Dyson-Schwinger equations II

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les Houches

A recursion for γ_k

Write $G(x,L) = 1 - \sum \gamma_k L^k$

Looking for a recursion for γ_k in terms of lower γ_ℓ

Essentially this is the RGE.

→ derive it directly from RG

or → use $\underline{S} * Y$

The renormalization group equation

For a vertex v

$$\left(\frac{\partial}{\partial L} + \beta(x)\frac{\partial}{\partial x} - \sum_{e \text{ adjacent to } v} \gamma^e(x)\right) x^{(\mathrm{val}(v)-2)/2} G^v(x, L) = 0$$

For an edge e

$$\left(\frac{\partial}{\partial L} + \beta(x)\frac{\partial}{\partial x} - 2\gamma^e(x)\right) G^e(x, L) = 0$$

where

$$\beta(x) = \partial_L x \phi_R(Q)|_{L=0} = \sum_{j \in \mathcal{R}} |s_j| \gamma_1^j(x) x$$

$$\gamma^e(x) = -\frac{1}{2}\partial_L G^e(x, L)|_{L=0} = \frac{1}{2}\gamma_1^e$$

Expanding the vertex

$$\left(\frac{\partial}{\partial L} + \beta(x)\frac{\partial}{\partial x} - \sum_{e \sim v} \gamma^e(x)\right) x^{\frac{\mathrm{val}(v)-2}{2}} G^v(x,L) = 0$$

$$= x^{\frac{\mathrm{val}(v)-2}{2}} \frac{\partial}{\partial L} G^v(x,L) + \beta(x)\frac{\mathrm{val}(v)-2}{2} x^{\frac{\mathrm{val}(v)-4}{2}} G^v(x,L)$$

$$+ x^{\frac{\mathrm{val}(v)-2}{2}} \beta(x)\frac{\partial}{\partial x} G^v(x,L) - x^{\frac{\mathrm{val}(v)-2}{2}} \frac{1}{2}\sum_{e\sim v} \gamma_1^e(x)\, G^v(x,L)$$

$$= x^{\frac{\mathrm{val}(v)-2}{2}} \left(\frac{\partial}{\partial L} G^v(x,L) + \beta(x)\frac{\partial}{\partial x} G^v(x,L) + \left[\cancel{\frac{\mathrm{val}(v)-2}{2}}\, \cancel{\frac{1}{\mathrm{val}(v)-2}}\left(\gamma_1^v + \cancel{\sum_{e\sim v}\gamma_1^e}\right) - \cancel{\frac{1}{2}\sum_{e\sim v}\gamma^e}\right] G^v(x,L)\right)$$

$$= x^{\frac{\mathrm{val}(v)-2}{2}} \left(\frac{\partial}{\partial L} G^v(x,L) + \beta(x)\frac{\partial}{\partial x} G^v(x,L) + \gamma_1^v G^v(x,L)\right)$$

$$0 = \left(\frac{\partial}{\partial L} + \beta(x)\frac{\partial}{\partial x} + \gamma_1^v(x)\right) G^v(x, L)$$

$$0 = \left(\frac{\partial}{\partial L} + \beta(x)\frac{\partial}{\partial x} - \gamma_1^e(x)\right) G^e(x, L)$$

$$\left(\frac{\partial}{\partial L} + \sum_{j\in\mathcal{R}} |s_j|\gamma_1^j(x)x\frac{\partial}{\partial x} + \operatorname{sign}(s_r)\gamma_1^r(x)\right) G^r(x, L) = 0.$$

The γ_k recursion

Extracting the coefficient of L^{k-1} and rearranging gives

$$G^r(x,L) = 1 \pm \sum \gamma_k^r L^k$$

$$k\gamma_k^r + \sum_j |s_j| \gamma_1^j(x) \times \frac{\partial}{\partial x} \gamma_{k-1}^r + \operatorname{sgn}(s_r)\gamma_1^r(x)\gamma_{k-1}^r = 0$$

so $$\gamma_k^r = \frac{1}{k}\Big(\gamma_1^r(x)\operatorname{sgn}(s_r) - \sum_j |s_j|\gamma_1^j(x)\, x\partial_x\Big)\gamma_{k-1}^r$$

Specializing to the single equation case gives

$$\gamma_k = \frac{1}{k}\gamma_1(x)(\text{sign}(s) - |s|x\partial_x)\gamma_{k-1}(x).$$

$S \star Y$ – some definitions

Some definitions

- Let S be the antipode of the Hopf algebra.
- Let Y be the grading operator. $Y($  $) = 3$ 
- Let

$$\sigma_1 = \partial_L \phi_r(S \star Y)|_{L=0}$$

and

$$\sigma_n = \frac{1}{n!} m^{n-1} (\underbrace{\sigma_1 \otimes \cdots \otimes \sigma_1}_{n \text{ times}}) \Delta^{n-1}$$

$S \star Y$ – some lemmas

- $S \star Y$ is 0 on products
-

$$\Delta([x^k]X^r) = \sum_{j=0}^{k} [x^j]X^r Q^{k-j} \otimes [x^{k-j}]X^r$$

$$\Delta([x^k]X^r Q^\ell) = \sum_{j=0}^{k} [x^j]X^r Q^{k+\ell-j} \otimes [x^{k-j}]X^r Q^\ell$$

Walter's formula in a different

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pulling out coeff → like a derivative.

$$(P_{\text{lin}} \otimes \text{id})\Delta X^r = X^r \otimes X^r - \sum_{j \in \mathcal{R}} s_j X^j \otimes x\partial_x X^r$$

$$(P_{\text{lin}} \otimes \text{id})\Delta X = X \otimes X - sX \otimes x\partial_x X$$

for the single equation case

The scattering type formula

The scattering type formula captures the renormalization group in a Connes-Kreimer framework. In our notation it says

$$\sigma_n(X^r) = \mathrm{sign}(s)\gamma_n^r(x)$$

(the DSE is

$$X = \mathbb{1} \pm \sum_k x^k B_+^k(X Q^k)$$

in single eqn case)

The γ_k recursion

Using

$$\begin{aligned}\sigma_n(X) &= \mathrm{sign}(s)\gamma_n\\ \sigma_1 &= \partial_L\phi_r(S\star Y)|_{L=0}\\ \sigma_n &= \frac{1}{n!}m^{n-1}(\underbrace{\sigma_1\otimes\cdots\otimes\sigma_1}_{n\text{ times}})\Delta^{n-1}\\ (P_{\mathrm{lin}}\otimes\mathrm{id})\Delta X &= X\otimes X - sX\otimes x\partial_x X\end{aligned}$$

Calculate

$$\begin{aligned}\gamma_n = \mathrm{sgn}(s)\,\sigma_n(X) &= \mathrm{sgn}(s)\frac{1}{n!}m^{n-1}(\sigma_1\otimes\dots\otimes\sigma_1)\Delta^{n-1}X\\ &= \mathrm{sgn}(s)\frac{1}{n}m(\sigma_1\otimes\sigma_{n-1})\Delta X\\ &= \mathrm{sgn}(s)\frac{1}{n}m(\sigma_1\otimes\sigma_{n-1})(P_{\mathrm{lin}}\otimes\mathrm{id})\Delta X\end{aligned}$$

$$= \operatorname{sgn}(s)\, \frac{1}{n}\, m(\sigma_1 \otimes \sigma_{n-1})(X \otimes X - sX \otimes x\partial_x X)$$

$$= \frac{1}{n}\left(\gamma_1 \gamma_{n-1} - s\gamma_1\, x\partial_x \gamma_{n-1}\right)$$

$$\gamma_n = \frac{1}{n}\gamma_1 \left(1 - s\, x\partial_x\right)\gamma_{n-1}$$

And similarly for systems.

Notes

1. As series in x, γ_k has lowest term x^k

2. Because we're renormalizing by subtraction there's no γ_0 term

Trading ρ for x

$\sum \gamma_k \partial_{-\rho}^k$

Let $D = \text{sign}(s)\gamma \cdot \partial_{-\rho}$ and $F_k(\rho) = \sum_{i=0}^{t_k} F_{k,i}(\rho)$ so the Dyson-Schwinger equation reads

$$\gamma \cdot L = \sum_{k \geq 1} x^k (1-D)^{1-sk} (e^{-L\rho} - 1) F_k(\rho) \bigg|_{\rho=0}$$

What is the lowest possible degree of x in

$$x^k (1-D)^{1-sk} \rho^\ell|_{\rho=0}$$

$k +$ (lowest degree of x in $\gamma_{k_1} \cdots \gamma_{k_n}$)
where $k_1 + \cdots + k_n = \ell$

$= k + \ell$.

Reduction to geometric series

So, there exists unique $r_k, r_{k,i} \in \mathbb{R}$, $k \geq 1$, $1 \leq i < k$ such that

good part

hack part

$$\sum_k x^k (1-D)^{1-sk} (e^{-L\rho} - 1) F_k(\rho) \Big|_{\rho=0}$$

$$= \sum_k x^k (1-D)^{1-sk} (e^{-L\rho} - 1) \left(\boxed{\frac{r_k}{\rho(1-\rho)}} + \boxed{\sum_{1 \leq i < k} \frac{r_{k,i} L^i}{\rho}} \right) \Bigg|_{\rho=0}$$

Note

Mysterious series

The $r_{i,j}$ are mysterious. With $s = 2$ and a single primitive at 1 loop we get.

with $F(\rho) = \frac{f_{-1}}{\rho} + f_0 + f_1\rho + \ldots$

$$
\begin{aligned}
r_1 &= f_{-1} \\
r_2 &= f_{-1}^2 - f_{-1}f_0 \\
r_{2,1} &= 0 \\
r_3 &= 2f_{-1}^3 + f_{-1}^2(-4f_0 + f_1) + f_{-1}f_0^2 \\
r_{3,1} &= -f_{-1}^3 + f_{-1}^2 f_0 \\
r_{3,2} &= 0 \\
r_4 &= 2f_{-1}^4 + f_{-1}^3(-12f_0 + 6f_1 - f_2) + f_{-1}^2(9f_0^2 - 3f_0f_1) - f_{-1}f_0^3 \\
r_{4,1} &= -f_{-1}^4 + f_{-1}^3(6f_0 - 2f_1) - 3f_{-1}^2 f_0^2 \\
r_{4,2} &= \frac{7}{6}f_{-1}^4 - \frac{7}{6}f_{-1}^3 f_0 \\
r_{4,3} &= 0 \\
&\vdots
\end{aligned}
$$

Even particular coefficients and examples are mysterious.

$$F(\rho) = \frac{-1}{\rho(1-\rho)(2-\rho)(3-\rho)},$$

gives

$$\begin{aligned}
r_1 &= -\frac{1}{6} & & & & \\
r_2 &= -\frac{5}{6^3} & r_{2,1} &= 0 & & \\
r_3 &= -\frac{14}{6^5} & r_{3,1} &= \frac{-5}{6^4} & r_{3,2} &= 0 \\
r_4 &= \frac{563}{6^7} & r_{4,1} &= \frac{-173}{6^6} & r_{4,2} &= \frac{-35}{6^6} \\
r_5 &= \frac{13030}{6^9} & &\vdots & &\vdots \\
r_6 &= -\frac{194178}{6^{11}} & & & &
\end{aligned}$$

L and L^2

The Dyson-Schwinger equation has become

$$\gamma \cdot L = \sum_k x^k (1-D)^{1-sk} (e^{-L\rho} - 1) \left(\frac{r_k}{\rho(1-\rho)} + \sum_{1 \leq i < k} \frac{r_{k,i} L^i}{\rho} \right) \Bigg|_{\rho=0}$$

Take the coefficients of L and L^2

$$[L]: \qquad \gamma_1 = -\sum_k x^k (1-D)^{1-sk} \frac{\cancel{\rho}\, r_k}{\cancel{\rho}(1-\rho)} \Bigg|_{\rho=0}$$

$$[L^2]: \qquad 2\gamma_2 = \sum_k x^k (1-D)^{1-sk} \left(\frac{\rho\, r_k}{(1-\rho)} - 2 r_{k,1} \right) \Bigg|_{\rho=0}$$

$$= -\sum_k x^k (1-D)^{1-sk} \left(r_k + 2 r_{k,1} \right) \Big|_{\rho=0}$$

$$+ \sum_k x^k (1-D)^{1-sk} \left(\frac{r_k}{1-\rho} \right) \Bigg|_{\rho=0}$$

$$2\gamma_2 = -\sum_k x^k (1-D)^{1-sk} (r_k + 2r_{k,1})\Big|_{\rho=0} - \gamma_1$$

$$= -\sum_k x^k (r_k + 2r_{k,1})\Big|_{\rho=0} - \gamma_1$$

$$2\gamma_2 = -\gamma_1 - \sum_{k\geq 1} (r_k + 2r_{k,1}) x^k$$

The γ_1 recursion

Recall

$$\gamma_k = \frac{1}{k}\gamma_1(x)(\operatorname{sign}(s) - |s|x\partial_x)\gamma_{k-1}(x).$$

Let

$$P(x) = -\sum_{k\geq 1}(r_k + 2r_{k,1})x^k$$

Get $\gamma_1(x)(\operatorname{sgn}(s) - |s|x\partial_x)\gamma_1(x) = P(x) - \gamma_1(x)$

Philosophy

- In defining then r_k, $r_{k,i}$ I made it a geometric series in L^0 but $\frac{1}{\rho}$ for higher L
- If I used $\frac{1}{\rho}$ in all cases then one L deriv gives $\gamma_1(x) = \sum r_k x^k$ so no recursion
- If geometric series for all $r_{k,i}$ then the ∂_ρ don't straightforwardly disappear
- Geometric series keeps the conformal invariance

The goal is

Balace

- giving results
- putting too much info in P
- representing the underlying physics.

How to interpret $P(x)$

In the original Yukawa example of Dirk and David

$P(x) = x$ - no extra garbage.

→ reducing to one insertion place → already the case

→ $F(\rho)$ already a geometric series.

The reduction to one insertion place is just a case of building new primitives

The reduction to geometric series just rearranges coefficients still think of P as the function for the primitives

Systems

The shape of the final equations in the system case is

$$\gamma_1^r = \sum_{k\geq 1} p^r(k)x^k - \text{sign}(s_r)\gamma_1^r(x)^2 + \sum_{j\in\mathcal{R}} |s_j|\gamma_1^j(x)x\partial_x\gamma_1^r(x)$$

Summary of the big picture

The Broadhurst Kreimer example

$$G(x,L) = 1 - \frac{x}{q^2}\int d^4k \frac{k\cdot q}{k^2 G(x,\log k^2)(k+q)^2} - \cdots \Big|_{q^2=\mu^2}$$

where $L = \log(q^2/\mu^2)$.

The differential equation is

$$\gamma_1(x) = x - \gamma_1(x)(1 - 2x\partial_x)\gamma_1(x)$$

Solving (presented in the beautiful form given by Broadhurst)

$$\sqrt{\frac{x}{2\pi}} = \exp\left(\left(\frac{(\gamma_1+2)^2}{\sqrt{2x}}\right)^2\right)\mathrm{erfc}\left(\frac{(\gamma_1+2)^2}{\sqrt{2x}}\right)$$

A variant

Take $s = 2$ and

$$F(\rho) = \frac{-1}{\rho(1-\rho)(2-\rho)(3-\rho)},$$

$$\rho F(\rho) = -\frac{1}{6} + \rho^2 F(\rho) - \frac{11}{6}\rho^3 F(\rho) + \frac{1}{6}\rho^4 F(\rho)$$

This gives that

$$\gamma_1 = -x(1-\gamma\cdot\partial_{-\rho})^{-1}\rho F(\rho)|_{\rho=0}$$

$$= -x(1-\gamma\cdot\partial_{-\rho})^{-1}\left(-\frac{1}{6} + \rho^2 F(\rho) - \frac{11}{6}\rho^3 F(\rho) + \frac{1}{6}\rho^4 F(\rho)\right)\Big|_{\rho=0}$$

$$= \frac{x}{6} - 2\gamma_2 - 11\gamma_3 - 4\gamma_4$$

Concluding the variant

So

$$\gamma_1 = \frac{x}{6} - 2\gamma_2 - 11\gamma_3 - 4\gamma_4.$$

But we still have

$$\gamma_k = \frac{1}{k}\gamma_1(x)(1 - 2x\partial_x)\gamma_{k-1}(x),$$

So we get a fourth order differential equation for γ_1 which contains no infinite series and for which we completely understand the signs of the coefficients.

Bonus slides – Growth estimates

View

$$\gamma_1(x) = P(x) - \gamma_1(x)(\operatorname{sign}(s) - |s|x\partial_x)\gamma_1(x)$$

as a recursive equation. At the level of coefficients

$$\gamma_1(x) = \sum \gamma_{1,n} x^n$$

so $$\gamma_{1,n} = p(n) - \sum_{l=1}^{n-1} \gamma_{1,n-l} \left(\operatorname{sgn}(s) - |s| l \right) \gamma_{1,l}$$

$$= p(n) - \sum_{l=1}^{n-1} \left(\operatorname{sgn}(s) - \frac{|s| n}{2} \right) \gamma_{1,l} \gamma_{1,n-l}$$

Rewrite for $a(n)$

Assume $\gamma_{1,1} \neq 0$ and $f(x) = \sum \frac{p(n)}{n!} x^n$ has radius of convergence $\rho > 0$.

Let $a(n) = \frac{\gamma_{1,n}}{n!}$. The recursion becomes

$$a_n = \frac{p(n)}{n!} + \left(\frac{|s|n}{2} - \operatorname{sgn}(s) \right) \sum_{l=1}^{n-1} a_l a_{n-l} \binom{n}{l}^{-1}$$

$$=$$

How bad is the growth of γ_1?

Idea:

$$a(n) \text{ is approximately } \frac{p(n)}{n!} + |s| a_1 a_{n-1}$$

giving a radius of $\min\left\{\rho, \frac{1}{sa_1}\right\}$ for $\sum a_n x^n$. For nonnegative series implement the idea by bounding on each side.

Easy direction:

$$a_n \geq \frac{p(n)}{n!} + |s| \frac{n-2}{n} a_1 a_{n-1}$$

Messy direction: for any $\epsilon > 0$ there is an $N > 0$ such that for $n > N$

$$a_n \leq \frac{p(n)}{n!} + |s| a_1 a_{n-1} + \epsilon \sum_{j=1}^{n-1} a_j a_{n-j}$$