Preliminary Notes on:

Four-vertex Model and Random Tilings
N. M. Bogoliubov
arXiv: 0711.0030v1

[I] Warming Up to Lattice Configurations

Before trying any interpretation of the algebraic Bethe ansatz in terms of the enumeration of lattice configurations on plane partitions we need to understand its "algebraic" structure.

Here we do not care about the diagonalization of some Hamiltonian but we directly look for solutions of the conditions for integrability.

Let $N \in \mathbb{N}^*$ be a fixed integer and consider $\mathcal{H} = \mathbb{C}^{2\,\otimes N}$ the space of states of the system with $A = \mathbb{C}^2$ an auxiliary space.

Consider the R-operator $R_{a_1,a_2}(u,v)$, acting on $A \otimes A$ with u and v complex parameters, defined in the canonical basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$ by the matrix

$$R_{a_1,a_2}(u,v) \overset{\text{def}}{=} \begin{pmatrix} f(u,v) & 0 & 0 & 0 \\ 0 & g(u,v) & 1 & 0 \\ 0 & 0 & g(u,v) & 0 \\ 0 & 0 & 0 & f(u,v) \end{pmatrix}$$

with $f(u,v) = \dfrac{u^2}{u^2 - v^2}$ and $g(u,v) = \dfrac{uv}{u^2 - v^2}$.

In order to perform certain calculations it is useful to introduce the Pauli matrices, generators of $sl_2(\mathbb{C})$:

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In terms of these matrices the R-operator is expressed as:

$$\begin{aligned} R_{a_1,a_2}(u,v) = {} & \frac{1}{2}(id_{a_1} + \sigma^3_{a_1})\left[\frac{f(u,v)}{2}(id_{a_2} + \sigma^3_{a_2}) + \frac{g(u,v)}{2}(id_{a_2} - \sigma^3_{a_2}) \right] \\ & + \frac{1}{2}(id_{a_1} - \sigma^3_{a_1})\left[\frac{g(u,v)}{2}(id_{a_2} + \sigma^3_{a_2}) + \frac{f(u,v)}{2}(id_{a_2} - \sigma^3_{a_2}) \right] \\ & + \sigma^+_{a_1}\sigma^-_{a_2} \quad . \end{aligned}$$

<u>Proposition</u>: The R-operator is a solution of the Yang-Baxter equation on $\mathrm{End}(A \otimes A \otimes A)$.

<u>Proof</u>: It is a calculation. We need to show that the following equation holds:

$$R_{a_1,a_2}(u,v)R_{a_1,a_3}(u,w)R_{a_2,a_3}(v,w) = R_{a_2,a_3}(v,w)R_{a_1,a_3}(u,w)R_{a_1,a_2}(u,v).$$

It follows from the fact that operators acting on different factors in the tensor product $A \otimes A \otimes A$ commute:

$$\sigma^\circ_{a_i}\sigma^\circ_{a_j} = \sigma^\circ_{a_j}\sigma^\circ_{a_i} \quad \text{if } i \neq j.$$

Using the representation of the R-operator in terms of Pauli matrices, any of the 9 terms coming from the left hand side of Yang-Baxter equation is of the form:

$$\begin{aligned}
& {}^\bullet a_1 \, {}^\bullet a_2 \, {}^\bullet a_1 \, {}^\bullet a_3 \, {}^\bullet a_2 \, {}^\bullet a_3 && \text{since the components acting on the different spaces are factorized,} \\
= \; & {}^\bullet a_2 \, {}^\bullet a_1 \, {}^\bullet a_1 \, {}^\bullet a_3 \, {}^\bullet a_2 \, {}^\bullet a_3 && \\
= \; & \cdots && \text{simply rearranging with the use of the commutation property,} \\
= \; & {}^\bullet a_2 \, {}^\bullet a_3 \, {}^\bullet a_1 \, {}^\bullet a_3 \, {}^\bullet a_1 \, {}^\bullet a_2 \, .
\end{aligned}$$

As a consequence any term from the left hand side of Yang-Baxter equation can be rearranged in a term from the right hand side of the relation.

■

The L-operators $L_{a,n}(u) \in \mathrm{End}(A \otimes \mathcal{H})$ with $n \in [\![1,N]\!]$ and u a complex parameter are defined as:

$$\begin{aligned}
L_{a,n}(u) \overset{\text{def}}{=} & -\frac{u}{4}(\mathrm{id}_a + \sigma_a^z)(\mathrm{id}_n + \sigma_n^z) \\
& + \frac{u^{-1}}{4}(\mathrm{id}_a - \sigma_a^z)(\mathrm{id}_n + \sigma_n^z) \\
& + \sigma_a^+ \sigma_n^- + \sigma_a^- \sigma_n^+ \, .
\end{aligned}$$

Proposition: The L-operators satisfy the intertwining and ultralocality equation on $\mathrm{End}(A \otimes A \otimes \mathcal{H})$.

Proof: It is exactly the same argument as in the proof that the R-operator satisfies the Yang-Baxter equation.

■

The monodromy of the system is the product of the L-operators over all the sites:

$$T_a(u) \stackrel{\text{def}}{=} L_{a,N}(u) L_{a,N-1}(u) \cdots L_{a,1}(u) .$$

To apply the algebraic Bethe ansatz we use the decomposition $T_a(u) \in \mathrm{End}(A \otimes St) \cong M_2(\mathbb{C}) \otimes \mathrm{End}(St)$ to write the monodromy as a 2 by 2 matrix whose coefficients are operators acting on St:

$$T_a(u) \stackrel{\text{def}}{=} \begin{pmatrix} \mathcal{A}(u) & \mathcal{N}_+(u) \\ \mathcal{N}_-(u) & \mathcal{D}(u) \end{pmatrix}$$

and $\mathcal{Q}(u) \stackrel{\text{def}}{=} \mathrm{tr}_a(T_a(u)) = \mathcal{A}(u) + \mathcal{D}(u) \in \mathrm{End}(St)$.

From the first two propositions we know that the monodromy satisfies an intertwining equation:

$$R_{a_1,a_2}(u,v) T_{a_1}(u) T_{a_2}(v) = T_{a_2}(v) T_{a_1}(u) R_{a_1,a_2}(u,v) .$$

Next we need to understand which relations are linking the operators $\mathcal{A}$, $\mathcal{D}$, $\mathcal{N}_+$ and $\mathcal{N}_-$.

Lemma: The operators $\mathcal{A}, \mathcal{D}, \mathcal{N}_-, \mathcal{N}_+$ satisfy the following relations:

$$\mathcal{N}_+(u)\mathcal{N}_+(v) = \mathcal{N}_+(v)\mathcal{N}_+(u) \quad , \quad \mathcal{N}_-(u)\mathcal{N}_-(v) = \mathcal{N}_-(v)\mathcal{N}_-(u) ,$$

$$\mathcal{N}_-(u)\mathcal{N}_+(v) = g(v,u)\left[\mathcal{A}(u)\mathcal{D}(v) - \mathcal{A}(v)\mathcal{D}(u) \right] ,$$

$$\mathcal{A}(u)\mathcal{N}_+(v) = f(v,u)\,\mathcal{N}_+(v)\mathcal{A}(u) + g(u,v)\,\mathcal{N}_+(u)\mathcal{A}(v) ,$$

$$\mathcal{D}(u)\mathcal{N}_+(v) = f(u,v)\,\mathcal{N}_+(v)\mathcal{D}(u) + g(v,u)\,\mathcal{N}_+(u)\mathcal{D}(v) .$$

Proof: Since $\mathrm{End}(A\otimes A\otimes St) \cong M_4(\mathbb{C}) \otimes \mathrm{End}(St)$ the intertwining equation satisfied by the monodromy can be expressed as an equality between two 4 by 4 matrices whose coefficients are operators acting on St.

The relations of the lemma are simply the equalities of the coefficients.

$\blacksquare$

In the decomposition $\mathrm{End}(A\otimes St) \cong M_2(\mathbb{C}) \otimes \mathrm{End}(St)$ the L-operators are:

$$L_{a,m}(u) = \begin{pmatrix} \hat{a}_m(u) & \hat{\eta}_{+,m} \\ \hat{\eta}_{-,m} & \hat{d}_m(u) \end{pmatrix}$$

with
$$\hat{a}_m(u) = -\frac{u}{2}(id_m + \tau_m^3),$$
$$\hat{d}_m(u) = \frac{u^{-1}}{2}(id_m + \tau_m^3),$$
$$\hat{\eta}_{+,m} = \tau_m^- ,$$
$$\hat{\eta}_{-,m} = \tau_m^+ .$$

Acting on the elements of the canonical basis of $\mathbb{C}^2$ they give:

$$\begin{cases} \hat{a}_\circ(u)\,|\uparrow\rangle = -u\,|\uparrow\rangle \\ \hat{a}_\circ(u)\,|\downarrow\rangle = 0 \end{cases}, \qquad \begin{cases} \hat{\eta}_{+,\circ}\,|\uparrow\rangle = |\downarrow\rangle \\ \hat{\eta}_{+,\circ}\,|\downarrow\rangle = 0 \end{cases},$$

$$\begin{cases} \hat{\eta}_{-,\circ}\,|\uparrow\rangle = 0 \\ \hat{\eta}_{-,\circ}\,|\downarrow\rangle = |\uparrow\rangle \end{cases}, \qquad \begin{cases} \hat{d}_\circ(u)\,|\uparrow\rangle = u^{-1}\,|\uparrow\rangle \\ \hat{d}_\circ(u)\,|\downarrow\rangle = 0 \end{cases}.$$

Define the element of $\mathcal{St}$, $|\omega\rangle = |\uparrow\cdots\uparrow\rangle$ to be the state of the system with all the spins up.

Lemma: The vector $|\omega\rangle$ is a vacuum state.

Proof: By applying the product of the L-operators to $|\omega\rangle$ we get:

$$\mathbb{E}(u)|\omega\rangle = \alpha(u)|\omega\rangle, \quad \mathcal{D}(u)|\omega\rangle = \delta(u)|\omega\rangle, \quad \mathcal{N}_-(u)|\omega\rangle = 0$$

with $\alpha(u) = (-1)^N u^N$ and $\delta(u) = u^{-N}$. It is the definition of a vacuum state. ∎

For $K \in [\![1, N]\!]$, define the element of $\mathcal{St}$:

$$|u_1, \dots, u_K\rangle = \mathcal{N}_+(u_1)\cdots\mathcal{N}_+(u_K)\,|\omega\rangle .$$

Theorem: (Algebraic Bethe Ansatz) The vector $|u_1, \dots, u_K\rangle \in \mathcal{H}$ is an eigenvector of $\mathcal{T}(u)$ with eigenvalue

$$\alpha(u) \prod_{j=1}^{K} f(u_j, u) + \delta(u) \prod_{j=1}^{K} f(u, u_j)$$

if and only if for all $i \in [\![1, K]\!]$ the u_i's are solution of Bethe equation:

$$\frac{\alpha(u_i)}{\delta(u_i)} \prod_{\substack{j=1 \\ j \neq i}}^{K} \frac{f(u_j, u_i)}{f(u_i, u_j)} = 1 .$$

Proof: Since $\mathcal{T}(u) = \mathcal{A}(u) + \mathcal{D}(u)$ we look at the action of $\mathcal{A}(u)$ and $\mathcal{D}(u)$ on the state $|u_1, \dots, u_K\rangle$. From the lemmas we deduce that:

$$\mathcal{A}(u)|u_1, \dots, u_K\rangle = \alpha(u) \prod_{j=1}^{K} f(u_j, u) \prod_{k=1}^{K} \mathcal{N}_+(u_k)|w\rangle$$
$$+ \sum_{i=1}^{K} \alpha(u_i) g(u, u_i) \prod_{\substack{j=1 \\ j \neq i}}^{K} f(u_j, u_i) \prod_{\substack{k=1 \\ k \neq i}}^{K} \mathcal{N}_+(u_k)|w\rangle$$

and

$$\mathcal{D}(u)|u_1, \dots, u_K\rangle = \delta(u) \prod_{j=1}^{K} f(u, u_j) \prod_{k=1}^{K} \mathcal{N}_+(u_k)|w\rangle$$
$$+ \sum_{i=1}^{K} \delta(u_i)\, g(u_i, u) \prod_{\substack{j=1 \\ j \neq i}}^{K} f(u_i, u_j) \prod_{\substack{k=1 \\ k \neq i}}^{K} \mathcal{N}_+(u_k)|w\rangle .$$

So if we want $|u_1, \dots, u_K\rangle$ to be an eigenvector of $\mathcal{T}(u)$ with eigenvalue:

$$\alpha(u) \prod_{j=1}^{K} f(u_j, u) + \delta(u) \prod_{j=1}^{K} f(u, u_j) \quad ,$$

we need that for all $i \in [\![1, K]\!]$

$$\alpha(u_i)\, g(u, u_i) \prod_{\substack{j=1 \\ j \neq i}}^{K} f(u_j, u_i) + \delta(u_i)\, g(u_i, u) \prod_{\substack{j=1 \\ j \neq i}}^{K} f(u_i, u_j) = 0 .$$

But $g(u_i, u) = \dfrac{u_i u}{u_i^2 - u^2} = -g(u, u_i)$ so we can rewrite the condition as:

$$\alpha(u_i) \prod_{\substack{j=1 \\ j \neq i}}^{K} f(u_j, u_i) = \delta(u_i) \prod_{\substack{\ell=1 \\ \ell \neq i}}^{K} f(u_i, u_\ell) .$$

$\blacksquare$

The next step is to understand the properties of the eigenvectors of $\mathcal{Q}(u)$.

Let $s_1, \dots, s_K \in [\![1, N]\!]$ with $s_i < s_{i+1}$ represent a K-uplet of sites of the space of states (labelled 1 to N from the left to the right). Define $|s_1, \dots, s_K\rangle \in St$ to be a state of the system where the spins at the sites $s_1, \dots, s_K$ are down and up everywhere else, e.g. if $N=4$ and $K=1$ then $|s_1 = 3\rangle = |\uparrow\uparrow\downarrow\uparrow\rangle$.

Then we decompose an eigenvector of $\mathcal{Q}(u)$ as:

$$|u_1, \dots, u_K\rangle \overset{\text{def}}{=} \sum_{1 \leq s_1 < \dots < s_K \leq N} \chi_{s_1 \cdots s_K}(u_1, \dots, u_K)\, |s_1, \dots, s_K\rangle .$$

Proposition: The coefficients $X_{s_1 \cdots s_K}(u_1, \cdots, u_K)$ are non zero if and only if for all $i \in [\![1, K-1]\!]$, $s_{i+1} - s_i > 1$.

Proof: Since $|u_1, \cdots, u_K\rangle = \mathcal{N}_+(u_1) \cdots \mathcal{N}_+(u_K)|\omega\rangle$ a way to reformulate this proposition is to say that the action of $\mathcal{N}_+(u_1) \cdots \mathcal{N}_+(u_K)$ on $|\omega\rangle$ gives a linear combination of states where two spins down are separated by at least one spin up. Lets prove that.

Let $[L_{a,n}(u)]_{pq}$ denote the (p,q) coefficient of the matrix $L_{a,n}(u) \in M_2(\mathbb{C}) \otimes \mathrm{End}(\mathcal{H})$. Then taking explicitly the product of the L-operators we obtain:

$$\mathcal{N}_+(u) = \sum_{i_1, \cdots, i_{N-1} \in \{1,2\}} [L_{a,N}(u)]_{1 i_1} \cdots [L_{a,1}(u)]_{i_{N-1} 2} .$$

If we apply the $\mathcal{N}_+$ operator one time on $|\omega\rangle$ we get:

$$\mathcal{N}_+(u_1)|\omega\rangle = \sum_{i_1, \cdots, i_{N-1} \in \{1,2\}} [L_{a,N}(u)]_{1 i_1} [L_{a,N-1}(u)]_{i_1 i_2} \cdots [L_{a,1}(u)]_{i_{N-1} 2} |\uparrow \cdots \uparrow\rangle .$$

But the following action of the matrix element of the L-operators send the vector to zero:

$$[L_{a,\circ}(\circ)]_{21} |\uparrow\rangle = \hat{\eta}_{-,\circ} |\uparrow\rangle = 0 .$$

So the non zero terms of the sum are the one from which the sequence of matrix coefficients exclude $(i_m, i_{m+1}) = (2,1)$.

Moreover the sequence of matrix coefficients in any term of the sum has the form

$$(1 i_1), (i_1 i_2), \cdots\cdots, (i_{N-2}\, i_{N-1}), (i_{N-1}\, 2).$$

So going from 1 to 2 there is at least one coefficient $(i_m, i_{m+1}) = (1,2)$ which corresponds to the following action:

$$[L_{a,0}(0)]_{12} \, |\uparrow\rangle = \hat{\tau}_{+,0}(0) \, |\uparrow\rangle = |\downarrow\rangle.$$

But there cannot be more than one $(1,2)$ coefficient as if it was the case we would find the following pattern in the sequence of coefficients:

$$(1 i_1), \cdots, (12), \cdots, (12), \cdots, (i_{N-1}\, 2)$$

so going from 2 to 1 there must be a (21) coefficient which anhilates the term.

Thus the action of $\mathcal{N}_+$ on $|\omega\rangle$ generates all the states of the system with exactly one spin down. So the proposition is true for $K=1$.

To calculate the action of $\mathcal{N}_+(u_2)\,\mathcal{N}_+(u_1)$ on $|\omega\rangle$ we simply need to understand the action of $\mathcal{N}_+(u_2)$ on a typical state obtained from the action of $\mathcal{N}_+(u_1)$ on $|\omega\rangle$.

So we look at the action of $\mathcal{N}_+$ on $|s_1=p\rangle = |\uparrow\cdots\uparrow\downarrow\uparrow\cdots\uparrow\rangle$ i.e. the state of the system with only one spin down on site p.

The action of any coefficient of the L-operator on $|\downarrow\rangle$ gives zero except :

$$[L_{a,\circ}(\circ)]_{21}|\downarrow\rangle = |\uparrow\rangle .$$

Hence the nonzero terms of:

$$\mathcal{N}_+(u_2)|s_1=p\rangle = \sum_{i_1,\dots,i_{N-1}\in\{1,2\}} [L_{a,N}(u_2)]_{1 i_1}\cdots[L_{a,1}(u_2)]_{i_{N-1}2}\,|\uparrow\cdots\uparrow\downarrow\uparrow\cdots\uparrow\rangle$$

are the one for which the sequence of coefficients is of the form

$$(1\,i_1),\ \cdots\cdots,\ (21),\ \cdots,\ (i_{N-1}\,2)$$

withe $(i_p\, i_{p+1}) = (21)$. So we have two subsequences of coefficients $(1 i_1), \cdots, (21)$ and $(21), \cdots, (i_{N-1}\,2)$ both going from 1 to 2 hence applying the argument of $K=1$ there must be exactly one (12) coefficient in each of these subsequences.

Since the action of (21) produce a spin up between these two subsequences we conclude that the action of $\mathcal{N}_+$ twice on $|w\rangle$ produce all the states of the system with exactly two spins down separated by at least one spin up. This proves the proposition for $K=2$.

Now suppose the proposition is true for K. Applying $\mathcal{N}_+$ on the vector $|s_1 \cdots s_K\rangle$ requires in order to form a non zero term to have a sequence of coefficients of the form:

$$(1\, i_1), \cdots, (21), \cdots, (21), \cdots, (21), \cdots, (i_{N-1}\, 2)$$

with $(i_{s_1} i_{s_1+1}) = \cdots (i_{s_K} i_{s_K+1}) = (21)$. Using the argument of $K=1$ for each subsequence of the form $(21), \cdots, (21)$ plus $(1 i_1), \cdots, (21)$ and $(21), \cdots, (i_{N-1} 2)$ gives exactly $K+1$ spins down.

But because of the action of (21) at the separation of these subsequences, the spins down ~~are~~ are separated by at least one spin up.

Thus the action of $\mathcal{N}_+$ $K+1$ times on $|\omega\rangle$ produces all the states of the system with exactly $K+1$ spin down separated by at least one spin up.

This completes the induction.

■

Let $\tilde{\mathcal{P}}_N$ be the set of strictly increasing partitions $(s_1, \cdots, s_K)$ with $K \in [\![1, N]\!]$ and $s_{i+1} - s_i > 1$:

$$\tilde{\mathcal{P}}_N = \left\{ \pi = (s_1, \cdots, s_K) \;\middle|\; K \in [\![1,N]\!],\ 1 \leq s_1 < \cdots < s_K \leq N,\ s_{i+1} - s_i > 1 \right\}.$$

If $\tilde{\mathcal{P}}_N(K)$ denotes those partitions of length K then the algebraic Bethe ansatz tells us that an eigenvector of $\mathcal{A}(u)$ generates all the elements of $\tilde{\mathcal{P}}_N$ of a given length with coefficients $\chi_\pi(u_1, \cdots, u_K)$:

$$|u_1,\cdots,u_K\rangle = \sum_{\pi\in\tilde{\mathcal{P}}_N(K)} \chi_\pi(u_1,\cdots,u_K)\,|\pi\rangle .$$

Hence we have a map $\phi : St \longrightarrow \mathbb{N}$ which associates to each eigenvector of $\mathcal{A}(u)$ the number of elements of $\tilde{\mathcal{P}}_N(K)$:

$$\phi(|u_1,\cdots,u_K\rangle) = \lim_{\{\chi_\pi(u_1,\cdots,u_K)\}\to(1,\cdots,1)} \sum_{\pi\in\tilde{\mathcal{P}}_N(K)} \chi_\pi(u_1,\cdots,u_K) = \#\,\tilde{\mathcal{P}}_N(K)$$

since $\chi_\pi(u_1,\cdots,u_K)$ is nothing else but a characteristic function of the partition π.

Finally, Bogoliubov proposes an explicit form for the coefficients $\chi_{s_1\cdots s_K}(u_1,\cdots,u_K)$.

__Theorem:__ The coefficients $\chi_{s_1\cdots s_K}(u_1,\cdots,u_K)$ have the determinental form:

$$\chi_{s_1\cdots s_K}(u_1,\cdots,u_K) = (-1)^{s_1+\cdots+s_K}(u_1\cdots u_K)^{N+1-2K}\frac{\det(M)}{\prod_{1\le p<q\le K}(u_p^{-2}-u_q^{-2})}$$

with $M_{pq} = u_q^{-2(s_p+p-K)}$.

__Proof:__ It is a computation. Use the definition of the action of the $\mathcal{N}_+$ operators to show that this determinental form satisfies the inductive construction. ∎