Dyson-Schwinger equations I

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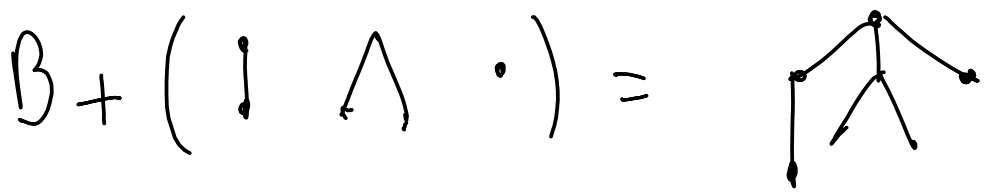
les Houches

B_+ for trees

In the Hopf algebra of rooted trees $B_+(F)$ constructs a tree by adding a new root with children the roots of each tree from the forest F.

For example



B_+ in rooted trees is a Hochschild 1-cocycle,

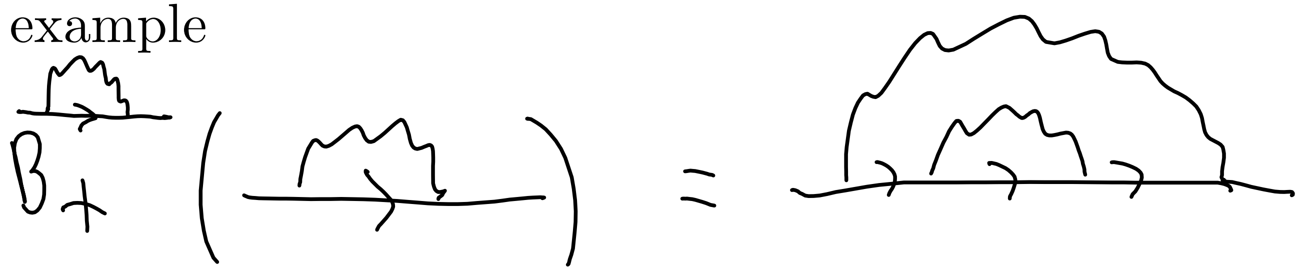
$$\Delta B_+ = (\mathrm{id} \otimes B_+)\Delta + B_+ \otimes \mathbb{I}.$$

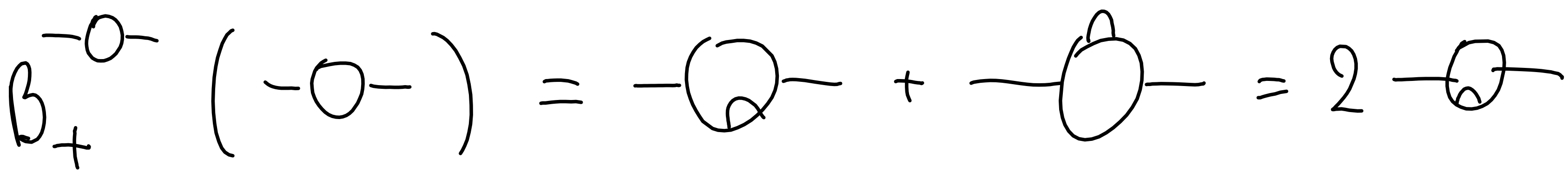
This 1-cocycle property is key

B_+ for graphs

In the Hopf algebra of divergent 1PI Feynman graphs from a given theory, write B_+^γ for insertion into the primitive graph γ.

For example



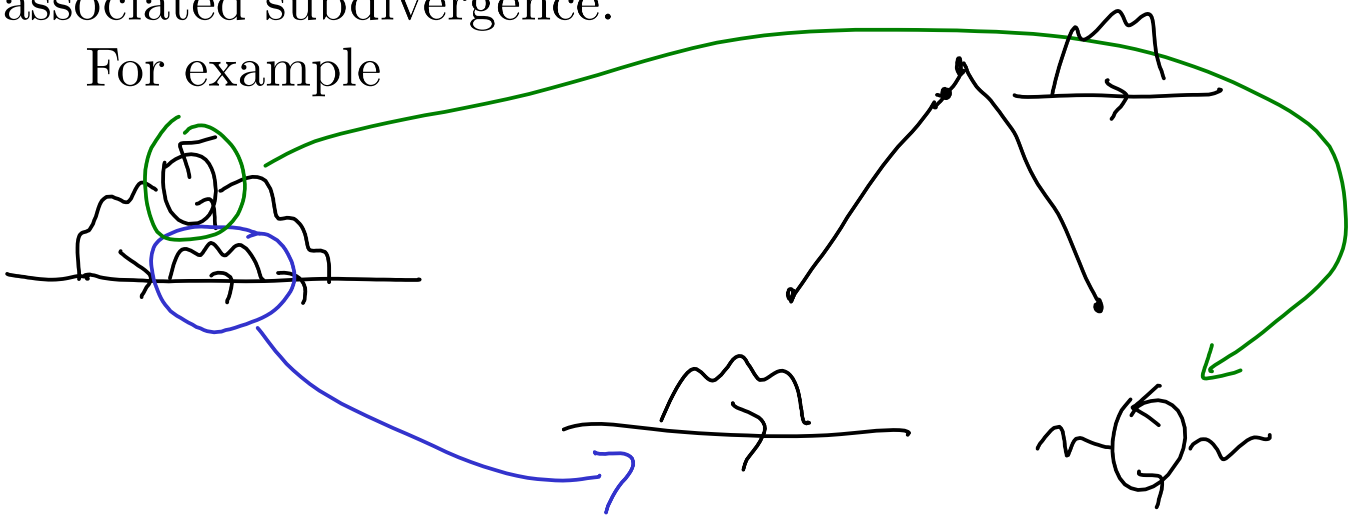


We want this to be a 1-cocycle too. Two things can go wrong.

Simple cases are just trees

In the simplest cases a 1PI Feynman graph can be uniquely represented by a rooted tree with labels on each vertex corresponding *just* to the associated subdivergence.

For example

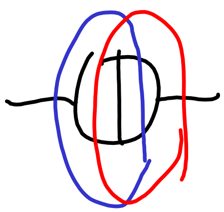


In such cases B_+^γ is automatically a 1-cocycle.

Overlapping divergences

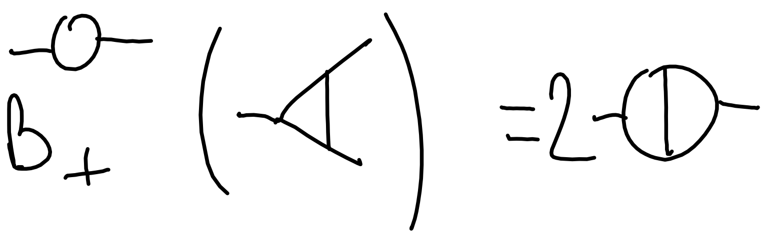
In general there are many possible ways to insert one graph into another so the tree must also contain the information of which insertion place to use.

Also



overlapping subdivergences

naively



The hairy coefficient

We can fix this by making the coefficient hairy. For γ primitive

$$B_+^\gamma(X) = \sum_{\substack{\Gamma \in \mathcal{H} \\ \Gamma \text{ connected}}} \frac{\mathbf{bij}(\gamma, X, \Gamma)}{|X|_\vee} \frac{1}{\mathrm{maxf}(\Gamma)} \frac{1}{(\gamma|X)} \Gamma$$

$\mathrm{maxf}(\Gamma)$: number of insertion trees for Γ,

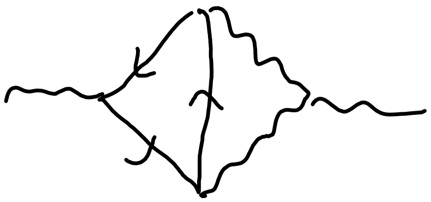
$|X|_\vee$: number of graphs from permuting the external edges of X,

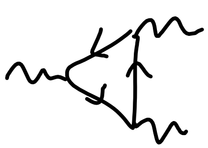
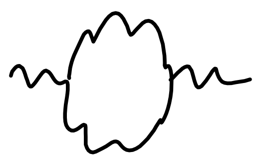
$\mathbf{bij}(\gamma, X, \Gamma)$: number of bijections of the external edges of X with an insertion place of γ giving Γ.

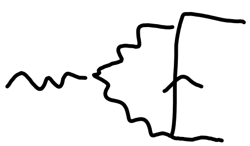
$(\gamma|X)$: number of insertion places for X in γ.

The coefficient assures that we do not double count graphs which can be made in more than one way.

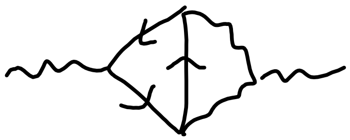
But even more can go wrong

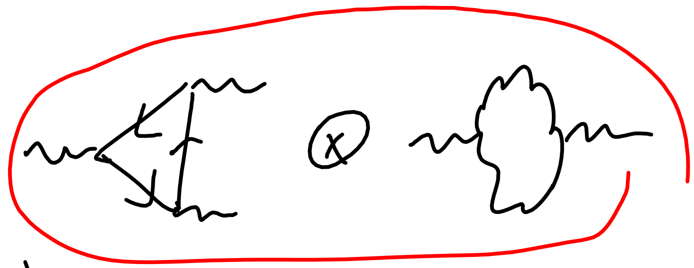
 can be made by inserting

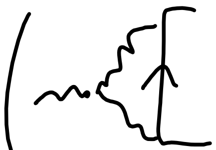
 into  or by inserting

 into  .

This makes it impossible for every B_+^γ to be a 1-cocycle.



so there is a  in

ΔB_+  ()

but not in (B_+  $\otimes \mathbb{1} + (\mathrm{id} \otimes B_+$  $)\Delta$) ()

Saved by Ward

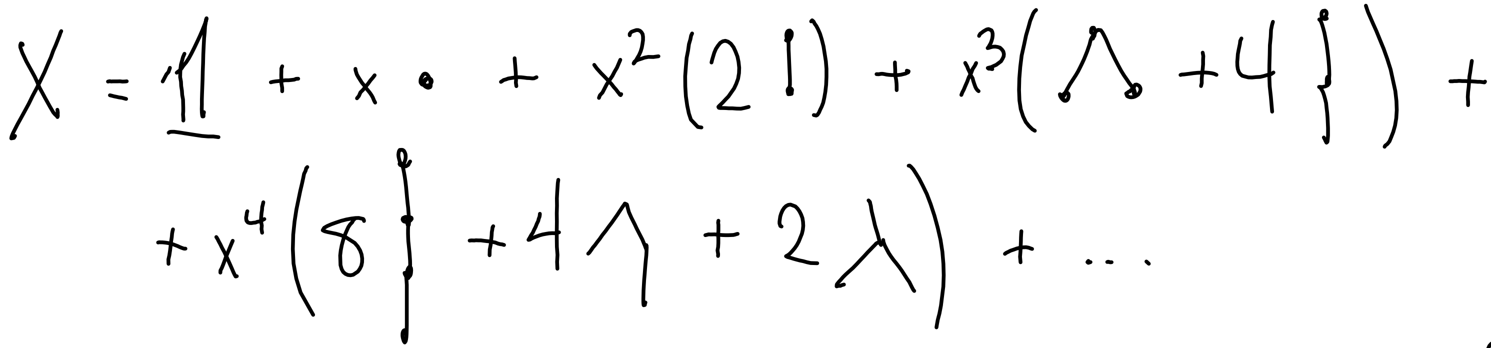
Recall van Suijlekom's Hopf ideal I. Then $\sum_{|\gamma|=k,\mathrm{res}\gamma=r} B_+^\gamma$ is a 1-cocycle in $\mathcal{H}/I$.

Unfolding some recursive equations

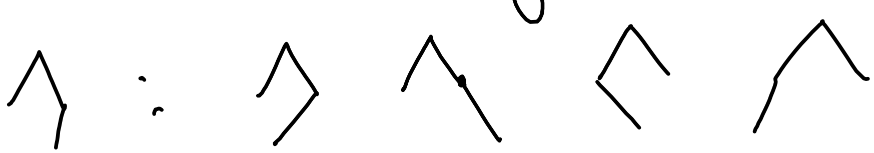
Lets get our intuition going in the Hopf algebra of rooted trees

$$X = \mathbb{I} + xB_+(X^2)$$

What does this count?



binary trees → computer science binary trees (distinct left and right children) and then forget

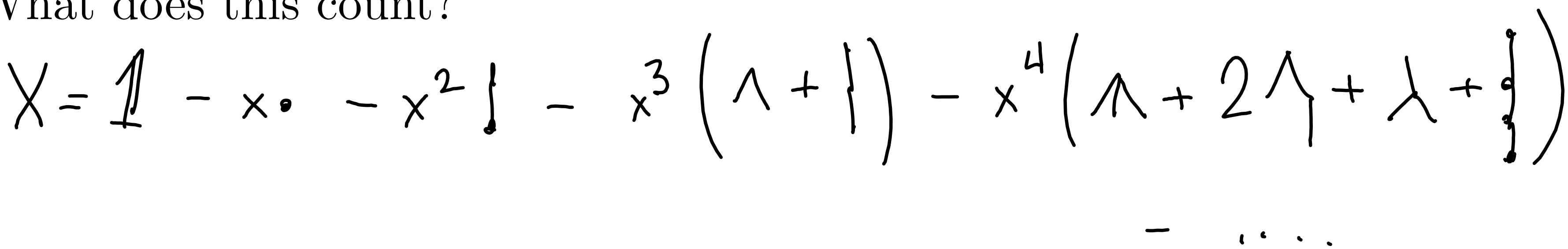


$$X = \mathbb{I} + xB_+(X^3)$$

What does this count?

$$X = \mathbb{I} - xB_+\left(\frac{1}{X}\right)$$

What does this count?



counts plane trees then forget

Combinatorial Dyson-Schwinger equations

Back to the Hopf algebra of 1PI divergent Feynman graphs in a given theory. The combinatorial Dyson-Schwinger equation is

$$X(x) = \mathbb{I} \pm \sum_{k \geq 1} x^k B_+^k(XQ^k).$$

k is the loop order we're inserting into.

where $Q(x) = X(x)^{-s}$.

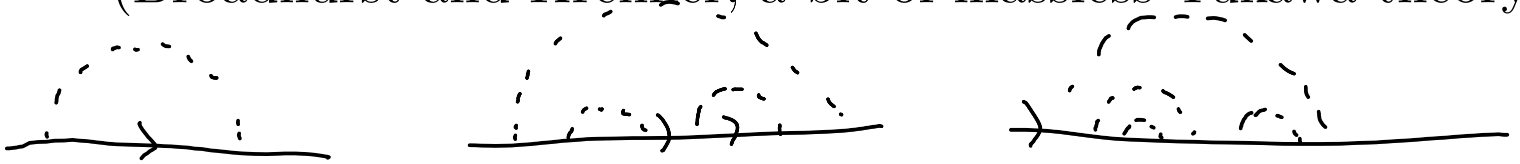
For systems

$$X^r(x) = \mathbb{I} \pm \sum_{k \geq 1} x^k B_+^{k,r}(X^r Q^k).$$

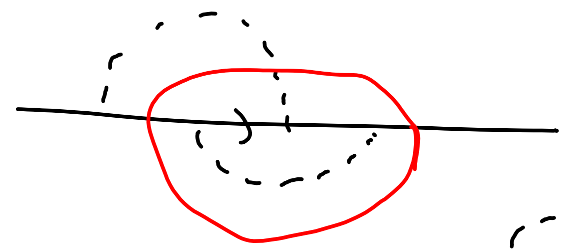
where $Q = \Bigg(\underbrace{\frac{(X^v)^2}{\prod_{i=1}^{n}(X^{e_i})^{m_i}}}_{\text{Walter's } \Upsilon_v}\Bigg)^{1/(\mathrm{val}(v)-2)}$ in $\mathcal{H}/I$ as van Suijlekom discussed.

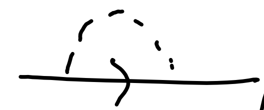
For example

(Broadhurst and Kreimer; a bit of massless Yukawa theory).



not



$X = \mathbb{1} - x B_+$  $\left(\frac{1}{X} \right) = \mathbb{1} - x B_+$  (XQ)

$$Q = X^{-2}$$

Analytic Dyson-Schwinger equations

Analytic Dyson-Schwinger equations are the result of applying the Feynman rules to combinatorial Dyson-Schwinger equations. We renormalize by subtracting at fixed values of the external momenta.

1. The recursive structure of the DSE takes of the recursive structure of renormalize
2. The counting var x becomes the coupling constant,
3. These are the DSEs of physics.
4. Now we also have variables from the external momenta
5. In the one scale case let $L = \log(q^2/\mu^2)$

Continuing the example

In the Broadhurst-Kreimer Yukawa example

same evaluated at

$$G(x,L) = 1 - \underbrace{\frac{x}{q^2}\int d^4k\,\frac{k\cdot q}{k^2 G(x,\log k^2)(k+q)^2}}_{B_+} - \cdots \Big|_{q^2=\mu^2}$$

where $L = \log(q^2/\mu^2)$.

This has the same recursive structure as the combinatorial equation.

The Mellin transform of the primitive

For a primitive graph γ

Get a formal integral expression

Regularize by raising propagators to $1+\rho_i$

analytic regularization

set external to 1

Call $F_\gamma(\rho_1 \cdots\cdots \rho_k)$

$F_\gamma(\rho_1, \ldots, \rho_n)$

Disentangling the analytic part

We had

$$G(x,L) = 1 - \frac{x}{q^2}\int d^4k \frac{k\cdot q}{k^2 G(x,\log k^2)(k+q)^2} - \cdots \bigg|_{q^2=\mu^2}.$$

Rewrite using the usual tricks

- plug in the Ansatz $G(x,L) = \cancel{\sum \gamma_k(x) L^k}$ $= 1 - \sum_{k\geq 1} \gamma_k L^k$
- use $\partial_\rho^k x^{-\rho}|_{\rho=0} = (-1)^k \log^k(x)$
- switch the order of $\int$ and ∂

$$\sum_{n\geq 1} \gamma_n L^n = \frac{x}{q^2}\int d^4k \frac{k\cdot q}{k^2\left(1-\sum_n \gamma_n (\log k^2)^n\right)(k+q)^2} - \cdots \bigg|_{q^2=\mu^2}$$

$$= \frac{x}{q^2}\int d^4k \frac{k\cdot q}{k^2(k+q)^2} \sum_j \left(\sum_n \gamma_n (\log k^2)^n\right)^j - \cdots \bigg|_{q^2=\mu^2}$$

$$= \frac{x}{q^2} \int d^4k \, \frac{k\cdot q}{k^2 (k+q)^2} \sum_m \sum_{j_1+\dots+j_s=m} \gamma_{j_1} \cdots \gamma_{j_s} \left(\log k^2\right)^m - \dots \Bigg|_{q^2=\mu^2}$$

$$= \frac{x}{q^2} \int d^4k \, \frac{k\cdot q}{k^2 (k+q)^2} \underbrace{\sum_{j_1+\dots+j_s=m} \gamma_{j_1} \cdots \gamma_{j_s} (-1)^m \partial_\rho^m} k^{2-\rho} - \dots \Bigg|_{q^2=\mu^2} \Bigg|_{\rho=0}$$

$$= \frac{x}{q^2} \underbrace{\sum_{j_1+\dots+j_s=m} \gamma_{j_1} \cdots \gamma_{j_s} (-1)^m \partial_\rho^m}_{\text{The geometric series}} \underbrace{\int d^4k \, \frac{k\cdot q}{(k^2)^{1+\rho} (k+q)^2} - \dots \Bigg|_{q^2=\mu^2}}_{\text{to be } F_\gamma(\rho)} \Bigg|_{\rho=0}$$

$$= x \left(\frac{1}{1 - \sum \gamma_n \partial_\rho^n} \right) \left(e^{-\log \frac{q^2}{\mu^2} \rho} - 1 \right) F_\gamma(\rho)$$

$$\gamma \cdot L = x(1 - \gamma \cdot \partial_{-\rho})^{-1}(e^{-L\rho} - 1)F(\rho)\Big|_{\rho=0}$$

where $\gamma \cdot U = \sum \gamma_k U^k$.

The example all together

$$X(x) = \mathbb{I} - xB_+ \left(\frac{1}{X(x)}\right),$$

$$F(\rho) = \frac{1}{q^2} \int d^4k \frac{k \cdot q}{(k^2)^{1+\rho}(k+q)^2}\bigg|_{q=1}.$$

Combine to get

$$G(x, L) = 1 - \frac{x}{q^2} \int d^4k \frac{k \cdot q}{k^2 G(x, \log k^2)(k+q)^2} - \cdots \bigg|_{q^2=\mu^2}$$

where $L = \log(q^2/\mu^2)$. Rearrange to

$$\gamma \cdot L = x(1 - \gamma \cdot \partial_{-\rho})^{-1}(e^{-L\rho} - 1)F(\rho)\bigg|_{\rho=0}$$

where $\gamma \cdot U = \sum \gamma_k U^k$.

Dyson-Schwinger equations – our setup

The combinatorial Dyson-Schwinger equation is

$$X(x) = \mathbb{I} - \mathrm{sign}(s) \sum_{k \geq 1} \sum_{i=0}^{t_k} x^k B_+^{k,i}(XQ^k).$$

where $Q(x) = X(x)^{-s}$. Associate with each $B_+^{k,i}$ a Mellin transform

$$F^{k,i}(\rho_1, \ldots, \rho_n).$$

Then the analytic Dyson-Schwinger equation is

$$G(x, L) = 1 - \mathrm{sign}(s) \sum_{k \geq 1} \sum_{i=0}^{t_k} x^k G(x, \partial_{-\rho_1})^{-\mathrm{sign}(s)} \cdots G(x, \partial_{-\rho_{n_k}})^{-\mathrm{sign}(s)}$$

$$(e^{-L(\rho_1 + \cdots + \rho_{n_k})} - 1) F^{k,i}(\rho_1, \ldots, \rho_{n_k}) \Big|_{\rho_1 = \cdots = \rho_{n_k} = 0}$$

where $n_k = \mathrm{sign}(s)(sk - 1)$.

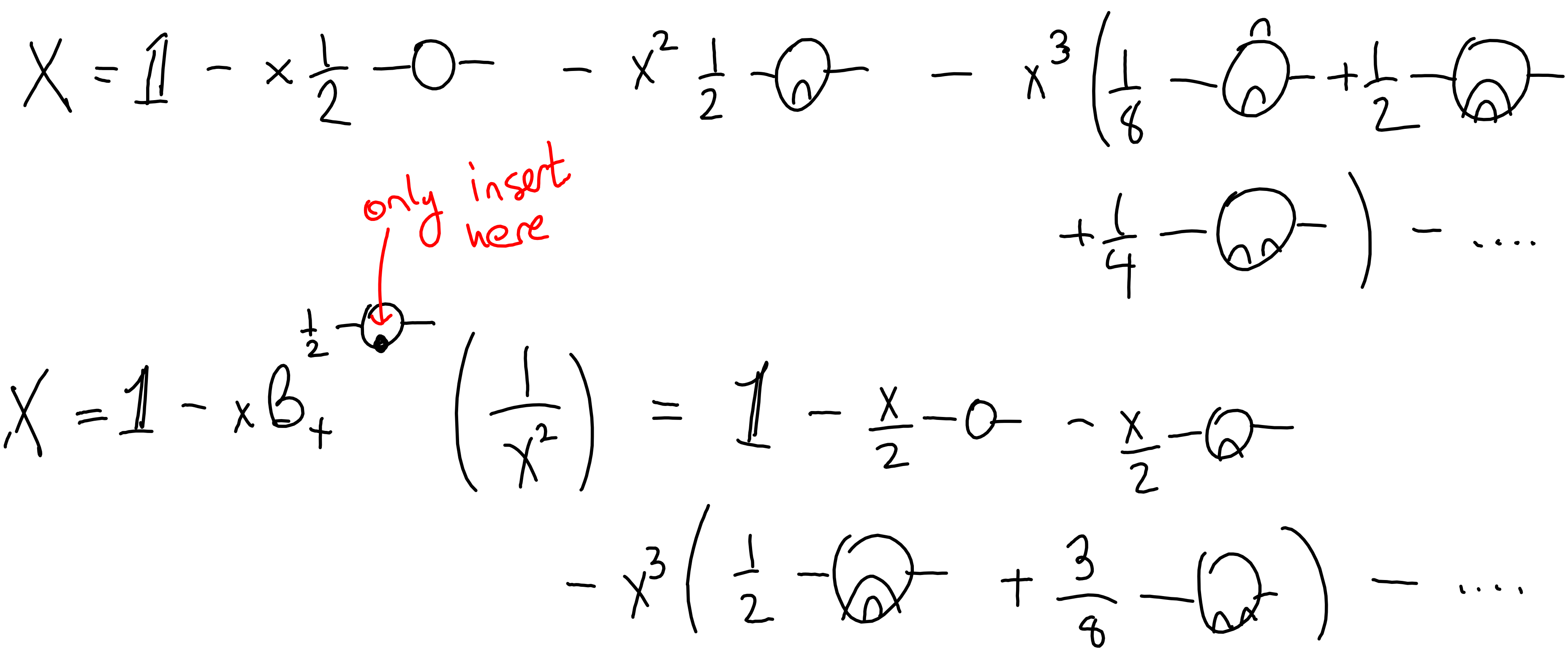
Systems of equation are similar but messier.

Reduction to one insertion place

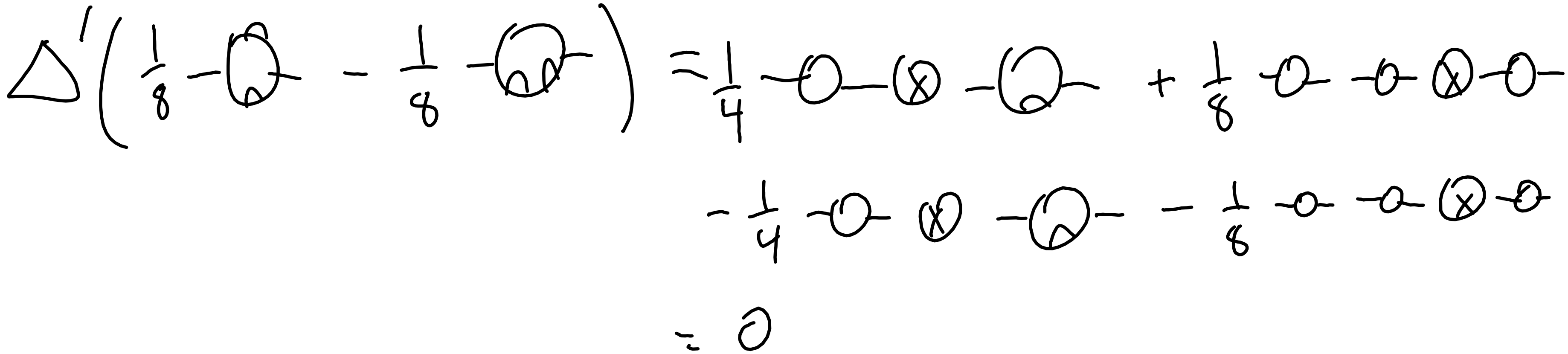
Use new primitives to account for the error when only inserting in one insertion place. For example

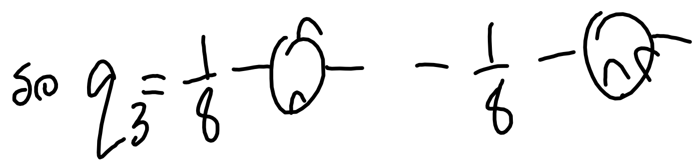
$$B_+^{\gamma}(\mathbb{1}) = \gamma$$

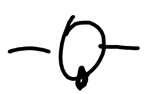
$$X = 1 - xB_+^{\frac{1}{2}-\bigcirc-}\left(\frac{1}{X^2}\right)$$



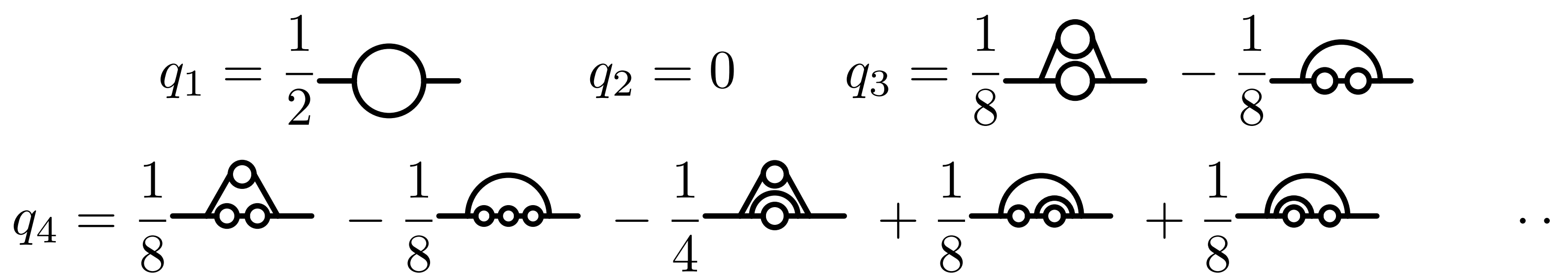
difference 



so  is primitive and hence a valid superscript for B_+



$$X = \mathbb{1} - xB_+(XQ) - x^3 B_+^{q_3}(XQ^3) \qquad Q = X^{-3}$$



Check primitivity

Return to rooted trees

In general the insertions we need aren't possible.

Let G be a 1PI Feynman graph; let $F(G)$ be the forest of insertion trees which give G.

F is an injective Hopf algebra morphism.

Extend this situation by colouring edges to encode different rules for insertion

- red insertion (symmetric insertion)
- black insertion (usu. insertion)

same algebraic structure carries over

Reduction to symmetric insertion

Use R_+ for red insertion

Switch from B_+ to R_+ by at each loop order defining a new primitive which is the difference between what we have already built with R_+ and what we had originally.

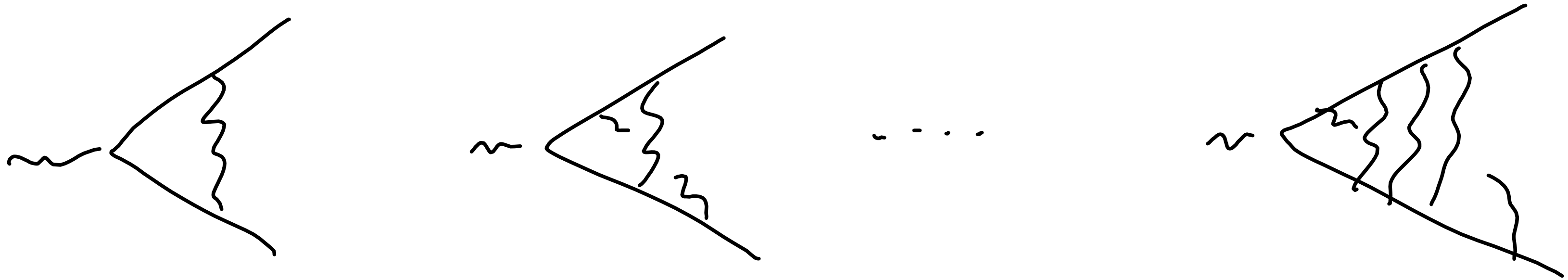
We had

$$X(x) = \mathbb{I} - \mathrm{sign}(s) \sum_{k \geq 1} \cancel{\sum_{i=0}^{t_k}} x^k B_+^{k\cancel{;i}}(XQ^k).$$

$$X = \mathbb{1} - x c_1 - \dots \qquad q_1 = c_1$$

$$X(x) = \mathbb{1} - \mathrm{sgn}(s)\, R_+^{q_1}(XQ^k) - \dots$$

$$q_2 = [x^2] X - [x^2]\left(- \mathrm{sgn}(s)\, R_+^{q_1}(XQ^k)\right)$$ my next primitive.



def If $A = \sum a_n x^n$ is a formal series then $[x^k]A = a_k$

$$q_n = -\mathrm{sign}(s)[x^n]X + \mathrm{sign}(s)\sum_{k=1}^{n-1} R_+^{q_k}([x^{n-k}]XQ^k)$$

$$X = 1 - \mathrm{sign}(s_r)\sum_{k\geq 1} x^k R_+^{q_k}(XQ^k).$$

Each q_n is primitive, inductively.

Consequence

Symmetric insertion means a single insertion place which means univariate Mellin transforms.

So the Dyson-Schwinger equation simplifies from

$$G(x,L) = 1 - \mathrm{sign}(s) \sum_{k\geq 1} \sum_{i=0}^{t_k} x^k G(x,\partial_{-\rho_1})^{-\mathrm{sign}(s)} \cdots G(x,\partial_{-\rho_{n_k}})^{-\mathrm{sign}(s)}$$

$$(e^{-L(\rho_1+\cdots+\rho_{n_k})} - 1) F^{k,i}(\rho_1,\ldots,\rho_{n_k}) \Big|_{\rho_1=\cdots=\rho_{n_k}=0}$$

where $n_k = \mathrm{sign}(s)(sk-1)$, to

$$G(x,L) = 1 - \mathrm{sign}(s) \sum_{k\geq 1} \cancel{\sum_{i=0}^{t_k}} x^k G(x,\partial_{-\rho})^{1-sk}(e^{-L(\rho)} - 1) F^{k,\cancel{i}}(\rho) \Big|_{\rho=0}$$

Bonus slide – symmetric insertion

For the purposes of symmetric insertion define use the Mellin transform

$$F_p(\rho) = (q^2)^\rho \int \mathrm{Int}_p(q^2) \left(\frac{1}{|p|} \sum_{i=1}^{|p|} (k_i^2)^{-\rho} \right) \prod_{i=1}^{|p|} d^4 k_i,$$

where $\mathrm{Int}_p(q^2)$ is the integrand determined by p.

We'll renormalize by subtraction at $q^2 = \mu^2$; let

$$\mathrm{Int}_p^-(q^2) = \mathrm{Int}_p(q^2) - \mathrm{Int}_p(\mu^2).$$

So for symmetric insertion we have

$$\phi_R(R_+^p(X))(q^2/\mu^2) = \int \mathrm{Int}_p^-(q^2) \left(\frac{1}{|p|} \sum_{i=1}^{|p|} \phi_R(X)(-k_i^2/\mu^2) \right) \prod_{i=1}^{|p|} d^4 k_i.$$