Patterns in denominators of Feynman integrals

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Special Session on Mathematics Related to Feynman Diagrams
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Set up

Consider a Feynman integral in Schwinger parametric form. That is, use

$$\int_0^\infty da\, e^{-ak^2} = \frac{1}{k^2}$$

on propagators; swap the order of integration; integrate the Gaussian.

If we started with a scalar Feynman graph G in 4-dimensions we get

$$\int_{e_i \geq 0} \frac{\delta(e_1 + \cdots + e_n) \prod de_i}{\Psi^2}$$

where Ψ is the Kirchhoff polynomial of G,

$$\Psi = \sum_{\substack{T \text{ spanning} \\ \text{tree of } G}} \prod_{e \notin T} a_e$$

If our graph has some tensor structure or we want some other terms from the epsilon expansion then we get extra stuff in the numerator. But today I'm interested in **denominators**.

What was that? – again for graph theorists

Let H be a 4-regular graph. Let G be H with a vertex removed. Again form

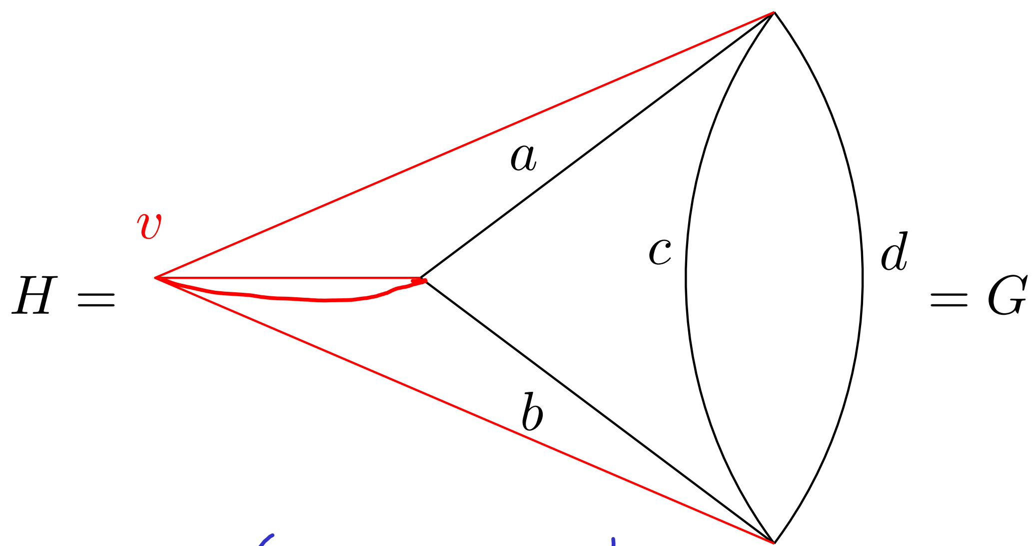
$$\int_{e_i \geq 0} \frac{\delta(e_1 + \cdots + e_n) \prod de_i}{\Psi^2}$$

where Ψ is the Kirchhoff polynomial of G,

$$\Psi = \sum_{\substack{T \text{ spanning} \\ \text{tree of } G}} \prod_{e \notin T} a_e$$

- This will converge provided all proper subgraphs of $G \smallsetminus v$ have more than twice as many edges as independent cycles.
- That this is independent of the choice of removed vertex is a theorem, but it is not known how to see it using this representation of the integral.
- That Schwinger parametrization gives this is the matrix-tree theorem.

Example



Spanning trees of $G = \{ac, ad, ab, bc, bd\}$

$$\Psi_G = bd + bc + cd + ad + ac$$

$$\int \frac{\delta(a+b+c+d)}{((c+d)(a+b)+cd)^2}$$

will diverge as c and d get large.

$$\int_{a=0}^{\infty} \frac{1}{(Aa+B)^2} = \frac{1}{AB}$$

A naive approach

Consider

$$\int_{e_i \geq 0} \frac{\prod de_i}{\Psi^2}$$

one edge variable at a time (Francis Brown).
So long as there is always a variable e so that the denominator is a product of two linear polynomials in e,

$$(Ae + B)(Ce + D),$$

then we can do the e integration next, getting explicit, increasingly complex polylogarithms in the numerator and

$$AD - BC$$

in the denominator.

If the denominator is the square of a linear polynomials in e,

$$(Ae + B)^2,$$

then we can again do the e integration. This time the weight of the polylogarithms in the numerator does not increase.

If all is nice we will end up evaluating some polylogarithms at 1. This gives multiple zeta values.

Multiple zeta values

$$\zeta(s_1, \ldots, s_n) = \sum_{a_1 > \cdots > a_n \geq 1} \frac{1}{a_1^{s_1} \ldots a_n^{s_n}}$$

The **weight** of $\zeta(s_1, \ldots, s_n)$ is $s_1 + \cdots + s_n$.

Multiple zeta values

- generalize special values of the Riemann zeta function
- have an interesting algebra structure and relations
- are the periods of moduli spaces
- ...

Consequences

Everything is controlled by some combinatorics of polynomials

- We know exactly how things go bad – when the polynomial does not factor. We can understand combinatorial criteria for this to happen, or be avoided.
- We will get a weight drop when the denominator has a factor which is a square or one of the edge variables is missing entirely.
- The basic story works even with stuff in the numerator.

The Dodgson polynomials

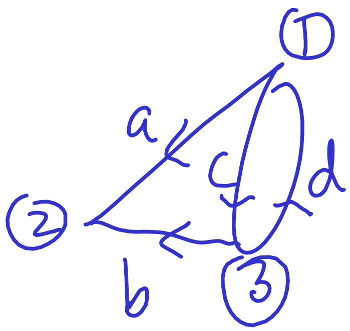
The main tool for understanding the denominators are some polynomials

$$\Psi^{I,J}_{K,G}$$

which we can understand **graphically** or via **matrices**. Each viewpoint has its uses.

The incidence matrix

We need the (oriented) incidence matrix of a graph. For example



	a	b	c	d
1	1	0	1	−1
2	−1	−1	0	0
3	0	1	−1	1

Dodgsons by matrices

Suppose G has n vertices and m edges. Let $\widehat{E}$ be the incidence matrix with one row removed. Build the matrix

$$M = (-1)^{n+1} \left[\begin{array}{ccc|c} a_1 & & & \\ & \ddots & & \widehat{E}^T \\ & & a_m & \\ \hline & \widehat{E} & & 0 \end{array} \right]$$

Then

$$\Psi_G = \det(M)$$

Let I, J, K be sets of edges of G with $|I| = |J|$. Let $M_G(I,J)_K$ be the matrix obtained from M_G by removing the rows of I, the columns of J, and setting $\alpha_e = 0$ for all $e \in K$. Then

$$\Psi^{I,J}_{G,K} = \det M_G(I,J)_K \ .$$

Spanning forest polynomials

Let $P = P_1 \cup \ldots \cup P_k$ be a set partition of a subset of the vertices of G. Define

$$\Phi_G^P = \sum_F \prod_{e \notin F} \alpha_e$$

where the sum runs over spanning forests $F = T_1 \cup \ldots \cup T_k$ where each tree T_i of F contains the vertices in P_i. Trees consisting of a single vertex are permitted.

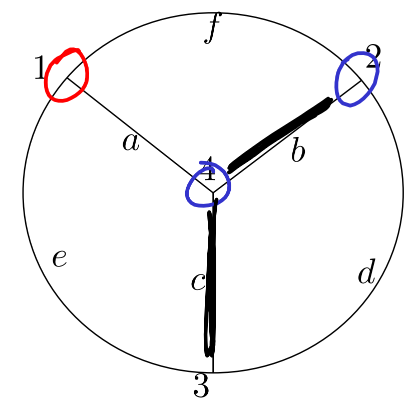
Then

$$\Psi_{G,K}^{I,J} = \sum \pm \Phi^P_{G \setminus I \cup J \cup K}$$

where P runs over partitions of the vertices adjacent to edges in I and J so that the resulting terms are trees after

- cutting I and contracting J **and**
- cutting J and contracting I.

Example

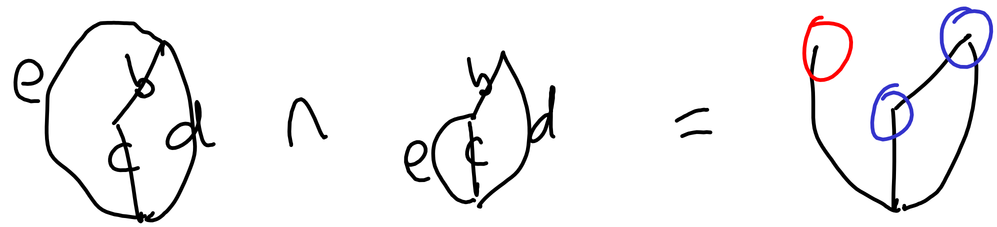


bc, bd, be, cd

Let $P = \{1\}, \{2, 4\}$. Then

$$\Phi_G^P = af(ed + ec + cd + be)$$

What is $\Psi_G^{a,f}$?



Structure in the denominators

The first few integrations look like:

$$\int \frac{1}{\Psi_G}$$

$$\int \frac{1}{\Psi_G^{1,1}\Psi_{G,1}}$$

$$\int \frac{\text{logs}}{(\Psi_G^{1,2})^2}$$

$$\int \sum \frac{\text{logs}}{\text{stuff}}$$

$$\int \frac{\text{dilogs}}{\Psi_G^{12,34}\Psi_G^{13,24}} + \frac{\text{dilogs}}{\Psi_G^{12,34}\Psi_G^{14,23}} + \frac{\text{dilogs}}{\Psi_G^{13,24}\Psi_G^{14,23}}$$

$$\int \frac{\text{trilogs}}{{}^5\Psi_G(1,2,3,4,5)}$$

The 5-invariant

The denominator after five integrations is given by

$$^5\Psi_G(1,2,3,4,5) = \pm \det \begin{pmatrix} \Psi_{G,5}^{12,34} & \Psi_G^{125,135} \\ \Psi_{G,5}^{13,24} & \Psi_G^{135,245} \end{pmatrix}$$

Up to sign it doesn't depend on order.

4 and 6 and onward

It would be nice to have a 4-invariant too. We have

$$\Psi_G^{12,34}\Psi_G^{13,24}$$
$$\Psi_G^{12,34}\Psi_G^{14,23}$$
$$\Psi_G^{13,24}\Psi_G^{14,23}$$

We'd like to think of **any one** of these as a 4-invariant. This is justified because any one of these gives the 5-invariant at the next integration.

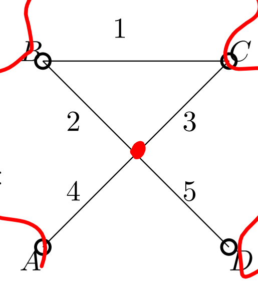
We'd also like to have 6-invariants, 7-invariants, etc. This is not always possible. The 5-invariant may not factor, or it may, but the 6 may not, ...
Write

$$D_G^n(i_1, \ldots, i_n)$$

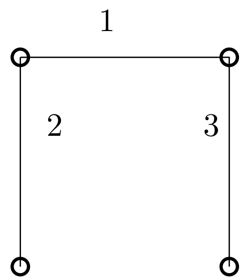
for the nth denominator when it exists.

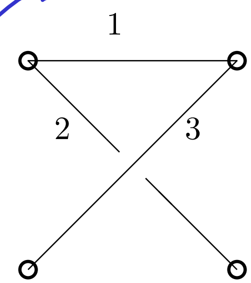
Denominator identities I

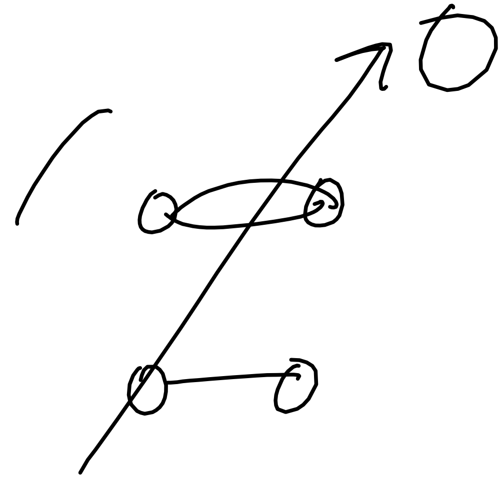
Let

$J =$ 

Let

$K_1 =$ 

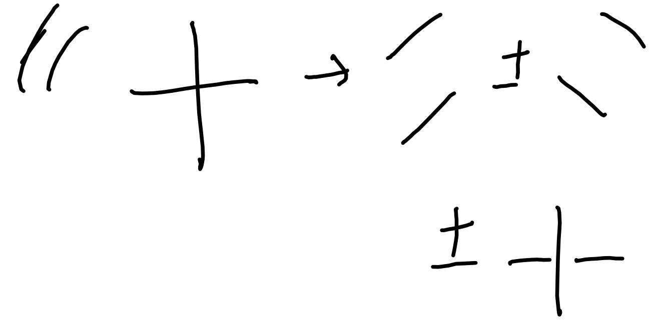
$K_2 =$ 



with the same remaining graph connecting at the circled vertices. Pick any 6th edge from among the remaining edges.

Theorem 1

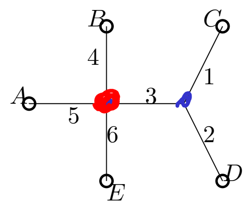
$$D_J^6 = \pm D_{K_1}^4 \pm D_{K_2}^4$$



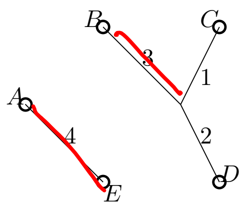
Denominator identities II

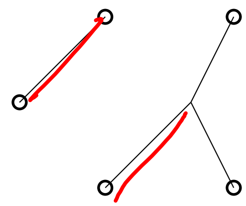
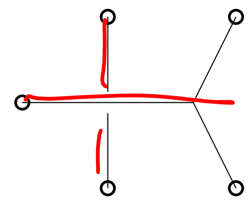
Let

$$G = $$



Let

$H_1 =$ 

$H_2 =$ 
$H_3 =$ 

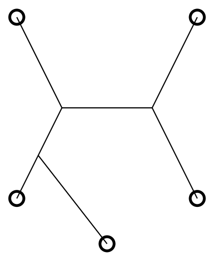
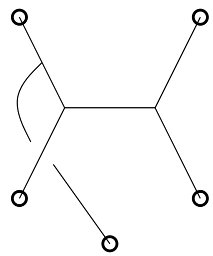
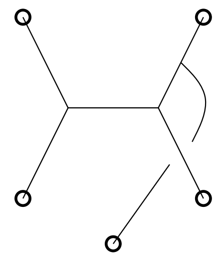
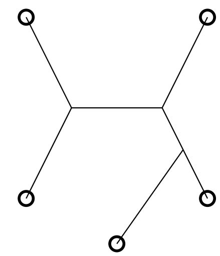
with the same remaining graph connecting at the circled vertices.

Theorem 2

$$D_G^6 = \pm D_{H_1}^4 \pm D_{H_2}^4 \pm D_{H_3}^4$$

Denominator identities III

We can play the same games even if the graph is not almost 4-regular. Let

$L_1 =$  $L_2 =$  $L_3 =$  $L_4 =$ 

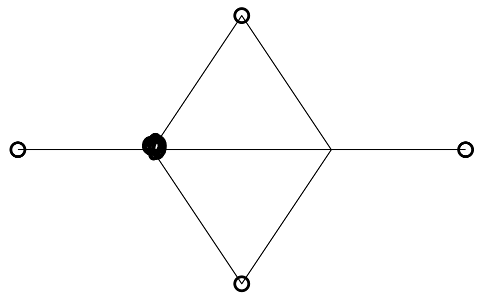
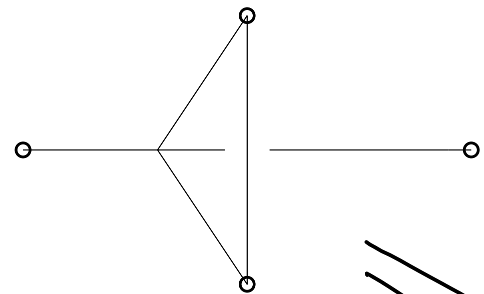
with the same remaining graph connecting at the circled vertices.

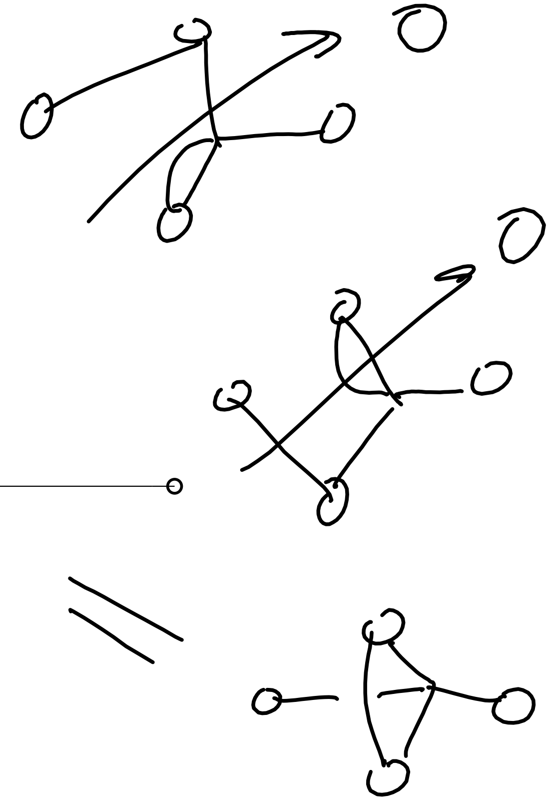
Theorem 3

$$D^7_{L_1} \pm D^7_{L_2} \pm D^7_{L_3} \pm D^7_{L_4} = 0$$

A special case – double triangle

Let

$M =$  $N =$ 



Then

$$D_M^7 = D_N^5$$

Double triangle is a special case of Theorem 1

Double triangle is special because it says that one denominator is the same as another, and since denominators determine the weight drops, it says that **one graph has weight drop if another, simpler graph does.**

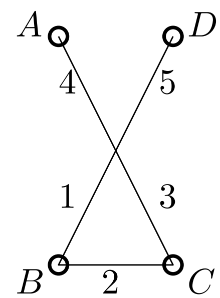
Proofs

All of these theorems are proved by manipulating the Dodgson polynomials and spanning forest polynomials.

Simpler but along the same lines is the direct proof of the double triangle identity. It will be most instructive to show it here.

Proof of double triangle – 1 triangle

Let

$G =$ 

with circles to indicate where the rest of the graph is attached. Let K be the rest of the graph.
Calculate

$$^5\Psi_G(1,2,3,4,5) = \pm\Psi_G^{123,245}\Psi_{G,2}^{14,35}.$$

$$\Psi_{G,2}^{14,35} =$$

$$=$$

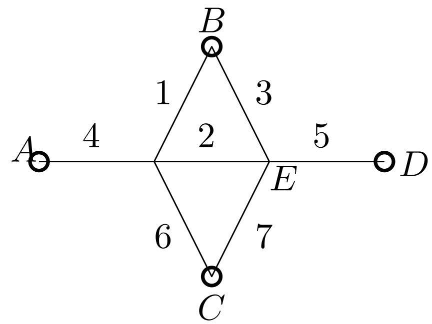
$$\Psi_G^{123,245} =$$

$$=$$

So

$$^5\Psi_G(1,2,3,4,5) = \pm\Phi_K^{\{A,D\},\{B\},\{C\}} \left(\Phi_K^{\{A,B\},\{C,D\}} - \Phi_K^{\{A,C\},\{B,D\}}\right)$$

Proof of double triangle – 2 triangles

$G' =$ 

Let K again be the rest of the graph.
By the above applied to edges $1, 3, 2, 4, 6$ we know that

$$^5\Psi_{G'}(1,2,3,4,6) = \pm\Phi^{\{A,C\},\{B\},\{E\}}_{K\cup\{5,7\}} \left(\Phi^{\{A,B\},\{C,E\}}_{K\cup\{5,7\}} - \Phi^{\{A,E\},\{B,C\}}_{K\cup\{5,7\}}\right)$$

The two ends of edge 7 are in different parts of $\{A,C\},\{B\},\{E\}$ so

$$\Phi^{\{A,C\},\{B\},\{E\}}_{K\cup\{5,7\}} = \alpha_7\Phi^{\{A,C\},\{B\},\{E\}}_{K\cup 5}$$

So we can easily continue the denominator reduction with edge 7.

$$ {}^{6}\Psi_{G'}(1,2,3,4,6,7) = \pm\Phi^{\{A,C\},\{B\},\{E\}}_{K\cup 5}\Phi^{\{A,B\},\{C\}}_{K\cup 5} $$

$$ = $$

From the pictures we can read off the contractions and deletions of edge 5 and deduce that the reduction with respect to edge 5 is

$$ {}^{7}\Psi_{G'}(1,2,3,4,5,6,7) $$

$$ = $$

We have

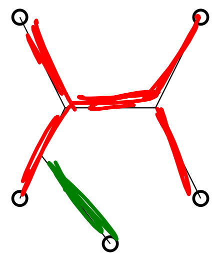
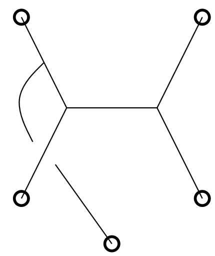
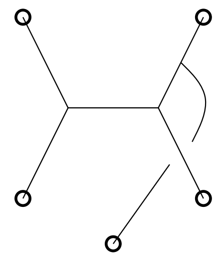
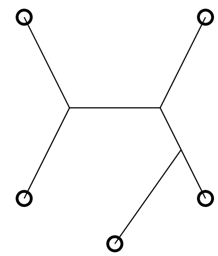
$$
\begin{aligned}
&{}^{7}\Psi_{G'}(1,2,3,4,5,6,7) \\
&= \pm\left(\Phi_K^{\{A,B\},\{C\},\{D\}}\Phi_K^{\{A,C\},\{B\}} - \Phi_K^{\{A,C\},\{B\},\{D\}}\Phi_K^{\{A,B\},\{C\}}\right)
\end{aligned}
$$

But this is itself a five-invariant, ${}^{5}\Psi_G(1,2,3,4,5)$ so by the previous calculation

$$D_7(G') = \pm\left(\Phi_K^{\{A,B\},\{C,D\}} - \Phi_K^{\{A,C\},\{B,D\}}\right)\Phi_K^{\{A,D\},\{B\},\{C\}}$$

Does this mean anything?

Recall Theorem 3:

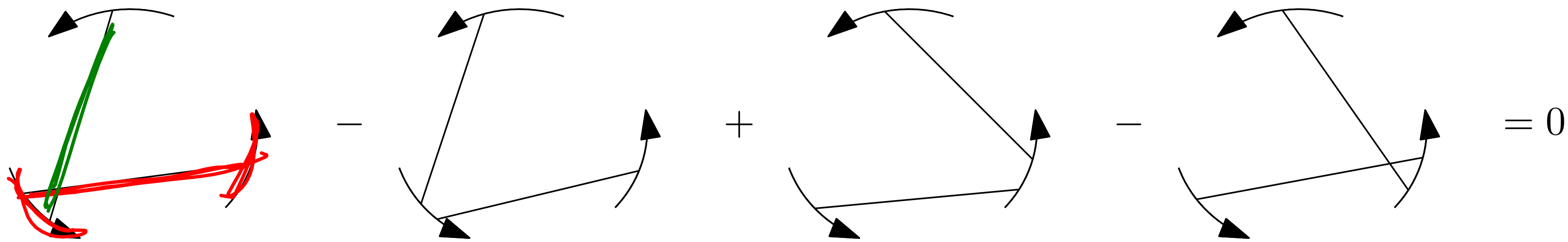
$L_1 =$  $L_2 =$  $L_3 =$  $L_4 =$ 

$$D^7_{L_1} \pm D^7_{L_2} \pm D^7_{L_3} \pm D^7_{L_4} = 0$$

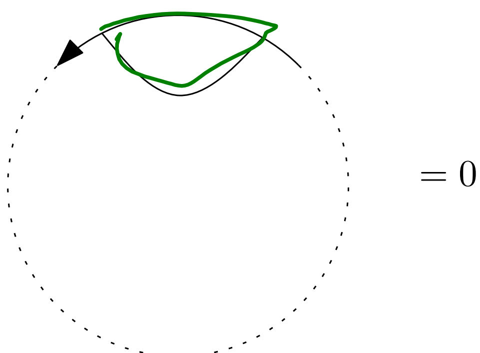
The signs come from the choice of order of vertices and edges and direction of edges.

4TR for chord diagrams

Compare the four-term relation for chord diagrams



Along with the one-term relation



these are exactly the identities satisfied by Vassiliev invariants.

Similarities and differences

- The shape of the identities in both contexts is exactly the same.
- We also have a one-term relation of denominators because double edges give zero denominators once we integrate them.

but

- There is no “outer cycle”
- Sums of denominators are garbage

Is there some precise connection here or is it just a coincidence?

Matroids

What else do denominators suggest?

Go back to momentum space, and suppose we are trying to remove tensor structure from numerators using simple tricks like.

$$\frac{r \cdot s}{(r-s)^2} = -\frac{1}{2} + \frac{r^2 + s^2}{2(r-s)^2}$$

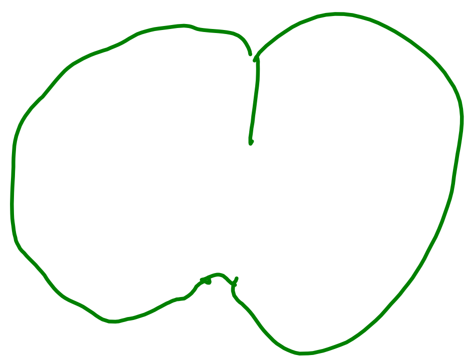
If there is no edge involving only momenta r and s then we're out of luck.

If we toss in such a factor, then the denominator may no longer come from a graph ... but it does come from a matroid.

Definition

A **matroid** consists of a finite set E and a set $\mathcal{C}$ of subsets of E satisfying

1. $\emptyset \notin \mathcal{C}$
2. If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$ then $C_1 = C_2$.
3. If $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$ and $e \in C_1 \cap C_2$ then there is a $C_3 \in \mathcal{C}$ with $C_3 \subseteq (C_1 \cup C_2) \setminus e$.



For a graph,

E: set of edges

C: the set of cycles of the graph (**circuits**).

But matroids are much more general.

Can also define matroids in terms of E and **bases** which correspond to spanning trees of a graph. And many other ways.

Incidence matrices and representable matroids

Graphs have incidence matrices, recall our example

$$\begin{array}{c} \\ ① \\ ② \\ ③ \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & -1 \\ -1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \end{array}$$

In the other direction, define a matroid from a matrix

- E is the set of columns
- a circuit is a set of columns which is linearly dependent but with every proper subset linearly independent

A matroid which comes from a matrix is called **representable**.

Row reduction and unique representability

None of the following change the linear dependence relations of columns

- Elementary row operations
- Removing rows of zeros
- Scaling columns
- Field automorphisms

So none of these change the matroid.

If we label the columns and keep track of the labels we can also swap columns.

None-the-less a matroid may have **inequivalent** representations.

Duality

Every matroid has a dual which generalizes the graph dual for planar graphs. For us representable matroids will suffice.

Take a representable matroid with matrix M. Row reduce M swapping columns and removing zero rows until it has the form

$$(I_n|D)$$

Then the dual matroid is represented by the matrix

$$(-D^T|I_m).$$

Matroids which are the duals of graphic matroids are called **cographic** matroids.

1PI, contraction and deletion

1PI is available for matroids (even though connectedness in the graph-sense is not!). It is the property of **bridgelessness**, that is every $e \in E$ is in at least one circuit.

One can also contract and delete elements of a matroid, as for edges of a graph. For a representable matroid

- cut by removing a column
- contract by row reducing until the column contains only one nonzero entry, then remove the corresponding row and column.

The opposite of contraction is **coextension** it is not unique.

Lets do something with all this

Tensor integrals can be resolved into scalar matroid integrals in the following way.

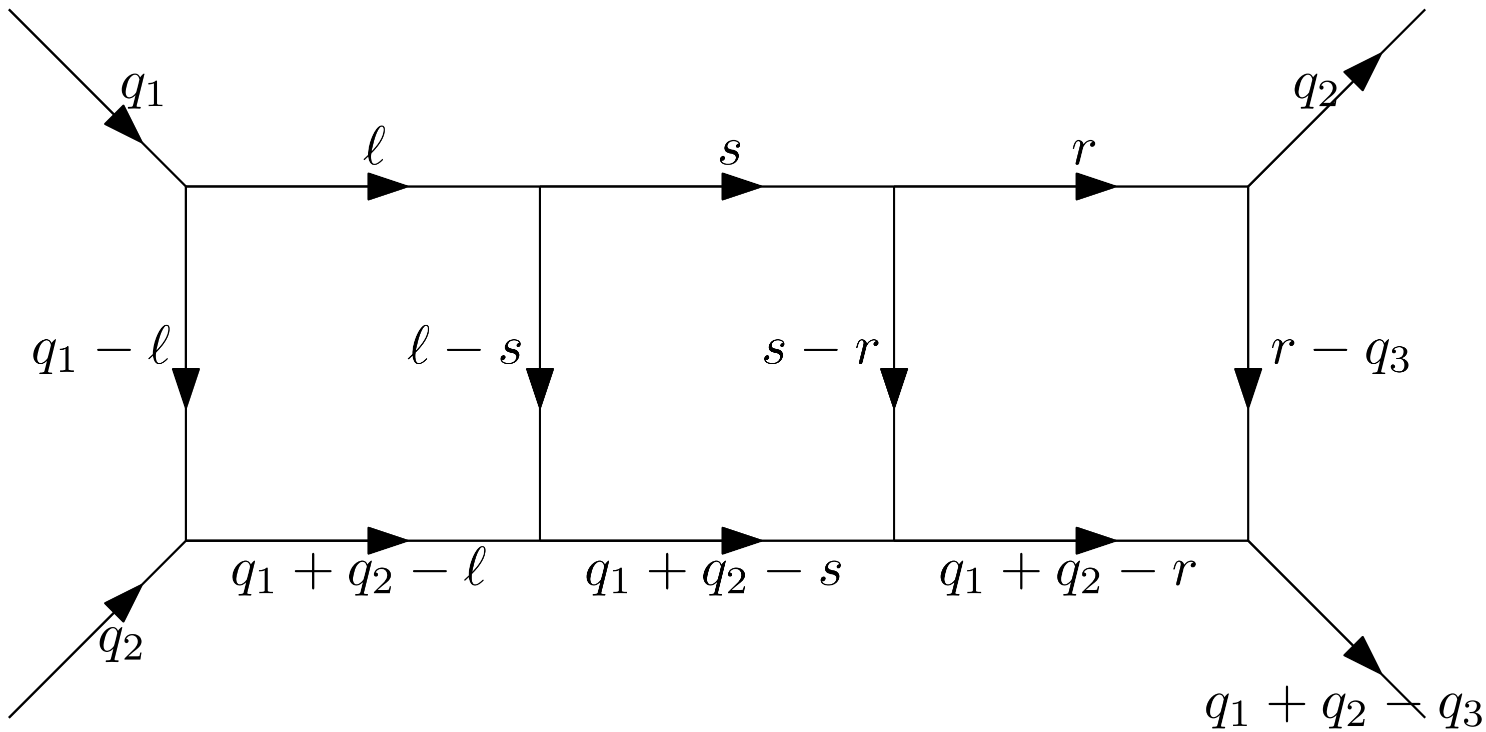
1. Begin with the matroid of the original graph.
2. Write the matroid in the form $(I_{\mathrm{rk}G}\, C)$ with all entries of C are 0 or ± 1.
3. Take a pair of edges of the graph.
4. Coextend with a new row which is nonzero in the columns of the two edges chosen above, and a new column which is nonzero only in the new row. By choosing appropriately the matrix can remain in the form
$$\begin{pmatrix} I_{\mathrm{rk}G} & 0 & C \\ 0 & I_r & D \end{pmatrix}$$
where all entries of C and D are 0 or ± 1.

5. This corresponds to adding a factor to the denominator of the Feynman integral with momentum a linear combination of the momenta of the pair of edges.

6. Continue until we have appropriate factors to clear all the tensor structure from the numerator.

7. The matroid of the resulting denominator may not be uniquely representable, but the nice representation above will be unique up to row operations etc. This is the scalar Feynman integral of the matroid.

Example



Suppose we have an $\ell \cdot r$ in the numerator. We get a scalar integral

$$\int \frac{d^4\ell d^4 s d^4 r}{\ell^2(\ell - s)^2 s^2 (s - r)^2 r^2 (q_1 - \ell)^2 (q - \ell)^2 (q - s)^2 (q - r)^2 (r - q_3)^2 (\ell - r)^2}$$

where $q = q_1 + q_2$.

This corresponds to

$$
\begin{array}{ccccccccccccccc}
\ell & \ell - s & s & s - r & r & q_1 - \ell & q - \ell & q - s & q - r & r - q_3 & q_1 & q_2 & q_3 & q - q_3 & \ell - r
\end{array}
$$

$$
\begin{pmatrix}
-1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

It is not a graph, but it is dual to a graph.

Pro-matroid evidence

- Matroids capture the redundancy of graphs compared to Feynman integrals.
- Allowing matroids, every graph has a dual.
- Whenever a graph result can be stated in terms of contraction and deletion only, it ought to be a matroid result.
- Matroid integrals are a natural tool to decompose tensor integrals.

Matroids suggest a hierarchy of difficulty.

1. Planar graphs: Both they and their duals are graphs.

2. General graphs and cographs: Cographs are a natural next term for any series which begins with a planar piece and doesn't continue with graphs themselves.

3. Regular matroids: Regular matroids are nice in many ways, notably they are uniquely representable over every field and they have a matrix-tree theorem identical to that of graphs; typically one's graph based intuition is valid.

4. The matroids we need: They always have a nice representation in the form $(I|D)$ with D having entries $0, 1, -1$.

5. General matroids: can be quite hairy.

What was all that – I got lost

Summary

If we integrate one variable at a time in parametric form the denominators have **nice combinatorial interpretations** and **satisfy identities**.

Some of the identities let us **determine the transcendental weight** of some graphs. Some of the identities look like **chord diagram identities**.

Matroids let us capture more denominators than graphs do, including ones which come up when resolving **tensor integrals**. They also contain **exactly** the information a Feynman integral needs.