Deriving Explicit Runge–Kutta Pairs of Lower Stage-order *

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Runge–Kutta pairs are effective if the difference of a pair gives an accurate error estimate, and the propagated value has a small error. While better pairs than those derived by Fehlberg [2] have been found [4], the number of stages per step has remained almost unreduced. Recently, Khashin [3] developed a *numerical* approach to solving the order conditions by formulating their solution set as an algebraic variety. Among new families he obtained, that of order $p = 9$, requires only 13 stages. The present work attempts a *direct* approach to deriving such methods using linear dependence of subquadrature expressions. The latter have been utilized in Butcher's simplifying conditions [1] for solving the order equations. The results admit rapid computation of the coefficients, and matched methods to obtain Runge–Kutta pairs. The stage-orders of these new pairs are lower, and unlike known pairs with stage-order $p - 5$, these new pairs yield reliable error estimates for all types of problems. The focus of this work is the direct solution of the order conditions, and it may be possible to apply the techniques developed to derive coefficients for other types of explicit methods for treating initial value problems in ordinary differential equations.

References

1. J.C. Butcher, On Runge–Kutta processes of high order, J. Austral. Math. Soc. **4** (1963), 179–194.

2. E. Fehlberg, Classical fifth-, sixth-, seventh-, and eighth-order Runge–Kutta formulas with stepsize control, NASA Technical Report NASA TRR–287, (1968), 82 pages.

3. S. Khashin, A symbolic-numeric approach to the solution of the Butcher equations, Can. Appl. Math. Quarterly, **17** (2009), 555–569.

4. J.H. Verner, Explicit Runge–Kutta methods with estimates of the local truncation error, SIAM J. Numer. Anal, **15** (1978), 772–790.

* This development of this approach has been motivated by discussions with John Butcher over many years.