Mathematics 251–3

Old Mid-Term Exam from Dr. Ryeburn

Second Mid-Term Exam Answers Monday, November 4, 1996

1. The upper nappe of an elliptical cone is defined by $z = \sqrt{x^2 + 4y^2}$.
The point $(3, 2,5)$ is on this surface. Find an equation for the tangent plane at $(3, 2,5)$ on the surface $z = \sqrt{x^2 + 4y^2}$. The arithmetic is easy and must be done completely.

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + 4y^2}} \quad \text{so} \quad \frac{\partial z}{\partial x}(3,2) = \frac{3}{5}.$$

$$\frac{\partial z}{\partial y} = \frac{4y}{\sqrt{x^2 + 4y^2}} \quad \text{so} \quad \frac{\partial z}{\partial y}(3,2) = \frac{8}{5}.$$

The tangent plane has equation $z = 5 + \frac{3}{5}(x - 3) + \frac{8}{5}(y - 2)$, or
$3x + 8y - 5z = 0.$

Alternatively the cone is a level surface $g(x, y, z) = 0$ for the function
$g(x, y, z) = x^2 + 4y^2 - z^2$ whose gradient is $g = 2x\,\mathbf{i} + 8y\,\mathbf{j} - 2z\,\mathbf{k}$.
The vector $g(3, 2, 5) = 6\,\mathbf{i} + 16\,\mathbf{j} - 10\,\mathbf{k}$ must be normal to the tangent plane at $(3, 2,5)$. Therefore the tangent plane must have equation
$\left(6\,\mathbf{i} + 16\,\mathbf{j} - 10\,\mathbf{k}\right) \left((x - 3)\mathbf{i} + (y - 2)\mathbf{j} + (z - 5)\mathbf{k}\right) = 0$, or
$6x + 16y - 10z = 0.$

2. (a) Use the differential to approximate the value of z on the elliptical cone nappe $z = \sqrt{x^2 + 4y^2}$ (the surface of Question 1) at the point where $x = 3.01$ and $y = 1.98$. The arithmetic is easy and must be done completely.
Calculators may not be used on this examination.

$$dz(x, y, dx, dy) = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = \frac{x}{\sqrt{x^2+4y^2}}\,dx + \frac{4y}{\sqrt{x^2+4y^2}}\,dy$$

from Question 1.

$dz(3, 2, 0.01, -0.02) = \frac{3}{5}\,(0.01) + \frac{8}{5}\,(-0.02) = -0.026.$

When $x = 3.01$ and $y = 1.98$, $z \approx 5 - 0.026 = 4.974.$

My TI–36 calculator gives $\sqrt{(3.01)^2 + 4\,(1.98)^2} \approx 4.974102934.$

(b) Use the differential to approximate the value of z on the elliptical cone nappe $z = \sqrt{x^2 + 4y^2}$ (the surface of Question 1) at the point where $x = 3.03$ and $y = 2.02$. The arithmetic is easy and must be done completely.
Calculators may not be used on this examination.

As in part (a), $dz(x, y, dx, dy) = \frac{\partial z}{\partial x}\,dx + \frac{\partial z}{\partial y}\,dy = \frac{x}{\sqrt{x^2+4y^2}}\,dx + \frac{4y}{\sqrt{x^2+4y^2}}\,dy$

so $dz(3, 2, 0.03, 0.02) = \frac{3}{5}\,(0.03) + \frac{8}{5}\,(0.02) = 0.05.$

When $x = 3.03$ and $y = 2.02$, $z \approx 5 + 0.05 = 5.05.$

(c) A dishonest student uses a calculator anyway and finds that $\sqrt{(3.03)^2 + 4\,(2.02)^2} = 5.050000000$. His calculator displays 10 significant figures, and he is surprised that the last 7 of them are zeros. Those aren't the only zeros that he will see associated with this examination!

The student later discusses the question with other students who followed instructions and used the differential to answer part (b). He is amazed that his answer was the same as theirs. Explain why the answer obtained using the differential is the same as that obtained by using a calculator. (What is so special about the values $x = 3.03$ and $y = 2.02$?)

The cone nappe contains a half-line of the form $x = 3t$, $y = 2t$, $z = 5t$, $t \ge 0$ since $\sqrt{(3t)^2 + 4\,(2t)^2} = \sqrt{25t^2} = 5t$ when $t \ge 0$.
This half-line lies in the tangent plane. Note that moving from $x = 3$, $y = 2$ to $x = 3.03$, $y = 2.02$ is equivalent to moving from $t = 1$ to $t = 1.01$. In this direction the surface does not curve away from its tangent plane so the differential causes no error whatsoever.

3. If $f(x,y) = x^2 + 2xy + 2y^3 + 4y^2 + 10$, find and classify all critical points of $f(x,y)$. Show your work! If your method is correct but your algebra is missing and was incorrect, we will think your method was incorrect.

$\frac{\partial z}{\partial x} = 2x + 2y$ and $\frac{\partial z}{\partial y} = 2x + 6y^2 + 8y$.
These exist everywhere so the critical points are found by equating them to zero.
If $\frac{\partial z}{\partial x} = 0$ then $2x + 2y = 0$ so $y = -x$.
Substituting this into $2x + 6y^2 + 8y = 0$, $2x + 6x^2 - 8x = 0$, so
$6x^2 - 6x = 0$, and thus $6x(x-1) = 0$, so $x = 0$ or $x = 1$.
The critical points are $(0, 0)$ and $(1, -1)$.
$\frac{\partial^2 z}{\partial x^2} = 2$, $\frac{\partial^2 z}{\partial y^2} = 12y + 8$, and $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y} = 2$, so

$$D(x, y) = \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial y \partial x}\right)^2 = 24y + 12.$$

$D(0, 0) = 12 > 0$, $\frac{\partial^2 z}{\partial x^2}(0, 0) = 2 > 0$, and $\frac{\partial^2 z}{\partial y^2}(0, 0) = 8 > 0$,
so $f(0, 0) = 10$ is a local minimum value.
$D(1, -1) = -12 < 0$, so $(1, -1, 11)$ is a saddle point.

You can see this algebraically as follows.
$f(x,y) = x^2 + 2xy + 2y^3 + 4y^2 + 10 = (x + y)^2 + (2y + 3)y^2 + 10$
and since $2y + 3 > 0$ when (x, y) is near $(0, 0)$,
we clearly have a local minimum at $(0, 0)$.
$f(x,y) = x^2 + 2xy + 2y^3 + 4y^2 + 10 = \left[(x-1) + (y+1)\right]^2 + (2y-1)(y+1)^2 + 11$
and since $2y - 1 < 0$ when (x, y) is near $(1, -1)$,
we have a saddle point at $(1, -1)$.
If we move away from that point along the line $x + y = 0$, the term $\left[(x-1) + (y+1)\right]^2$ remains zero while the term $(2y-1)(y+1)^2$ causes z to decrease on either side of $(1, -1)$, at least for a while.
If we move away from that point along the line $y = -1$, the term $(2y-1)(y+1)^2$ remains zero while the term $\left[(x-1) + (y+1)\right]^2$ causes z to increase on either side of $(1, -1)$, no matter how far we go.

4. Consider the function $f(x,y) = x + 16y + 2$. The branch of the hyperbola $xy = 4, x > 0, y > 0$ although closed is not a bounded set, so the theorem guaranteeing the existence of a maximum and a minimum for a continuous function on a closed bounded set does not apply.

(a) Explain briefly, using words and not calculus, why $f(x,y)$ does **not** have a constrained maximum value along the hyperbola branch $xy = 4$, $x > 0$, $y > 0$.

You can make $f(x,y) = x + 16y + 2$ as large as you like, along the hyperbola branch $xy = 4$, $x > 0$, $y > 0$, by taking one of x or y very large and positive and choosing the other so that their product is 4. Consequently no constrained maximum can exist.

(b) The function $f(x,y)$ **does** have a constrained minimum value along the hyperbola branch $xy = 4, x > 0, y > 0$. Find it, using the method of Lagrange multipliers. No credit will be given if any other method is used.

Let $g(x, y) = xy$. Our constraint condition is $g(x, y) = 4$.

$g(x, y) = y\mathbf{i} + x\mathbf{j} \neq \mathbf{0}$ along $g(x, y) = 4$ so we determine where

$f(x, y) = \lambda \; g(x,y)$ along $g(x, y) = 4$.

$f(x, y) = \mathbf{i} + 16\mathbf{j}$ so we solve $\mathbf{i} + 16\mathbf{j} = \lambda(y\mathbf{i} + x\mathbf{j})$.

$\lambda y = 1$ and $\lambda x = 16$ so $\lambda^2 xy = 16$.

But $xy = 4$, so $\lambda^2 = 4$ and $\lambda = \pm 2$.

If $\lambda = -2$ then $(x, y) = (-8, -0.5)$ which is on the wrong branch of the hyperbola.

If $\lambda = 2$ then $(x, y) = (8, 0.5)$ and the required constrained minimum value is $f(8, 0.5) = 8 + 16 \; 0.5 + 2 = 18$.

The level curve $x + 16y + 2 = 18$ of $f(x,y) = x + 16y + 2$ passes through $(8, 0.5)$ and is tangent to the curve $xy = 4$ there.
On either side of that point, the hyperbola lies above and to the right of the tangent line and the values of $f(x,y) = x + 16y + 2$ are larger.

5. Evaluate the double integral $\int_0^1 \int_{5x}^5 e^{-y^2}\,dy\,dx$.

No elementary function is an antiderivative of the function e^{-y^2} so we cannot evaluate the integral as it stands.

The graph to the right shows the region of integration. Changing the order of integration,

$$\int_0^1 \int_{5x}^5 e^{-y^2}\,dy\,dx = \int_0^5 \int_0^{y/5} e^{-y^2}\,dx\,dy =$$

$$= \int_0^5 xe^{-y^2}\Big]_0^{y/5} dy = \int_0^5 \frac{y}{5} e^{-y^2}\,dy =$$

$$= -\frac{1}{10} e^{-y^2}\Big]_0^5 = \frac{1-e^{-25}}{10}.$$