Math 251, Calculus III

J. Hebron, Spring 2000

Final Examination

Monday, April 10th, 2000

This is a closed-book, 3-hour exam. No calculators are allowed. Students are allowed one 8.5 by 11 inch formula sheet which may be filled on both sides.

Fill out all the required information on the front of the exam booklet, including your name, student ID number, and signature.

Write all your answers in the exam booklet provided. You may keep the exam questions after you are done. Hand-in only your answer booklet.

The total points for each question are shown in square brackets [like this]. The exam is out of a total of [**120**] points.

Do not open this booklet until told to do so.

1. **(a)** Find the equations, *in parametric form,* of the line of intersection of the planes $x - z = 1$ and $y + z = 1$. [4]

(b) Find the equation of the plane that contains this line of intersection and is perpendicular to the plane $x + y = 1$. [6]

2. Let a space curve be defined by $\vec{\mathbf{r}} = \langle \cos(t^2), \sin(t^2), \sqrt{3}t^2 \rangle$, where $t \geq 0$.

(a) Find the velocity vector and its magnitude. [4]
(b) Find the acceleration vector. [3]
(c) What is the unit tangent vector $\vec{\mathbf{T}}$? [2]
(d) What is the unit normal vector $\vec{\mathbf{N}}$? [2]
(e) What is the curvature of the curve? [2]
(f) What is the magnitude of the tangential component of the acceleration vector? [2]
(g) What is the magnitude of the normal component of the acceleration vector? [2]
(h) What is the arc length from $t = 0$ to $\sqrt{2\pi}$? [2]

3. What is the equation, in cylindrical coordinates, of a sphere of radius *a* centered on the origin? [2]

4. What is the equation, in spherical coordinates, of a cylinder of radius *b* having the z-axis as its cylindrical axis? [3]

5. Let $f(x,y) = \dfrac{(x+1)^2 y^2}{(x+1)^3 + y^6}$. Does $\lim\limits_{(x,y)\to(-1,0)} f(x,y)$ exist? If so, what is its value? [6]

6. The shape of a volcano is given by $z = he^{-(x^2+3y^2)}$, where $h > 0$ is its maximum height. The volcano erupts and lava is flowing. In what direction, in the xy-plane, is the lava flowing at x=4, y=1? Express the direction as a unit vector in the xy-plane. [8]

7. Find and classify all critical points of $f(x,y) = e^{-(|x-1|+|y+1|)}$. [8]

8. Find and classify all critical points of $f(x,y) = \left(x^3 - 3x - 18\right)\left(y^2 - 2y + 2\right)$.
The following factorization may come in handy: $\left(x^3 - 3x - 18\right) = (x-3)\left(x^2 + 3x + 6\right)$. [10]

9. Using the method of **Lagrange Multipliers** *(no points will be given for any other method)*, find the volume of the largest rectangular box with edges parallel to the axes that can be inscribed in the ellipsoid $x^2 + \dfrac{y^2}{4} + \dfrac{z^2}{9} = 3$. [8]

10. Evaluate $\int_0^1 \int_{x^2}^1 x^3 \sin\left(y^3\right) dy dx$. [8]

11. Find the moment of inertia about a diameter of the base of a solid homogeneous hemisphere of radius *a*. (Let the hemisphere be sitting with its base in the *xy*-plane and calculate I_y. Use spherical coordinates to evaluate the triple integral.) [16]

12. What is the Jacobian of the transformation from Cartesian coordinates to spherical coordinates? [2]

13. Find the work done by the force field $\vec{\mathbf{F}}(x,y,z) = z\hat{\mathbf{i}} + 2yz\hat{\mathbf{j}} + \left(x + y^2\right)\hat{\mathbf{k}}$ in moving an object along the space curve in problem #**2** from $t = 0$ to $\sqrt{2\pi}$. [10]

14. Evaluate $\oint_C \left(2y - \cos(\sin(\ln(x^2 + 3x^4 + 1)))\right) dx + \left(x^2 + \tan^{-1}(y)\sinh(y)\right) dy$, where *C* is the boundary of the region enclosed by the parabola $y = x^2$, the line $y = 1$, and the line $x = 0$, in the first quadrant. [8]

15. What is $\vec{\nabla} \times \vec{\mathbf{F}}$, where $\vec{\mathbf{F}}$ is the **gradient** of ϕ, and $\phi = \left(\dfrac{x^3 y + e^{xy}\cos(yz) - \tan^{-1}(xy)}{\ln(1 + x^2 + z^2) + y^2 + 1} \right)$? [2]