{array}{rcl} S_* : \mathcal{H}^* & \to & \mathcal{H}^* \\ f & \mapsto & fS \end{array}$$

How should we think about the restricted dual of $\mathcal{T}$? Let F be a forest in $\mathcal{T}$. Define $Z_F : \mathcal{T} \to \mathbb{K}$ by

$$Z_F(F') = \begin{cases} 1 & \text{if } F = F', \\ 0 & \text{otherwise.} \end{cases}$$

4 The Structure of $\mathcal{P}(\mathcal{T}^*)$

Let $Z_F \in \mathcal{T}^*$ be as above where $F \in \mathcal{T}_i$. We know that $Z_F(F) = 1$ and $\Delta_*(Z_F) = Z_F \otimes \mathbb{1}_* + \mathbb{1}_* \otimes Z_F$. Thus $\Delta_* Z_F(F_i \otimes F_j)$ is nonzero precisely when either $F_i = F$ and $F_j = \mathbb{1}$, or $F_i = \mathbb{1}$ and $F_j = F$. On the other hand,

$$\begin{array}{rcl} \Delta_*(m(F_i \otimes F_j)) & = & \Delta_*(F_i F_j) \\ & = & \begin{cases} 1 & \text{if } F_i F_j = F, \\ 0 & \text{otherwise.} \end{cases} \end{array}$$

Thus $Z_F \in \mathcal{P}(\mathcal{T}^*)$ if and only if F is a single tree. Therefore,

$$\mathcal{P}(\mathcal{T}^*) = \operatorname{span}(Z_T \mid T \text{ a tree in } \mathcal{T}).$$

Renormalization Hopf Algebras II: The Insertion Lie Algebra and B_+

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Before we begin, recall that we stated (without full justification) that

$$\mathcal{P}(\mathcal{T}^*) = \operatorname{span}(Z_T \mid T \text{ a tree in } \mathcal{T}).$$

Let us complete the proof of this. We showed last time that

$$\operatorname{span}(Z_T \mid T \text{ a tree in } \mathcal{T}) \subseteq \mathcal{P}(\mathcal{T}^*).$$

In the other direction, let $Z \in \mathcal{P}(\mathcal{T}^*)$. As the forests form a basis of $\mathcal{T}$, it suffices to check how Z acts on forests. Since $Z \in \mathcal{P}(\mathcal{T}^*)$, by definition we must have

$$\Delta_*(Z) = Z \otimes \mathbb{1}_* + \mathbb{1}_* \otimes Z.$$

Moreover, it is easy to check that for any forests F and F' we have

$$m_*(Z_F \otimes Z_{\mathbb{1}})(F') = Z_F(F').$$

This implies that $\mathbb{1}_* = Z_{\mathbb{1}}$. Thus $\Delta_*(Z)(F_1 \otimes F_2) = 0$ unless either $F_1 = \mathbb{1}$ or $F_2 = \mathbb{1}$.

On the other hand, by definition we have $\Delta_*(Z)(F_1 \otimes F_2) = Z(F_1F_2)$. Thus $Z(F) = 0$ if F is a forest which is not a single tree. In other words, there exist $c_T \in \mathbb{Q}$ such that

$$Z = \sum_{T \text{ a tree}} c_T Z_T.$$

Exercise. Let F be primitive. Show that Z_F is indecomposable. That is, show Z_F cannot be written as a linear combination of products of elements in $\ker \varepsilon_*$.

1 More on the Structure of $\mathcal{P}(\mathcal{T}^*)$

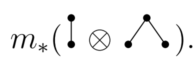
How do we see the structure coming from Δ (i.e., m_*) on $\mathcal{P}(\mathcal{T}^*)$? In particular, what is $m_*(Z_{T_1} \otimes Z_{T_2})$ for trees T_1 and T_2? Well, let T_3 be a tree. Then

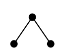
$$m_*(Z_{T_1} \otimes Z_{T_2}) = Z_{T_1} \otimes Z_{T_2}(\Delta(T_3)).$$

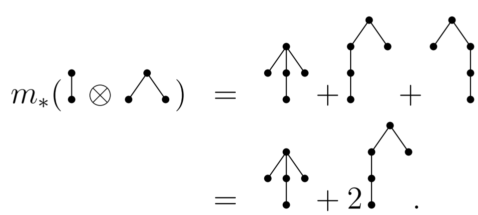
This is nonzero precisely when T_3 has a term $T_1 \otimes T_2$ in $\Delta(T_3)$, i.e, when we can (admissably) cut T_3 to get T_1 and T_2. Another way of phrasing this is that we can *graft* T_1 onto T_2 to obtain T_3. In this case, the coefficient is the number of different such graftings. In conclusion $m_*(Z_{T_1} \otimes Z_{T_2})$ is the sum over all possible graftings of T_1 onto T_2.

Note. For the remainder of this section, we'll often (but not always) abuse notation and write (or draw) the tree T to represent the map Z_T.

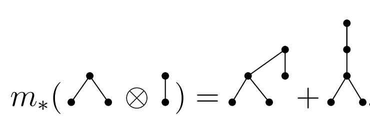
Example. Let us calculate



As there are three vertices in , we may graft onto it in three ways. Thus



On the other hand,



It is easy to check that $m_*(Z_{T_1} \otimes Z_{T_2})$ is, in fact, a *pre-Lie product.* So we may turn $\mathcal{P}(\mathcal{T}^*)$ into a Lie algebra with Lie bracket given by

$$[Z_{T_1}, Z_{T_2}] := m_*(Z_{T_1} \otimes Z_{T_2}) - m_*(Z_{T_2} \otimes Z_{T_1}).$$

Aside. Recall that a $\mathbb{K}$-vector space $\mathfrak{g}$ is called a *Lie algebra* if it is equipped with a *Lie bracket* $[\cdot, \cdot] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ satisfying the following properties:

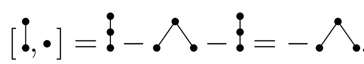
1. The Lie bracket is bilinear;
2. The Lie bracket is antisymmetric (i.e., $[a, b] = -[b, a]$ for all $a, b \in \mathfrak{g}$);
3. The Lie bracket satisfies the Jacobi identity
$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$$
for all $a, b, c \in \mathfrak{g}$.

(Note that the second property implies $[a, a] = 0$ for all $a \in \mathfrak{g}$.)

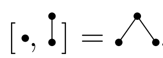
In our case, the Jacobi identity is implied by the coassociativity of $\mathcal{T}$.

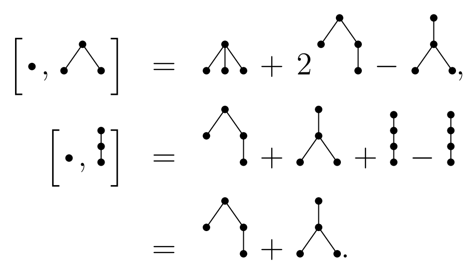
Exercise. Give a combinatorial proof of the Jacobi identity for $\mathcal{P}(\mathcal{T}^*)$.

Example.

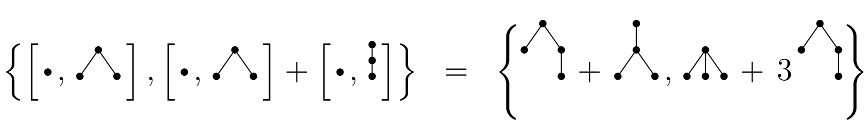


Consider the *commutator subspace*
$$[\mathcal{P}(\mathcal{T}^*), \mathcal{P}(\mathcal{T}^*)] := \mathrm{span}\{[f, g] \mid f, g \in \mathcal{P}(\mathcal{T}^*)\}.$$

We have already shown that  This is (up to scalar multiplication) the only $Z_T \in [\mathcal{P}(\mathcal{T}^*), \mathcal{P}(\mathcal{T}^*)]$ with T of size 3. (Recall that the size of a tree is simply its number of vertices.) How about those elements of the commutator subspace that are linear combinations of trees of size 4? Well, it is easy to see that a basis may be formed by the two elements



Of course, a different basis for the size 4 part (with positive coefficients) can be formed by



Exercise. 1. Calculate a basis for the 5-vertex part of $[\mathcal{P}(\mathcal{T}^*), \mathcal{P}(\mathcal{T}^*)]$

2. What happens in general?

Research Problem. Describe the lower central series of $\mathcal{P}(\mathcal{T}^*)$, where the lower central series is the sequence of subspaces $L_1, L_2, \ldots$ defined recursively by $L_1 := [\mathcal{P}(\mathcal{T}^*), \mathcal{P}(\mathcal{T}^*)]$ and, for $k > 1$, $L_k := [\mathcal{P}(\mathcal{T}^*), L_{k-1}]$. This should be related to the function "P" from my (Karen Yeats') thesis. It should also say something about transcendental content (a la Drinfel'd associatior).

2 The Big Picture

The idea is that $\mathcal{T}$ and $\mathcal{P}(\mathcal{T}^*)$ contain the same information. This is true in a precise sense because of

Theorem 2.1 (Milnor-Moore)**.** *If $\mathcal{H}$ is a graded, connected, commutative or cocommutative Hopf algebra of finite type, then $\mathcal{H} \simeq \mathcal{U}(\mathcal{P}(\mathcal{H}))$ (as Hopf algebras).*

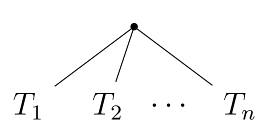
Aside. Here $\mathcal{U}(\mathfrak{g})$ denotes the *universal enveloping algebra* of a Lie algebra $\mathfrak{g}$ defined as follows. If $\mathfrak{g}$ is a Lie algebra, let $T(\mathfrak{g})$ be the tensor algebra (i.e., $T(g) = \bigoplus_{k\geq 0} \mathfrak{g}^{\otimes k}$). Then

$$\mathcal{U}(\mathfrak{g}) = T(\mathfrak{g})/\langle [a, b] - a \otimes b + b \otimes a \mid a, b \in \mathfrak{g}\rangle.$$

In our case, $\mathcal{T}^* \simeq \mathcal{U}(\mathcal{P}(\mathcal{T}^*))$. Our general outlook is to first find interesting Hopf algebras $\mathcal{H}$ from renormalization and then to look at the corresponding $\mathcal{P}(\mathcal{H}^*)$.

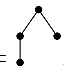
3 B_+

The obvious tree operation that we're missing is that of adding a root. We'll denote this operation by B_+. Thus,

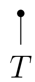
$B_+(T_1T_2\cdots T_n) =$ 

.

Example.

$B_+($  $) =$  .

How does B_+ relate to the Hopf algebra? What is ΔB_+? Let T be a tree. Then

$\Delta B_+(T) = \Delta($  $)$

$$\begin{aligned} &= B_+(T) \otimes \mathbb{1} + T \otimes \bullet + \sum_{\text{adm cuts c}} P_C(T) \otimes B_+(R_C(T)) \\ &= B_+(T) \otimes \mathbb{1} + (\mathrm{id} \otimes B_+)\Delta(T). \end{aligned}$$

A similar calculation for forests yields

$$\begin{aligned} \Delta B_+(F) &= B_+(F) \otimes \mathbb{1} + (\mathrm{id} \otimes B_+)\left(\prod_{T \text{ a tree of } F}\left(T \otimes \mathbb{1} + \sum_{\text{adm cuts } C} P_C(T) \otimes R_C(T)\right)\right) \\ &= B_+(F) \otimes \mathbb{1} + (\mathrm{id} \otimes B_+)\Delta(F). \end{aligned}$$

Thus $\Delta B_+ = B_+ \otimes \mathbb{1} + (\mathrm{id} \otimes B_+)\Delta$.

What is the fancy name for this? Let $\mathcal{H}$ be a bialgebra. Let $L : \mathcal{H} \to \mathcal{H}^{\otimes n}$ be a linear map. Define the map b that takes L to the map $bL : \mathcal{H} \to \mathcal{H}^{\otimes(n+1)}$ where

$$bL = (\mathrm{id} \otimes L)\Delta + \sum_{i=1}^{n}(-1)^i \Delta_i L + (-1)^n L \otimes \mathbb{1},$$

where

$$\Delta_i = \mathrm{id} \otimes \mathrm{id} \cdots \mathrm{id} \otimes \Delta \otimes \mathrm{id} \cdots \mathrm{id}$$

(with Δ in the ith slot).

Exercise. Check that $b^2 = 0$.

We can build a cohomology theory called Hochschild cohomology. Here is a three line summary of cohomology:

1. You need a map b taking objects of one size to objects of the next size, such that $b^2 = 0$.

2. Take quotients $\ker(b)/\mathrm{Im}(b)$.

3. Use these to understand your original objects.

For $n = 1$, $bL = 0$ says $0 = bL = (\mathrm{id} \otimes L)\Delta - \Delta L + L \otimes \mathbb{1}$. This is the B_+ identity, i.e., B_+ is a *1-cocycle*. This is the cohomology we will use here.

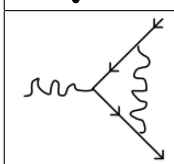
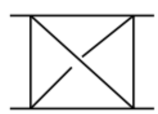
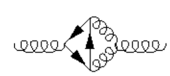
Renormalization Hopf Algebras III: The Hopf Algebras of Feynman Graphs

Karen Yeats
(Scribe: Sophie Burrill)

May 13, 2010

1. Feynman graphs

We begin by considering some examples of Feynman graphs that arise in each of quantum electrodynamics (QED), scalar field theory, and quantum chromodynamics (QCD):

QED	scalar field theory	QCD
		

We then axiomatize by half edges:

- A **graph** is a set of half edges, a set of vertices, a set of half-edge half-edge adjacencies, and a set of half-edge vertex adjacencies.
- A pair of adjacent half edges is called an **internal edge**.
- Other half edges are called **external edges**.

The only inputs that we will need from the physical theory are:

(0) a set of half edges

(1) a set of permissible types for the half-edge half-edge adjacencies (i.e. a set of edge types)

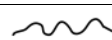
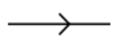
(2) a set of permissible types for half-edges adjacent to a vertex (i.e. a set of vertex types)

(3) an integer weight for each edge-type and each vertex-type (i.e. "power counting weights")

(4) an integer dimension of space time (usually 4)

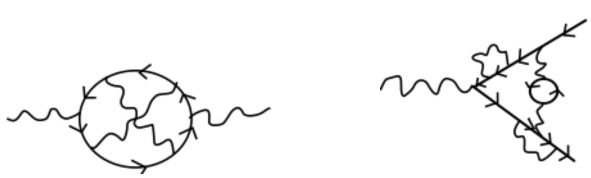
Then given such information, we are interested in graphs with types that conform to **(1)** and **(2)**, that is graphs with a set of edge types and vertex types.

Example. We give an example from QED where there are 2 types of permissible edges and 1 permissible vertex type. We illustrate them below with their power counting weights:

permissible	picture	weight
edge		2
edge		1
vertex		0

In this case the dimension, $D = 4$. We also note that the first illustrated edge type is referred to as a photon.

Then, two graphs that satisfy these conditions are:



Example. From scalar field theory, ϕ^4 we can have the following edge and vertex permissible with $D = 4$:

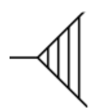
permissible	picture	weight
edge		2
vertex		0

Two graphs that satisfy these conditions are:

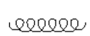
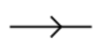
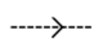
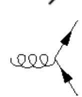
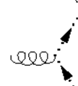
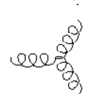
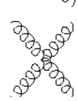
Example. Also in scalar field theory with ϕ^4 with $D = 6$:

permissible	picture	weight
edge		2
vertex		0

An example of a graph that satisfies these conditions is:



Example. In QCD with $D = 4$:

permissible	picture	weight
edge		2
edge		1
edge		1
vertex		0
vertex		0
vertex		-1
vertex		0

Labelled vs. Unlabelled counting

Traditionally in the Quantum Field Theory (QFT) community the external edges are taken as unlabelled, that is:

 $\neq$ 

while the internal edges are taken as unlabelled. However, each graph appears with a factor of $\frac{1}{|Aut|}$ (where $|Aut|$ is defined to be the number of automorphisms) so in reality we are doing labelled counting. In particular we have the usual exponential relationship between connected and not necessarily connected.

Vocabulary: *One particle irreducible* (1PI) means 2 -edge connected (i.e. a connected graph remains connected with any one edge removed).

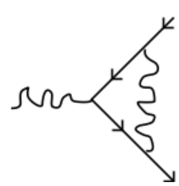
Definition 1. Let l be the loop number of a graph. The **superficial degree of divergence** of a graph G in a theory is

$$w(G) := Dl - \sum_{e\ internal} w(e) - \sum_{v} w(v).$$

If $w(G) \geq 0$ we say G is **divergent**. We interested in 1PI and divergent graphs.

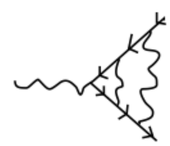
We care about these divergent graphs because in physics they are the ones that need fixing, since their Feynman integrals diverge.

Example. Calculate the degree of divergence of the following QED:



Answer: In QED, $D = 4$, $l = 1$, there are 3 internal edges, two with weight 1 and one with weight 4, and the weight of all vertices is 0. Thus $w(G) = 4 * 1 - 4 - 0 = 0$. This is a graph that we are interested in.

Example. Calculate the degree of divergence of the following QED:

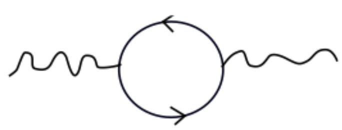


Answer: In this case there are two loops so $l = 2$, and there are 2 internal edges weighted 2 and 4 internal edges weighted 1. Thus $w(G) = 4 * 2 - 8 - 0 = 0$. Again we are interested in this graph.

Example. Calculate the degree of divergence of the following QED:

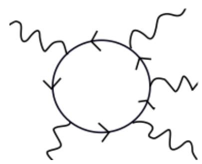
Answer: Here we have $w(G) = 4 * 2 - 8 - 0 = 0$. Again we are interested in this graph.

Example. Calculate the degree of divergence of the following QED:



Answer: Here we have only two internal edges, each weighted 1. Thus: $w(G) = 4 * 1 - 2 - 0 = 2 > 0$ and we are again interested in this graph.

Example. Calculate the degree of divergence of the following QED:



Anwer: Here there are 5 internal edges each weighted 1. Thus: $w(G) = 4 * 1 - 5 - 0 = -1$ and the graph is not divergent.

Exercise. Show that for the 4 theories given the superficial degree of divergence depends only on the external edge types.

2. The Hopf Algebra

We pick a theory T. The Hopf algebra $\mathcal{H}_T$ as a vector space is the $\mathbb{Q}$ span of disjoint unions of 1PI divergent graphs in T. Recall that $\mathbb{1}$ is the empty graph, which we identify with the single vertex with no cycles graph. Then:

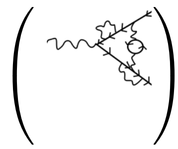
- m is a disjoint map
- unit map: $\mathbb{Q} \to \mathcal{H}_T$

  $$1 \mapsto \mathbb{1}$$
- co-unit map: $\mathcal{H}_T \to \mathbb{Q}$

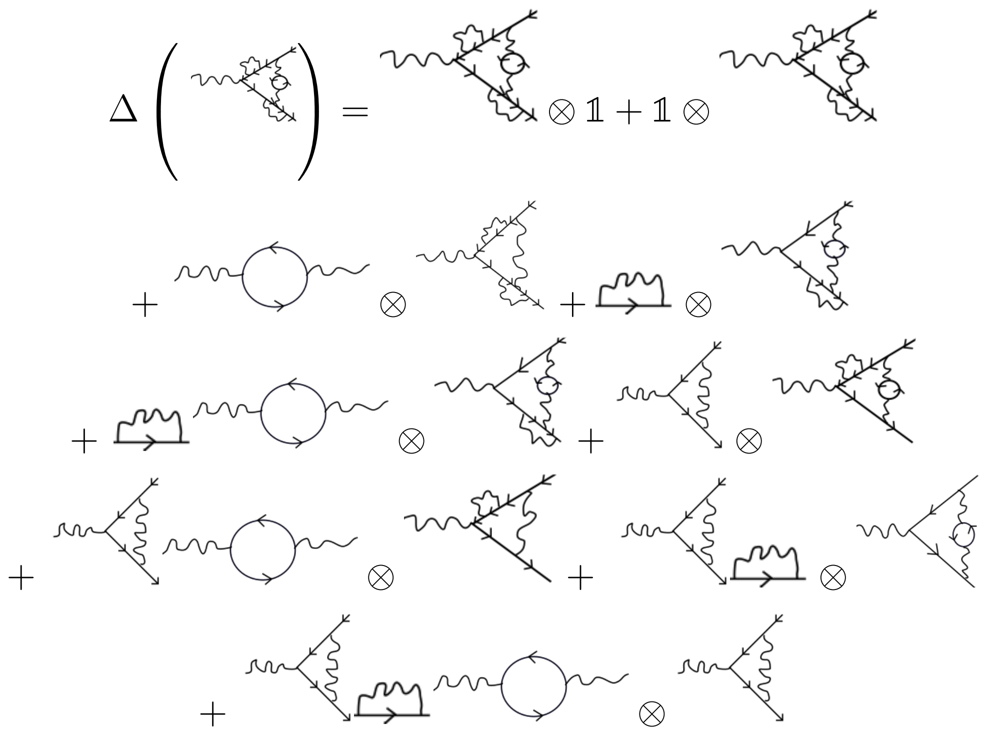
  $$\mathbb{1} \mapsto 1$$

  $$\mathbb{G} \mapsto 0$$
- coproduct: $\Delta(G) = G \otimes \mathbb{1} + \mathbb{1} \otimes G + \sum_{\gamma \subseteq G} \gamma \otimes G \backslash \gamma$

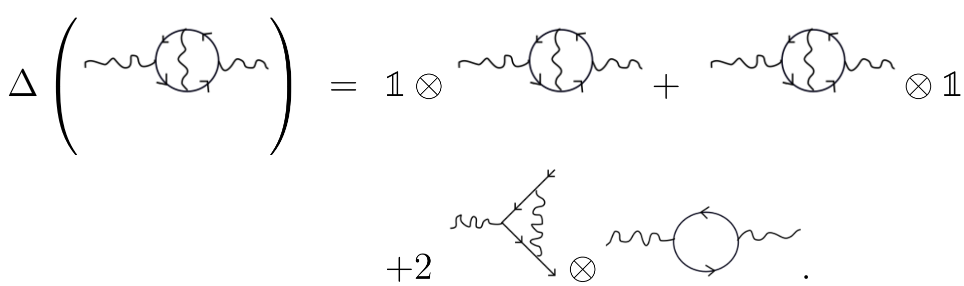
Where G is non-trivial, and γ is also the disjoint union of 1PI and divergent graphs, and $G\backslash\gamma$ is the contraction of connected components to an edge or vertex in the theory.

Example. Compute Δ .

Answer:



Example. We compute the following comultiplication which contains overlapping subdivergencies:

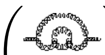


Finally, the antipode comes recursively as before:

$$S(G) = -G - \sum_{\gamma} S(\gamma) G \backslash \gamma$$

Also, as before, we can also consider primitives, Milnor-Moore and Hochschild cohomology. With regards to B_+, there are some subtleties which must be examined.

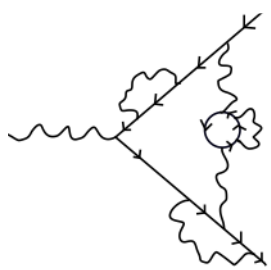
Exercise. Calculate Δ () in ϕ^3.

Exercise. Calculate Δ () in QCD.

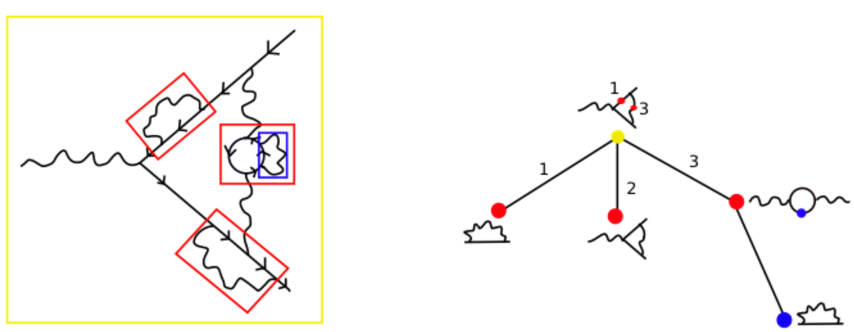
3. Graphs to trees and back again

We illustrate how the graphs are represented with trees via a few examples.

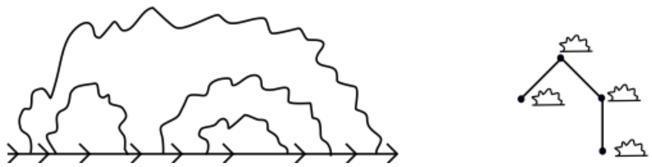
Example. Consider the following graph in QED:



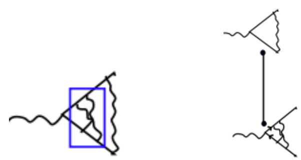
We break the above graph into pieces and form a tree as follows:



Example. Consider the following graph in QED and its tree representation:

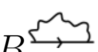


Example. Consider the following graph in QED and its tree representation:



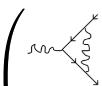
4. Insertion

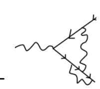
The above examples tell us what B_+ then has to be: *insertion*. For example:

Example. B_+ $\Big($$\Big) =$ .

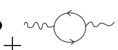
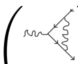
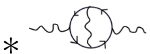
Note that inside the parentheses is the graph you want to insert, and in the superscript is the graph you want to insert it into. Furthermore, there are subtleties to this insertion:

1. There are different ways of inserting.

   Example. Consider B_+ $\Bigg($$\Bigg)$. There are three different ways the insertion could be performed. The solution to the problem is sum over all of them:

   B_+ $\Bigg($$\Bigg) =$  $+$  $+$ 

2. Double counting. Naively, for example, we compute:

B_+ $\left(\right.$$\left.\right) = 2 *$ 

This can be fixed with a messy fudge factor:

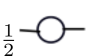
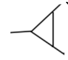
$$B_+^{\gamma}(X) = \sum_{\Gamma \in \mathcal{H}} \frac{bij(\gamma, X, \Gamma)}{|X|_{\vee}} \frac{1}{maxf(\Gamma)} \frac{1}{(\gamma|X)} \Gamma$$

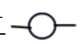
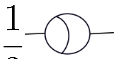
where

$maxf(\Gamma)$	:=	the number of insertion trees corresponding to Γ
$\lvert X\rvert_{\vee}$	:=	the number of distinct graphs coming from permuting external edges
$bij(\gamma, X, \Gamma)$	:=	the number of external edges of X to insertion places of γ
$(\gamma\lvert X)$	:=	the number of insertion places for X in γ

This fudge factor was constructed so that if we sum over all primitives with a given structre and insert into all places we get the series with each graph with its symmetry factor $\frac{1}{|Aut|}$. For a large example of this, see "Anatomy of a Gauge Theory" by D. Kreimer (arxiv.org).

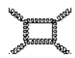
Example. Let x be a counting variable.

$xB_+^{\frac{1}{2}}$ $\left(\left(\mathbb{1} + x\right.\right.$  $\left.\left.\right)^2 + O(x^2)\right) =$

$x\frac{1}{2}$ $+ x^2\frac{1}{2}$ $+ O(x^2)$

Exercise. Define $X := \mathbb{1} - B_+^{\frac{1}{2}}$ $\left(\frac{1}{X^2}\right)$ in ϕ^3. Calculate X to $O(X^4)$. (The answer is an in K. Yeats' thesis).

Example. Calculate the following in QCD:

$$B_+^{\frac{1}{2}}$$ ( +  + )

The answer is in the "Anatomy" paper by Kreimer.

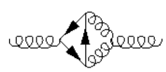
3. We want the Hochschild 1-cocycle property for B_+^γ, that is we want:

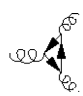
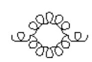
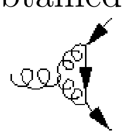
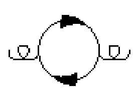
$$\Delta B_+ = (id \otimes B_+)\Delta + B_+ \otimes \mathbb{1}.$$

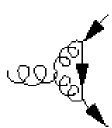
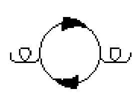
Thus, we need γ to be primitive since $\Delta B_+^\gamma(\mathbb{1}) = \Delta(\gamma)$

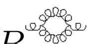
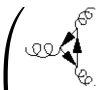
$(\Delta(B_+^\gamma(\mathbb{1}) = (id \otimes B_+^\gamma)(\mathbb{1} \otimes \mathbb{1}) + B_+(\gamma) \otimes \mathbb{1} = \mathbb{1} \otimes \gamma + \gamma \otimes \mathbb{1}.)$

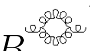
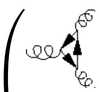
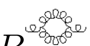
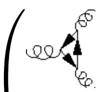
4. But this still is not enough. For example:

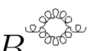
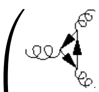
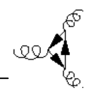
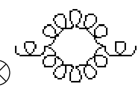
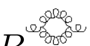
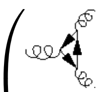


can be obtained by inserting  into ; and also by inserting  into . Thus,

 $\otimes$ 

appears in $\Delta \Big(B_+^{\,}$  $\big($  $\big)\Big)$ but can not appear in

$\big(id \otimes B_+$  $\big)\,\Delta\big($  $\big) + B_+$  $\big($  $\big) \otimes \mathbb{1}$

$= \mathbb{1} \otimes B_+$  $\big($  $\big) +$  $\otimes$  $+ B_+$  $\big($  $\big) \otimes \mathbb{1}.$

It is too much to ask for each B_+^γ to be a 1-cocyle. We want:

$$B_+^{k,r} = \sum_{\gamma\ primitive} B_+^\gamma$$

to be a 1-cocycle, where k is the loop, order, and $|\gamma| = k$ and the external structure of γ is r.

This still does not fix the problem, so we must go back to physics and get some identities between graphs.

Example. Slavnov Taylor identities.

The B_+ sub-Hopf-algebra

Karen Yeats
(Scribe: Guillaume Chapuy)

May 20, 2010

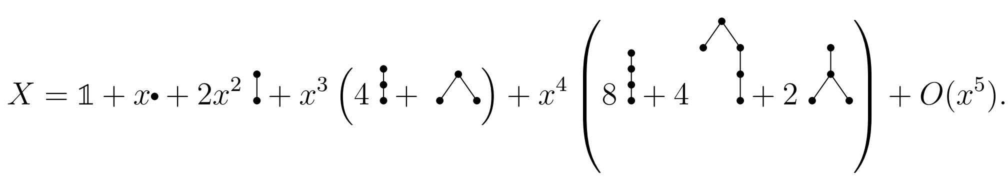
Today it is going to be a lot of gruesome examples, all of them dealing with the following problem: given a Hopf algebra $\mathcal{H}$ and a collection of elements $(c_i)_{i\geq 0}$ in $\mathcal{H}$, is the algebra $\mathcal{C}$ generated by the c_i's a Hopf-algebra? (for the induced structure). In this case, we will say that $\mathcal{C}$ is a *sub-Hopf-algebra*. In the case of the graded Hopf algebras we consider in this course, the only property we need for $\mathcal{C}$ to be a sub-Hopf-algebra is the "stability" of the coproduct: indeed, the stability of the antipode S will then follow from the induction formula we have already used several times.

1 A first example

We will focus in particular on structures defined recursively using (systems of) equations involving the B_+ operation. Our first example is the equation:

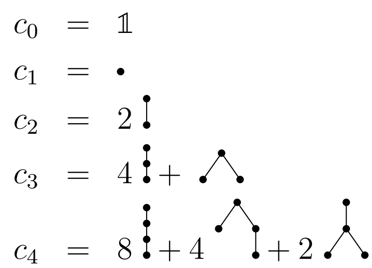
$$X = \mathbb{1} + xB_+(X^2)$$

in the Connes-Kreimer algebra $\mathcal{T}$ of rooted trees. Here, we have to think of x as a formal variable, and of $X = X(x)$ as a formal power series in x with coefficients in $\mathcal{T}$. Let's iterate to find the first few terms:



Side-remark: If we think about what this equation does, we understand it generates (computer scientists) binary trees, where we have forgotten the left-right ordering of the children of each node. For example, the term 4 corresponds to the 4 different ways of realizing the tree as a binary tree (2 left/right choices for each edge). Not suprisingly, we recognize here the sequence $1, 1, 2, 5, 14, \ldots$ of Catalan numbers.

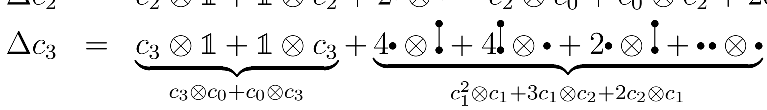
Back to our problem, let's write $X = \sum_{n\geq 0} c_n x^n$. The first terms are:



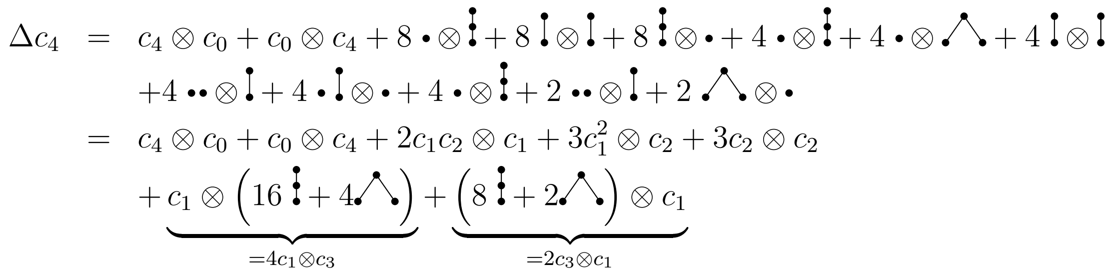
Now take the subalgebra $\mathcal{C}$ generated by the c_i's: is this a sub-Hopf-algebra of $\mathcal{T}$? Equivalently, is it true that for all $i \geq 0$, the coproduct Δc_i is a linear combination of terms of the form $c \otimes c'$, with $c, c' \in \mathcal{C}$?

Let's check on the first terms:

$$
\begin{aligned}
\Delta c_0 &= \mathbb{1} \otimes \mathbb{1} = c_0 \otimes c_0 \\
\Delta c_1 &= \bullet \otimes \mathbb{1} + \mathbb{1} \otimes \bullet = c_0 \otimes c_1 + c_1 \otimes c_0 \\
\Delta c_2 &= c_2 \otimes \mathbb{1} + \mathbb{1} \otimes c_2 + 2\bullet \otimes \bullet = c_2 \otimes c_0 + c_0 \otimes c_2 + 2c_1 \otimes c_1
\end{aligned}
$$



Observe that, in the four equations above, each term of the coproduct is "directly" of the form $c \otimes c'$, with $c, c' \in \mathcal{C}$. The next case is more interesting: we need to re-arrange the terms (by grouping some of them together) to obtain what we want. We have:



Will that always work? Yes! In fact we have:

Proposition 1.1. *For all $n \geq 0$ we have:*

$$\Delta c_n = \sum_{k=0}^{n} P_k^n \otimes c_k$$

where the (P_k^n)'s are defined by induction:

$$P_{k+1}^{n+1} = \sum_{i=0}^{n-k} P_0^i P_k^{n-i}.$$

It follows that the algebra generated by the c_i's is a sub-Hopf-algebra.

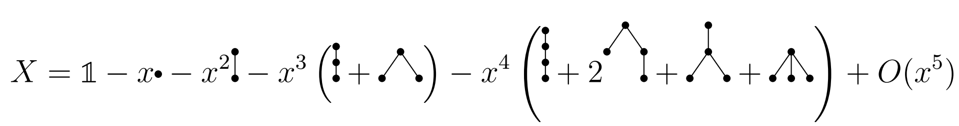
Observe that the proposition follows by an immediate induction.
Exercise: What is the base case of this induction?

2 Another example

This time we consider the equation:

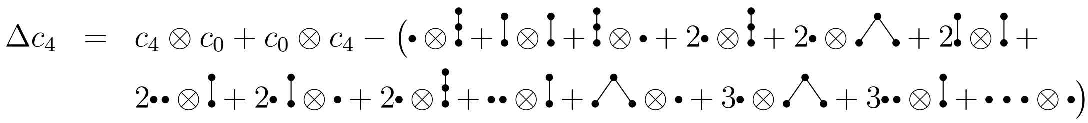
$$X = \mathbb{1} - xB_+\left(\frac{1}{X}\right).$$

Again, X is a formal power series in x with coefficients in $\mathcal{T}$. Observe that the series X defined in this way has constant coefficient $\mathbb{1}$ (and that there is no problem with the inverse $\frac{1}{X}$). Again we can compute by hand the first few terms:



Again we write $X = \sum_{n\geq 0} c_n x^n$ and we consider the subalgebra generated by the c_n's. Is it a sub-Hopf-algebra... $\rightarrow$ yes! For example, we have:

$$\begin{aligned}
\Delta c_0 &= c_0 \otimes c_0 \\
\Delta c_1 &= c_0 \otimes c_1 + c_1 \otimes c_0 \\
\Delta c_2 &= c_2 \otimes c_0 + c_0 \otimes c_2 - c_1 \otimes c_2 \\
\Delta c_3 &= c_3 \otimes c_0 + c_0 \otimes c_3 + (-c_2 + c_1^2) \otimes c_1 - 3c_1 \otimes c_2
\end{aligned}$$



$$= c_4 \otimes c_0 + c_0 \otimes c_4 - c_1^3 \otimes c_1 + 2c_1c_2 \otimes c_1 + 6c_1^2 \otimes c_2 - 3c_2 \otimes c_2 - 5c_1 \otimes c_3 - c_3 \otimes c_1$$

3 Some results

Now that we have seen two magical examples, let us state some results. Recall that in one previous lecture we have seen that the B_+ operator is a 1-cocycle for the Hochschild cohomology. We have:

Theorem 3.1 (Bergbauer, Kreimer, 2005)**.** *Let $\mathcal{H}$ be a connected graded Hopf algebra which is either free or free-commutative as an algebra. Let $(B_+^{d_n})_{n\in\mathbb{N}}$ be a collection of 1-cocycles fo the Hochschild cohomology*[1]*. Then the combinatorial Dyson-Schwinger equation:*

$$X = \mathbb{1} + \sum_{n=1}^{\infty} x^n w_n B_+^{d_n}(X^{n+1})$$

has a unique solution $X = X(x) = \sum_{n\geq 0} c_n x^n$ given by:

$$\begin{aligned}
c_0 &= \mathbb{1} \\
c_n &= \sum_{m=1}^{n} w_m B_+^{d_m} \left(\sum_{k_1+k_2+\cdots+k_{m+1}=n-m, k_i\geq 0} c_{k_1} c_{k_2} \ldots c_{k_{m+1}} \right)
\end{aligned}$$

and the c_n generate a sub-Hopf-algebra, with:

$$\Delta c_n = \sum_{k=0}^{n} P_k^n \otimes c_k$$

[1] in the two previous examples, we had only one operator, i.e. $B_+^{d_n} = B_+$ for all n.

where $P_k^n = \sum_{l_1+l_2+\cdots+l_{k+1}=n-k} c_{l_1}c_{l_2}\ldots c_{l_{k+1}}$.

Exercise: Find an elementary inductive proof of this theorem, which is less than 2 pages (the current proof is a complicated two-pages induction).

Theorem 3.2 (Foissy, 2007)**.** *Let* $P = \sum_{n=0}^{\infty} p_n x^n$ *be a formal power series with constant term* $p_0 = 1$*. Then the equation:*

$$X_P = B_+(P(X_P))$$

in the Connes-Kreimer algebra $\mathcal{T}$ *has a unique solution* $X_P = \sum_{n=0}^{\infty} a_n$*, where for all* $n \geq 0$*,* $a_n \in \mathcal{T}$ *is homogeneous of size* n*. Moreover, the following are equivalent:*

1. *The algebra generated by the* a_i*'s is a sub-Hopf-algebra;*
2. $\exists(\alpha, \beta) \in \mathbb{Q}^2$ *such that* $(1 - \alpha\beta x)P'(x) = \alpha P(x)$*;*
3. $\exists(\alpha, \beta) \in \mathbb{Q}^2$ *such that:*
   - *a.* $P(x) = 1$ *if* $\alpha = 0$*,*
   - *b.* $P(x) = e^{\alpha x}$ *if* $\beta = 0$*,*
   - *c.* $P(x) = (1 - \alpha\beta x)^{-\frac{1}{\beta}}$ *else.*

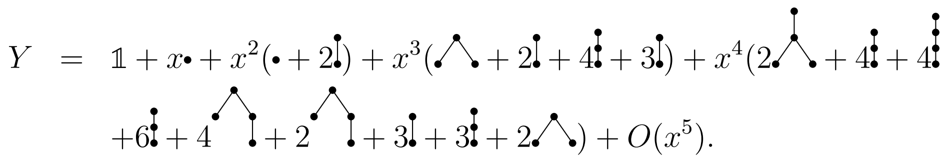
Moreover, there is an analogous statement in the non-commutative case.

Remark: Theorems 3.1 and 3.2 are different. In the first one, there is a variable x, whereas in the second one there is not (and the terms in the expansion are only grouped according to their *size*). **What difference does a counting variable make?** Let's consider an example:

Example-A: Consider the equation:

$$Y = \mathbb{1} + xB_+(Y^2) + x^2B_+(Y^3).$$

This equation is of the type of Theorem 3.1, so we know that we have a good sub-Hopf-algebra. Combinatorially, we can think of this equation as describing certain trees with "two types of nodes" (or two colours) to which we assign different weights (different powers of x). As before we can expand and find:



Exercise: check that if we note $Y = \sum c_k x^k$ then:

$$
\begin{aligned}
\Delta c_3 &= c_3 \otimes c_0 + c_0 \otimes c_3 + c_1^2 \otimes c_1 + 2c_2 \otimes c_1 + 3c_1 \otimes c_2, \\
\Delta c_4 &= c_4 \otimes c_0 + c_0 \otimes c_4 + 2c_1 c_2 \otimes c_1 + 2c_3 \otimes c_1 + 3c_1^2 \otimes c_2 + 2c_2 \otimes c_2 + 4c_1 \otimes c_3.
\end{aligned}
$$

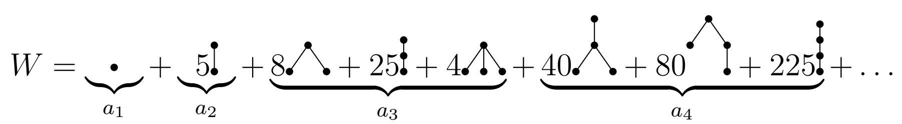
Example-B: Now we consider the same "combinatorial equation" as in Example-A, but just count by the number of vertices (in the spirit of Theorem 3.2). Let's write $Y = \mathbb{1} + 2W$ so that

$$\mathbb{1} + 2W = \mathbb{1} + B_+((\mathbb{1} + 2W)^2) + B_+((\mathbb{1} + 2W)^3).$$

Using the linearity of B_+, we obtain:

$$W = B_+(\mathbb{1} + 5W + 8W^2 + 4W^3),$$

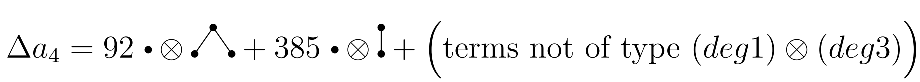
so from Theorem 3.2 we know that this is not the correct form to generate a sub-Hopf-algebra. This can also be checked directly as follows. First, let's write the expansion:



Now let us try to "check" as before all the Δc_i's. Clearly we have no problems for $i = 0, 1, 2$. For $i = 3$, we obtain

$$\Delta a_3 = a_3 \otimes \mathbb{1} + \mathbb{1} \otimes a_3 + \frac{41}{5} a_1 \otimes a_2 + 5 a_2 \otimes a_1,$$

so we still have no problem, even if the fraction $\frac{41}{5}$ already tells us that things are not very nice. Actually the first problem appears for $i = 4$. Indeed, let us consider the $(deg1) \otimes (deg3)$ part of Δa_4. We have after computation:



and we see that the two terms are *not* in the correct ratio to be a scalar factor of $a_1 \otimes a_3$. Therefore (as we already knew from Theorem 3.2) the algebra generated by the a_i's is *not* a sub-Hopf-algebra.

Exercise/Research problem: Clean this up! For example, give a Foissy-type theorem in the case of vertices with different weights.

4 Systems (of colored rooted trees, of Feynman graphs...)

I (Karen Yeats) am principally interested in systems of the form:

$$X^r = \mathbb{1} + \mathrm{sgn}(s_r) \sum_k x^k B_+^{k,r}\big(X^r Q^k\big) \tag{1}$$

where $Q = \prod_r (X^r)^{s_r}$ and (X^r) is a collection of formal power series in x. Typically, the index r will run over all possible external leg structures of certain Feynman graphs. The quantity Q is called the "invariant charge".

Note: The equation above is possibly considered *after taking some appropriate quotient* (in the spirit of the previous lecture).

Example: Consider the following system in the QED theory:

$$\bullet\ X^{\sim\!\sim\!<} = \mathbb{1} + xB_+\left(\frac{\left(X^{\sim\!\sim\!<}\right)^3}{\left(X^{\rightarrow}\right)^2 X^{\sim\sim}}\right) + xB_+(\dots) + \dots$$

$$= \mathbb{1} + \sum_{\substack{\gamma \text{ primitive with external} \\ \text{structure } \sim\!\sim\!<}} x^{|\gamma|} B_+^{\gamma}\left(\frac{\left(X^{\sim\!\sim\!<}\right)^{1+2|\gamma|}}{\left(X^{\sim\sim}\right)^{|\gamma|}\left(X^{\rightarrow}\right)^{2|\gamma|}}\right).$$

$$\bullet\ X^{\sim\sim} = \mathbb{1} - xB_+\left(\frac{\left(X^{\sim\!\sim\!<}\right)^2}{\left(X^{\rightarrow}\right)^2}\right)$$

$$\bullet\ X^{\rightarrow} = \mathbb{1} - xB_+\left(\frac{\left(X^{\sim\!\sim\!<}\right)^2}{X^{\sim\sim} X^{\rightarrow}}\right).$$

Note that for this system the invariant charge is: $Q = \dfrac{\left(X^{\sim\!\sim\!<}\right)^2}{X^{\sim\sim}\left(X^{\rightarrow}\right)^2}$

We finish this lecture by mentioning another theorem of Foissy that deals with systems of equations and coloured trees. We will not state the theorem exactly, rather give an idea of what it says:

Theorem 4.1 (Foissy, 2010). *We consider coloured trees, with n colours. We let B_+^d be the same operator B_+ as before except that it gives the colour* *d to the root. Consider the system of equations:*

$$\begin{cases} X_1 & = & B_+^1(F_1(X_1, X_2, \dots, X_n)) \\ X_2 & = & B_+^2(F_2(X_1, X_2, \dots, X_n)) \\ & & \dots \\ X_n & = & B_+^n(F_n(X_1, X_2, \dots, X_n)) \end{cases}$$

where $F_1, F_2, \dots F_n$ are formal power series in n indeterminates. Then [Foissy 2010] describes the cases in which the solution generates a sub-Hopf-algebra.

What about systems of the type of Equation (1)??

→ There are ideas in the "anatomy" paper by Kreimer.

→ Walter van Suijlekom has done things rigourously for scalar field theory, QED, QCD.

Exercise/Research problem:

→ A Foissy-type theorem for these systems?.

→ Give an "if and only if" condition for what Hopf ideals give sub-Hopf-algebra for B_+.

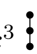
Renormalization Hopf Algebras V: The B_+ Lie Algebras

Karen Yeats
(Scribe: Pinar Colak)

May 27, 2010

1 The Linear Story

First consider the following example:

$X = \mathbb{1} + xB_+(X)$ $X = \mathbb{1} + x \bullet + x^2$ $+ x^3$  $+ \cdots = \sum_{k\geq 0} c_k x^k$. Thus c_k is the tree which is just a chain of k vertices and $\Delta c_k = \sum_{l=0}^{k} c_l \otimes c_{k-l}$.

This is called the ladder Hopf algebra $\mathcal{L}$ and is isomorphic to the Hopf algebra of symmetric functions. The Hopf algebra of symmetric functions is generated (for instance) by h_k, the complete symmetric functions. i.e., $h_2 = x_1^2 + x_1x_2 + x_2^2 + x_1x_3 + \cdots$ and has $\Delta(h_k) = \sum_{l=0}^{k} h_l \otimes h_{k-l}$.

Now let's play our standard game of looking at the Lie algebra of the primitive elements of the dual of an interesting Hopf algebra. Consider $P(\mathcal{T}^*)$. It is generated by $Z_k = Z_{c_k}$ and

$$\begin{aligned}
[Z_i, Z_j]c_k &= (Z_i \otimes Z_j) - (Z_j \otimes Z_i)\Delta c_k \\
&= (Z_i \otimes Z_j - Z_j \otimes Z_i) \sum_{l=0}^{k} c_l \otimes c_{k-l} = 0.
\end{aligned}$$

So it is an abelian Lie algebra, and not so interesting from this vantage point. However, you can still go somewhere interesting with this. You can add tree operation to the Lie algebra, like B_+, or the grading operator Y. For examples, please see arXiv: hep-th/0201157, math/0309042, math-ph/0408053.

2 The Foissy Story

We looked at Foissy's results on sub Hopf algebras last time, now consider the analogous Lie story in the cases most interesting to us. For Foissy's setup, please see arXiv:0709.1204. He gives necessary and sufficient conditions for the solution of the equation

$$W = \beta + (P(W))$$

to generate a sub Hopf algebra. The sub Hopf case which is most interesting to us is when

$$P(h) = (1 - \alpha\beta h)^{-1/\beta}.$$

The other cases can be found in the reference. We get the linear example of the previous section for $X = 1 + W$, $\alpha = 1$, $\beta = -1$.

Now consider $\alpha = 1$ and $\beta \neq -1$. We denote this Hopf subalgebra by $\mathcal{A}_{\alpha,\beta}$. We have two important special cases:

- $X + \mathbb{1} - xB_+(\frac{1}{X})$. To put this into Foissy's framework let $X = \mathbb{1} - W$ then $W = xB_+((\mathbb{1} - W)^{-1})$. So this is $\alpha = 1$, $\beta = 1$.

- $X = \mathbb{1} + xB_+(X^2)$. To put this into Foissy's framework let $X = \mathbb{1} + W$ then $W = xB_+((\mathbb{1} + W)^2)$. So this is $\alpha = 1$, $\beta = -1/2$.

Foissy says that all the $\mathcal{A}_{1,\beta}$ with $\beta \neq -1$ are isomorphic to the Faà di Bruno Hopf algebra.

Definition 1. Let $\mathcal{F}$ as a ring be the polynomial ring in countably many variables Y_i, $i \geq 1$. Let $\mathbf{Y} = 1 + \sum_{n=1}^{\infty} Y_n$ then $\Delta(\mathbf{Y}) = \sum_{n=1}^{\infty} (\mathbf{Y})^{n+1} \otimes Y_n$. This makes $\mathcal{F}$ into a Hopf algebra called the Faà di Bruno Hopf algebra.

This definition comes from the following. Let G be a ring of formal series over K in one variable h which are of the form $h + \sum_{n\geq 1} a_n h^{n+1}$. Then let $Y_i : G \to K$ be defined by $h + \sum_{n\geq 1} a_n h^{n+1} \mapsto a_i$. Define $\Delta(f)(P \otimes Q) = f(Q \circ P)$.

Exercise. Show that these two expressions for Δ are equivalent.

Now we calculate the bracket on $P(\mathcal{F}^*)$. First let $Z_i = Z_{Y_i}$. Note that $[Z_i, Z_j]$ is some multiple of Z_{i+j} by the grading. So we get

$$\begin{aligned}
[Z_i, Z_j](\mathbf{Y}) &= (Z_i \otimes z_j - Z_j \otimes Z_i)\Delta(\mathbf{Y}) \\
&= (Z_i \otimes z_j - Z_j \otimes Z_i)\sum_{n=0}^{\infty}(\mathbf{Y})^{n+1} \otimes Y_n \\
&= Z_i(\mathbf{Y})^{j+1} - Z_j(\mathbf{Y})^{i+1} = j + 1 - (i+1) = j - i.
\end{aligned}$$

Hence $[Z_i, Z_j] = (j - i)Z_{i+j}$.

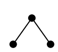
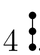
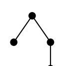
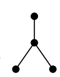
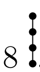
Foissy calls this the Faà di Bruno Lie algebra.

It is also half of the Witt algebra.

Definition 2. The Witt algebra is the Lie algebra of meromorphic vector fields on the Riemann sphere which are holomorphic except at 2 fixed points. It is generated by $L_n = -Z^{n+1}\frac{\partial}{\partial Z}$, where $n \in \mathbb{Z}$. For these generators we get $[L_m, L_n] = (m - n)L_{m+n}$.

Let's see this in our 2 examples:

For $X = \mathbb{1} + xB_+(X^2)$, recall

$c_0 = \mathbb{1},\ c_1 = \bullet,\ c_2 = 2$, $c_3 =$  $+ 4$ , $c_4 = 4$  $+ 2$  $+ 8$ , and

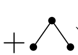
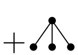
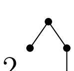
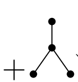
$$\begin{aligned}
&\triangle c_1 = c_1 \otimes c_0 + c_0 \otimes c_1, \\
&\triangle c_2 = c_2 \otimes c_0 + c_0 \otimes c_2 + 2c_1 \otimes c_1, \\
&\triangle c_3 = c_3 \otimes c_0 + c_0 \otimes c_3 + 3c_1 \otimes c_2 + (c_1^2 + 2c_2) \otimes c_1, \\
&\triangle c_4 = c_4 \otimes c_0 + c_0 \otimes c_4 + (2c_3 + 2c_1c_2) \otimes c_1 + (3c_1^2 + 3c_2) \otimes c_2 + 4c_1 \otimes c_3.
\end{aligned}$$

Let's calculate some brackets ($Z_i = Z_{c_i}$).

$$[Z_1, Z_2]c_3 = 3 - 2 = 1(= j - i), \quad [Z_1, Z_3]c_4 = 4 - 2 = 2, \quad \text{etc...}$$

Now let's consider the other example: $X = \mathbb{1} - xB_+(\frac{1}{X})$.

Recall

$c_0 = \mathbb{1},\ c_1 = -\bullet,\ c_2 = -$ , $c_3 = -($  $+$ $),\ c_4 = -($  $+$  $+ 2$  $+$ $)$, and

$$\triangle c_3 = c_3 \otimes c_0 + c_0 \otimes c_3 + (-c_2 + c_1^2) \otimes c_1 - 3c_1 \otimes c_2,$$
$$\triangle c_4 = c_4 \otimes c_0 + c_0 \otimes c_4 + (-c_3 - c_1^3 + 2c_2) \otimes c_1 + (-3c_2 + 6c_1) \otimes c_2 - 5c_1 \otimes c_3.$$

Then

$$[Z_1, Z_2]c_3 = -3+1 = -2 = (-2)(j-i), \ \ [Z_1, Z_3]c_4 = -5+1 = -4 = (-2)(j-i), \ \ \text{etc...}$$

This is the general picture for $\alpha = 1$, $\beta \neq 1$. Foissy gives explicit isomorphism $A_{1,\beta} \cong A_{1,\beta'}$, $\beta \neq 1$, $\beta' \neq 1$. He gives the coproducts explicitly. He also gives an explicit isomorphism with the Faà di Bruno Hopf algebra.

Exercise. Trace through the isomorphism for the two given examples to write Y_n as trees.

Research Problem. Answer the following questions.

- What sense can we make Z_i for $i < 0$ from the Witt algebra?
- What can the theory of Witt algebras say about trees or physics?
- Can trees say anything about Witt algebras?

3 The Generalized Witt Story

In Foissy's systems (arXiv:0909.0358) he gets 3 cases:

1. Path Lie algebras,
2. Iterated extensions of Faà di Bruno Lie algebras,
3. Iterated extensions of an abelian Lie algebra.

Foissy gives explicit expressions for the brackets. But Foissy's systems aren't really the sort of systems we're most interested in. Some playing around with some examples suggests that we may want to look at generalized Witt algebras. Without going into any details, the braket for these looks like

$$[t^x\partial_1, t^y\partial_2] = t^{x+y}(\partial_1(y)\partial_2 - \partial_2(x)\partial_1)$$

If we have only one ∂_i and it is the grading operator we get what we had before. In the more general case it should be something about the number of intertions of a particular sort, or the number of intertion places for a graph of a given external structure. Unfortunately, there was no time to show a worked example, or to even properly explain these objects.