Feynman graphs and a chord diagram expansion

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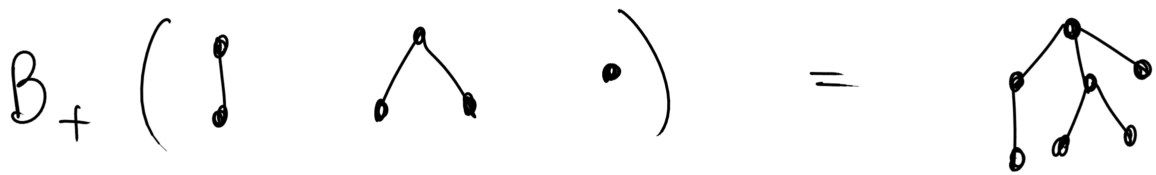
Building trees

Let $B_+(F)$ be the tree constructed by adding a new root above each tree from the forest F.
Eg:



my trees don't come with a planar embedding

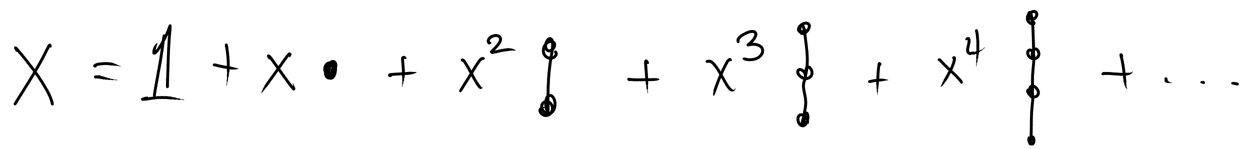
Tree recurrences

Let X be a formal power series with coefficients from the algebra of trees. What does

empty tree

$$X = \mathbb{I} + xB_+(X)$$

count?

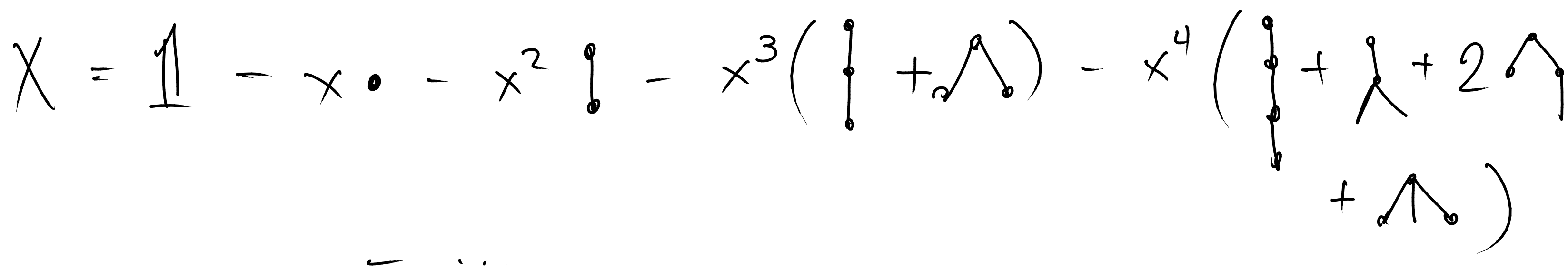


More tree recurrences

What does

$$X = \mathbb{I} - xB_+\left(\frac{1}{X}\right)$$

count?

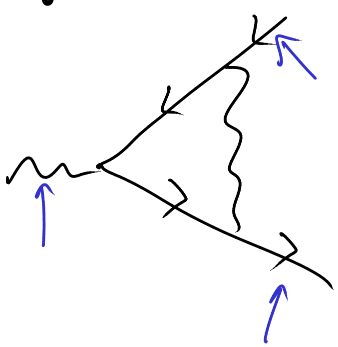


Feynman graphs

Feynman graphs describe interactions in particle physics. They are graphs built of half-edges with specified

- edge types (oriented and unoriented) and
- vertex types

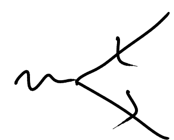
They may have *external edges*.
Eg: QED



edge types in QED

photon 

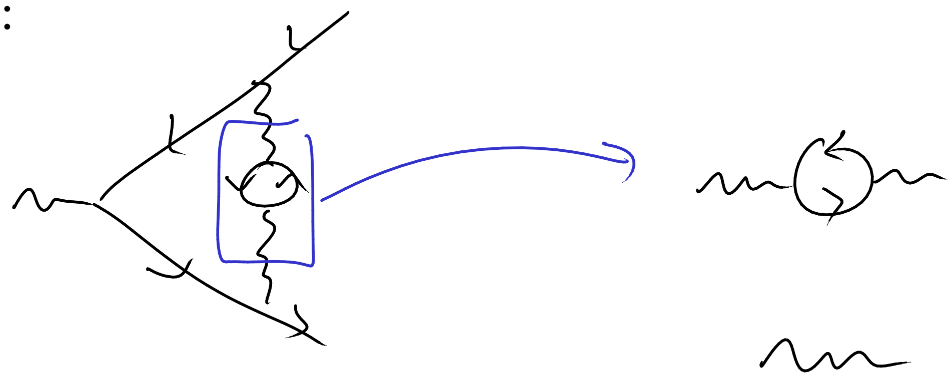
electron 

vertex types 

A Feynman graph is 1PI if it is 2-edge-connected.
Feynman rules map Feynman graphs to (formal) integrals.

Divergences

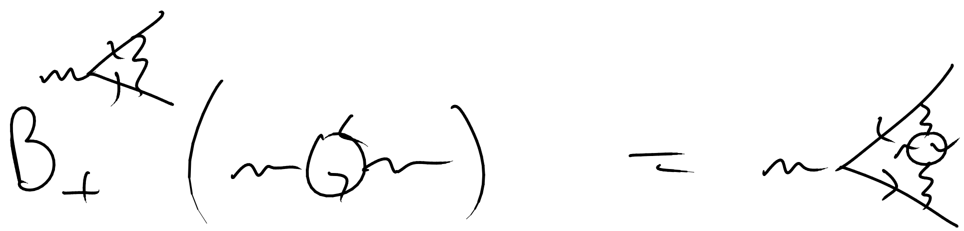
A Feynman graph is *divergent* if the associated integral diverges. If we have set up our types correctly, this will occur when the external edges of the graph give one of the edge or vertex types.
Eg:



A graph is *primitive* if it has no divergent subgraphs.

B_+ for graphs

Write B_+^γ for insertion into the primitive graph γ.
Eg:



By weighting the insertions by an appropriate combinatorial coefficient, and, where necessary, working in a quotient algebra (Ward identities...) we obtain that B_+ is a Hochschild 1-cocycle for the renormalization Hopf algebra.

$$\Delta B_+ = (\mathrm{id} \otimes B_+)\Delta + B_+ \otimes \mathbb{I}$$

Combinatorial Dyson-Schwinger equations

The recurrences in Feynman diagrams which describe how to build the graphs of a theory out of smaller graphs are the *combinatorial Dyson-Schwinger equations.* For today

primitive graph size k

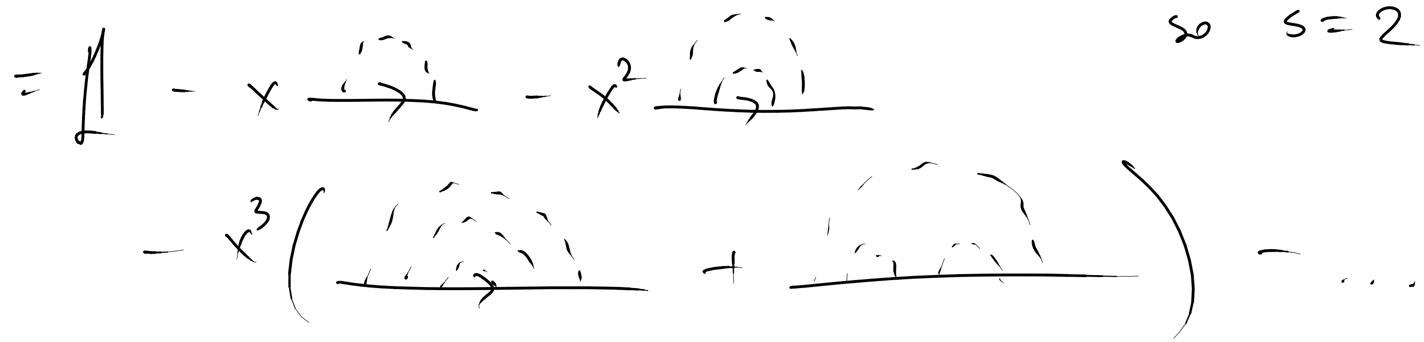
$$X = \mathbb{I} \pm \sum_{k \geq 1} x^k B_+^{\gamma_k}(XQ^k)$$

counting size

where $Q = X^{-s}$.
Eg (Broadhurst and Kreimer):

$$X = \mathbb{1} - x B_+\left(\frac{1}{X}\right) \qquad \frac{1}{X} = \frac{X}{X^2}$$

so $s = 2$



Analytic Dyson-Schwinger equations

Analytic Dyson-Schwinger equations are the result of applying Feynman rules to combinatorial Dyson-Schwinger equations.

- The recursive structure of the DSE takes care of the recursive structure of renormalization.
- The counting variable x becomes the coupling constant
- We get new analytic variables coming from the external momenta. For today just one variable L.
- X becomes the Green function $G(x, L)$.

After some manipulation we obtain

$$G(x, L) = 1 \pm \sum_{k \geq 1} x^k G(x, \partial_{-\rho})^{1-sk} (e^{-L\rho} - 1) F_k(\rho) \big|_{\rho=0}$$

Where $F_k(\rho)$ is the integral for γ_k regularized by a parameter ρ which marks the insertion place.

Now you can forget all that

Today we are looking at $s = 2$ and $k = 1$. That is

$$\boxed{G(x, L) = 1 - xG(x, \partial_{-\rho})^{-1}(e^{-L\rho} - 1)F(\rho)\big|_{\rho=0}}$$

where

$$\boxed{F(\rho) = \frac{f_0}{\rho} + f_1 + f_2\rho + f_3\rho^2 + \cdots}$$

Write

$$G(x, L) = 1 - \sum_{n \geq 1} \boxed{\gamma_n(x)} L^n$$

then the Dyson-Schwinger equation determines the γ_n in terms of the f_i, *but not in a nice way.*

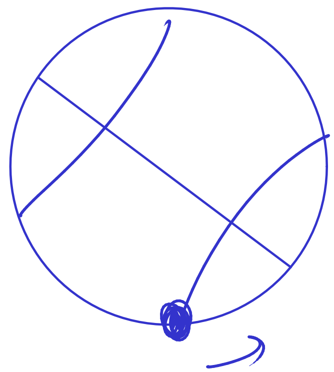
This talk will show a nice way to untangle this with an expansion indexed by chord diagrams. (Joint work with Dirk Kreimer and Nicolas Marie)

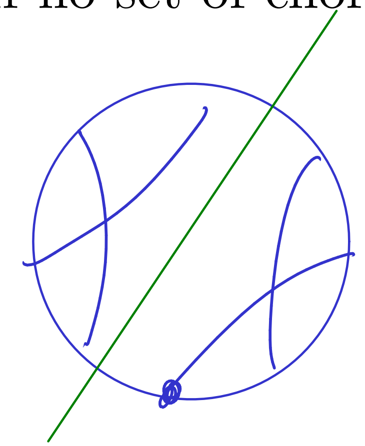
Rooted connected chord diagrams

A chord diagram is *rooted* if it has a distinguished vertex. oriented

A chord diagram is *connected* if no set of chords can be separated from the others by a line.

Eg:





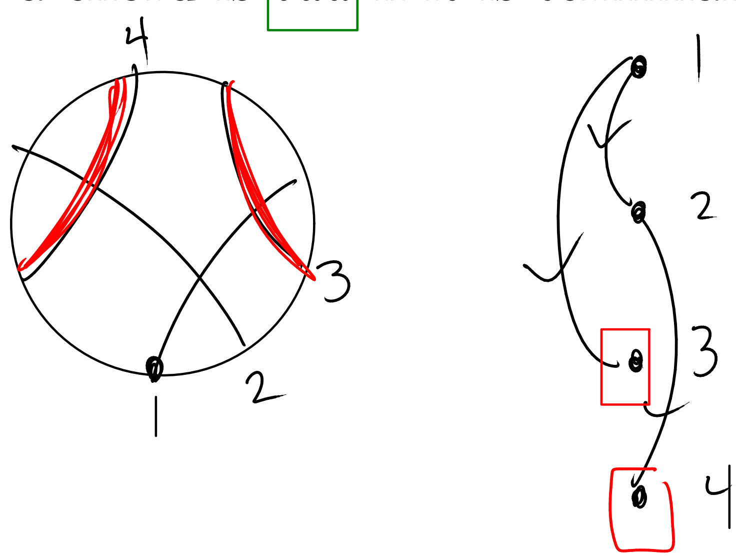
These are really just irreducible matchings of points along a line.

Intersection graphs and bad chords

The *intersection graph* of a chord diagram is the graph with

- **vertices:** the chords of the diagram
- **adjacencies:** vertices where the corresponding chords cross.

The root and counterclockwise order of the chord diagram let us direct the intersection graph.
Say a chord is *bad* if it is terminal in the directed intersection graph.
Eg:

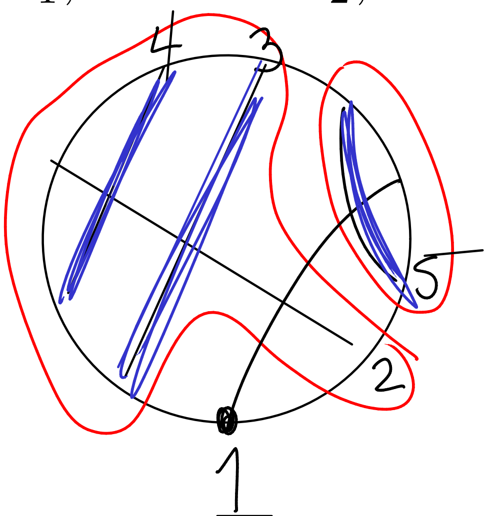


Recursive chord order

Let C be a connected rooted chord diagram. Order the chords recursively:

- c_1 is the root chord
- Order the connected components of $C \setminus c_1$ as they first appear running counterclockwise, $D_1, D_2, \ldots$. Recursively order the chords of D_1, then of D_2, and so on.

Eg:



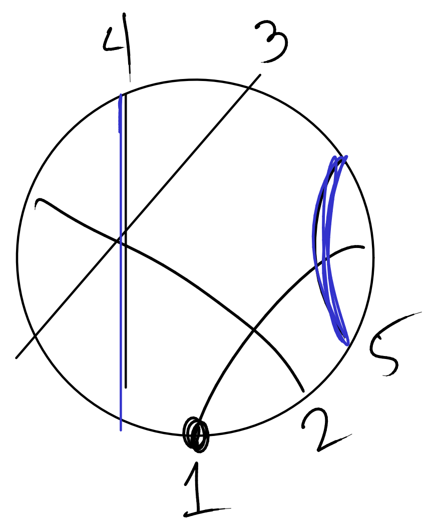
The bad chords come from applications of the base case: a diagram with only one chord.

Index lists

Let C be a connected rooted chord diagram. Define

- $w(C) = \{i : c_i \text{ is bad}\}$ (using the recursive chord order)
- $i(C)$ is the list of differences of successive elements in $w(C)$ padded with 0s to contain $|C| - 1$ elements.
- $b(C)$ is the minimum index of a bad chord.

Eg:



$w(C) = \{4, 5\}$

$i(C) = (0, 0, 0, 1)$

$b(C) = 4$

These will be our index lists: If I is a list of nonnegative integers let $f_I = \prod_{i \in I} f_i$.

Goal

Theorem 1

$$\gamma_i(x) = \frac{(-1)^i}{i!} \sum_{\substack{C \\ b(C) \geq i}} x^{|C|} \boxed{f_{i(C)}} f_{b(C)-i-1}$$

where C runs over rooted connected chord diagrams, solves the DSE

$$G(x, L) = 1 - xG(x, \partial_{-\rho})^{-1}(e^{-L\rho} - 1)F(\rho)\Big|_{\rho=0}$$

where

$$F(\rho) = \frac{f_0}{\rho} + f_1 + f_2\rho + f_3\rho^2 + \cdots$$
$$G(x, L) = 1 - \sum_{n \geq 1} \gamma_n(x) L^n$$

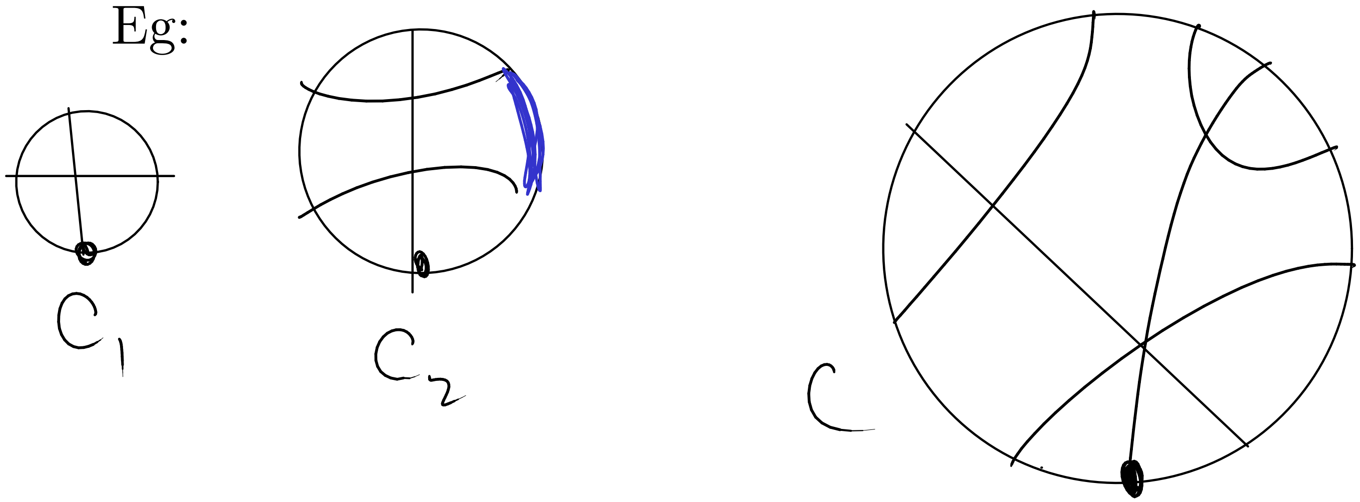
We will prove the theorem by proving two recurrences.

The root-share decomposition

We can insert a rooted connected chord diagram C_1 into another C_2, by

- choosing an interval of C_2 other than the one before the root
- putting the root of C_1 just before the root of C_2 and
- putting the rest of C_2 in the chosen interval

Eg:



Since the diagrams are connected C_1 and C_2 can be recovered. This is the *root-share decomposition.*

The first recurrence – chord diagrams

The root-share decomposition is classical. Nijenhuis and Wilf (1978) use it to prove the recurrence (originally due to Stein (1978) and rephrased by Riordan)

$$s_n = \sum_{k=1}^{n-1} (2k-1) s_k s_{n-k} \qquad \text{for } n \geq 2$$

where s_n is the number of connected rooted chord diagrams with n chords.

This recurrence can be extended to keep track of the bad chords. Let

$$g_{k,i} = \sum_{\substack{C \\ |C|=i \\ b(C)\geq i}} f_{i(C)} f_{b(C)-i-1}$$

where C runs over rooted connected chord diagrams. Then

$$\boxed{g_{k,i} = \sum_{\ell=1}^{i-1} (2\ell-1) g_{1,i-\ell} g_{k-1,\ell} \quad \text{for } 2 \leq k \leq i}$$

The first recurrence – DSEs

We had

$$g_{k,i} = \sum_{\ell=1}^{i-1}(2\ell-1)g_{1,i-\ell}g_{k-1,\ell} \quad \text{for } 2 \leq k \leq i$$

Let

$$\gamma_k = \frac{(-1)^k}{k!}\sum_{i\geq k} g_{k,i}x^i$$

then the recurrence becomes

$$\boxed{\gamma_k(x) = \frac{1}{k}\gamma_1(x)\left(-1+2x\frac{d}{dx}\right)\gamma_{k-1}(x) \quad \text{for } k \geq 2.}$$

which was known (Broadhurst and Kreimer 2000) to be true for γ_k satisfying the DSE.

Now we know the γ_k depend correctly on γ_1 for the theorem.

Binary trees

To obtain the second recurrence, we need another representation for the chord diagrams.
Let C be a rooted chord diagram. Build a binary tree with leaves labelled $1, 2, \ldots |C|$ as follows

- If $|C| = 1$ then the tree has one vertex labelled 1
- Otherwise let C_1 and C_2 be the root-share decomposition of C with the insertion into slot k, and t_1 and t_2 the corresponding trees.
  - Add 1 to each label of t_2
  - Add $|C_2|$ to each label of t_1 except for the label 1.
  - Find the kth vertex of t_2 in a preorder traversal, replace this vertex with a new vertex with t_1 as its right subtree and what had been there as its left subtree.

Binary tree example

The second recurrence

To prove the theorem it remains to show

$$\gamma_1 = x(1 - \sum_{k\geq 1} \gamma_k (\partial_{-\rho})^k)^{-1}(-\rho)F(\rho)\Big|_{\rho=0}$$

With a couple of pages of manipulations, we can check that it suffices to show

$$\boxed{\sum_{\substack{C \\ |C|=i+1 \\ b(C)=j+1}} f_{i(C)} = \sum_{k=1}^{i}\sum_{\ell=1}^{j} \binom{j}{\ell} \overset{\text{left child}}{\left(\sum_{\substack{C \\ |C|=k \\ b(C)\geq \ell}} f_{i(C)} f_{b(C)-\ell-1} \right)} \overset{\text{right child}}{\left(\sum_{\substack{C \\ |C|=i-k+1 \\ b(C)=j-\ell+1}} f_{i(C)} \right)}}$$

for $i \geq 1$ and $j \geq 1$, where the sums run over connected rooted chord diagrams with the indicated conditions.

Comments on the second recurrence

The second recurrence naturally comes by viewing a binary tree in terms of its left and right subtrees.

It is not apparent directly at the level of the chord diagrams. Eg:

Conclusions

We solve the Dyson-Schwinger equation to get the Green function as a sort of multivariate generating function for chord diagrams

$$G(x, L) = 1 - \sum_{i \geq 1} \frac{(-L)^i}{i!} \sum_{\substack{C \\ b(C) \geq i}} x^{|C|} f_{i(C)} f_{b(C)-i-1}$$

This is a new expansion for the Green function and it completely unwinds both the combinatorial and analytic sides of the Dyson-Schwinger equation.

The next steps are

- exploring further the objects and constructions we used
- more general Dyson-Schwinger equations, beginning with other values of s.