Mathematics 251–3 (Fall 1996)

Old Final Exam from Dr. Ryeburn

Thursday, December 5, 1996

1. Does $\lim\limits_{(x,y)\to(0,0)} \frac{x^4-y^4}{x^4+y^4}$ exist? If it exists, what is its value?

Do not give an ε-δ argument, but support your conclusions with convincing reasoning. You may use any theorems discussed in the course.

2. The distance A between $(2,1,1)$ and $(5,-5,-1)$ is exactly 7.
Use differentials to give a close approximation to the distance D between $(2.01, 1.02, 0.98)$ and $(5.03, -5.01, -1.02)$ involving indicated sums, differences, products, or quotients but **without** using square roots. Your answer should be ready for the use of a cheap calculator that can add, subtract, multiply, and divide, but you should not make any actual arithmetical calculations you would ordinarily do on such a calculator. **Think!** Don’t make this a six-variable question!

3. Let T be the closed bounded triangular region in the xy-plane whose vertices are $(0,0)$, $(-1,0)$, and $(0,-3)$. Let $f(x,y) = x^2 - 2xy + y^2 - 4x + 4y + 7$. Maximize and minimize $f(x,y)$ throughout T. Make your reasoning clear!

4. Use the method of Lagrange multipliers to find the absolute maximum and minimum values of the function $f(x,y) = x^2 + 4x + y^2 + 6y$ subject to the constraint $x^2 - 4x + y^2 = 21$. (The constraint condition defines a circle — a closed, bounded set.)

Note: There are many other ways to answer this question. No credit will be given unless the method of Lagrange multipliers is used.

5. If $f(x,y) = 2x^2 + 16xy - y^3 + 32y^2 + 300y$, find and classify all critical points of the function $f(x,y)$.

6. (a) In what direction does the function $f(x,y) = 2x^2 + 16xy - y^3 + 32y^2 + 300y$ of Question 5 increase most rapidly at the point $(0,10)$?

(b) How rapidly does $f(x,y) = 2x^2 + 16xy - y^3 + 32y^2 + 300y$ increase in its direction of most rapid increase, at $(0,10)$?

(c) How rapidly does $f(x,y) = 2x^2 + 16xy - y^3 + 32y^2 + 300y$ increase in the direction towards $(-3,14)$ at $(0,10)$?

7. I want to make a box, with bottom but no top, in the shape of a rectangular parallelepiped. The box is to have volume 375m^3. The material used for the bottom costs $12 per square metre and the material used for the four vertical faces costs $2 per square metre. What dimensions give the cheapest box?

(You should find only one critical point; you need not verify that it provides a minimum.)

8. Find the surface area of the portion of the paraboloid $z = 20 - x^2 - y^2$ between the planes $z = 4$ and $z = 11$.

9. Evaluate the integral $\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \left(x^2 + y^2\right)^{100} \, dy \, dx$.

10. Evaluate the integral $\int_0^3 \int_{2x}^6 xy\cos\left(y^4\right) dy\, dx$.

11. Let C be the closed curve consisting of the line segment from $(0,0,0)$ to $(1,1,5)$, followed by the line segment from $(1,1,5)$ to $(0,1,5)$, followed by the portion of the parabola $\mathbf{r}(t) = (1-t)\,\mathbf{j} + (5-5t^2)\,\mathbf{k}$ from $(0,1,5)$ to $(0,0,0)$.

Evaluate the line integral $\int_C e^{yz}\,dx + xze^{yz}\,dy + xye^{yz}\,dz$.

12. Let C be the closed curve consisting of the line segment from $(0,0)$ to $(5,0)$, followed by the quarter of the circle $x^2 + y^2 = 25$ from $(5,0)$ to $(0,5)$, followed by the line segment from $(0,5)$ to $(0,0)$.

Evaluate the line integral $\int_C \left(2xy^2 + y\right) dx + \left(2x^2 y - x\right) dy$.