Mathematics 251–3

Solutions to Dr. Ryeburn's Old Mid-Term Exam

First Mid-Term Test Answers | Monday, September 30, 1996

1. In this question $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is a three-dimensional vector-valued function of a scalar variable, and a is a real number such that $\mathbf{r}(t)$ is defined in an open interval containing a (i.e., at a, and both to the left and to the right of a).

(a) What does it mean to say that $\mathbf{r}(t)$ is continuous at $t = a$?

$\mathbf{r}(t)$ is continuous at $t = a$ means that $\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a)$.

Alternatively, it means that each of $f(t)$, $g(t)$, and $h(t)$ are continuous at $t = a$.

(b) What do we mean by the derivative $\mathbf{r}'(a)$ of $\mathbf{r}(t)$ at $t = a$?

$\mathbf{r}'(a) = \lim_{h \to 0} \dfrac{\mathbf{r}(a+h) - \mathbf{r}(a)}{h}$.

Alternatively, $\mathbf{r}'(a) = \lim_{t \to a} \dfrac{\mathbf{r}(t) - \mathbf{r}(a)}{t - a}$.

Alternatively, $\mathbf{r}'(a) = \langle f'(a), g'(a), h'(a) \rangle$.

2. (a) Find either a vector equation or a triple of scalar (parametric) equations for the line L through the points $A = (2,5,1)$ and $B = (3,7,-1)$.

A direction vector for the line is $\mathbf{w} = \overrightarrow{AB} = \langle 1,2,-2\rangle$.
The line has equation $\mathbf{r}(t) = \langle 2+t, 5+2t, 1-2t\rangle$.
Another correct answer is $\mathbf{r}(t) = \langle 3+t, 7+2t, -1-2t\rangle$,
obtained by starting at $B = (3,7,-1)$ instead of at $A = (2,5,1)$.
Other correct answers use nonzero scalar multiples of our direction vector or start at other points on the line.
Corresponding to the vector equation $\mathbf{r}(t) = \langle 2+t, 5+2t, 1-2t\rangle$
is the triple of scalar equations $x = 2+t$, $y = 5+2t$, $z = 1-2t$.
Corresponding to the vector equation $\mathbf{r}(t) = \langle 3+t, 7+2t, -1-2t\rangle$
is the triple of scalar equations $x = 3+t$, $y = 7+2t$, $z = -1-2t$.
Other correct vector equation answers have corresponding correct triples of scalar equations.

(b) How far is the point $(2,10,-3)$ from this line?

Choose a point on the line, for example $A = (2,5,1)$.
Let $\mathbf{v}$ be the vector $\langle 0,5,-4\rangle$ from A to the point $(2,10,-3)$.
If D is the distance from $(2,10,-3)$ to the line then $D = |\mathbf{v}||\sin\theta|$,
where θ is an angle between $\mathbf{v}$ and the line's direction vector $\mathbf{w} = \langle 1,2,-2\rangle$.
But $|\mathbf{v}\times\mathbf{w}| = |\mathbf{v}||\mathbf{w}||\sin\theta|$, and thus $|\mathbf{v}||\sin\theta| = \frac{|\mathbf{v}\times\mathbf{w}|}{|\mathbf{w}|}$.

So $D = \frac{|\mathbf{v}\times\mathbf{w}|}{|\mathbf{w}|} = \frac{|\langle 0,5,-4\rangle\times\langle 1,2,-2\rangle|}{|\langle 1,2,-2\rangle|} = \frac{|\langle -2,-4,-5\rangle|}{|\langle 1,2,-2\rangle|} = \frac{\sqrt{45}}{3} = \sqrt{5}$.

Alternatively, the square of the distance from an arbitrary point $(2+t, 5+2t, 1-2t)$ on the line to the point $(2,10,-3)$ is
$D^2 = t^2 + (2t-5)^2 + (4-2t)^2 = 9t^2 - 36t + 41 = 9(t-2)^2 + 5$,
which is minimized by taking $t = 2$. Thus the closest point on the line is $(4,9,-3)$ which is at distance $\sqrt{5}$ from $(2,10,-3)$.

3. (a) Find an equation for the plane P through the points $A = (7,0,0)$, $B = (0, -7, -7)$, and $C = (6, -2, 0)$.

The vectors $\overrightarrow{AB} = \langle -7, -7, -7 \rangle$ and $\overrightarrow{AC} = \langle -1, -2, 0 \rangle$ are in the plane and are not parallel, so their cross product $\langle -14, 7, 7 \rangle$ is normal to the plane, and so is its scalar multiple $\mathbf{w} = \langle -2,1,1 \rangle$. Thus the plane has equation $\langle -2,1,1 \rangle \; \langle x-7, y-0, z-0 \rangle = 0$, or $2x = y + z + 14$.

Here I used $A = (7, 0, 0)$ as starting point in the plane. I could have used $B = (0, -7, -7)$, $C = (6, -2, 0)$, or any other point in the plane. I used $\mathbf{w} = \langle -2,1,1 \rangle$ as normal vector to the plane. I could have used any non-zero multiple of it.

(b) How far is the point $(1, -9, 3)$ from this plane?

Choose a point in the plane, for example $A = (7, 0, 0)$. Let $\mathbf{v}$ be the vector $\langle -6, -9, 3 \rangle$ from A to the point $(1, -9, 3)$. If D is the distance from $\langle 1, -9, 3 \rangle$ to the plane then $D = |\mathbf{v}||\cos\theta|$ where θ is an angle between $\mathbf{v}$ and the plane's normal vector $\mathbf{w} = \langle -2,1,1 \rangle$.

But $\mathbf{v} \;\; \mathbf{w} = |\mathbf{v}||\mathbf{w}||\cos\theta|$, and thus $|\mathbf{v}||\cos\theta| = \frac{|\mathbf{v} \; \mathbf{w}|}{|\mathbf{w}|}$.

So $D = \frac{|\mathbf{v} \; \mathbf{w}|}{|\mathbf{w}|} = \frac{|\langle -6, -9, 3 \rangle \; \langle -2,1,1 \rangle|}{|\langle -2,1,1 \rangle|} = \frac{6}{\sqrt{6}} = \sqrt{6}$.

Alternatively, the plane has equation $2x - y - z - 14 = 0$, so

$$D = \frac{|2 \; 1 - (-9) - 3 - 14|}{\sqrt{2^2 + (-1)^2 + (-1)^2}} = \frac{|-6|}{\sqrt{6}} = \sqrt{6}.$$

4. Consider the helix defined by $\mathbf{r}(t) = \langle 3\cos t, 3\sin t, 4t\rangle$.

(a) Find the unit tangent vector $\mathbf{T}(t)$.

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -3\sin t, 3\cos t, 4\rangle$.

$v(t) = |\mathbf{v}(t)| = |\mathbf{r}'(t)| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = \sqrt{9+16} = 5$.

$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{v(t)} = \langle -0.6\sin t, 0.6\cos t, 0.8\rangle$.

(b) Find the unit normal vector $\mathbf{N}(t)$.

$\mathbf{T}'(t) = \langle -0.6\cos t, -0.6\sin t, 0\rangle$.

$|\mathbf{T}'(t)| = \sqrt{(-0.6\cos t)^2 + (-0.6\sin t)^2 + 0^2} = 0.6$.

$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \langle -\cos t, -\sin t, 0\rangle$.

(c) Find the curvature $\kappa(t)$.

$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{0.6}{5} = 0.12$.

Alternatively $\mathbf{r}''(t) = \langle -3\cos t, -3\sin t, 0\rangle$ so

$\mathbf{r}'(t) \times \mathbf{r}''(t) = \langle 12\sin t, -12\cos t, 9\rangle$ and

$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{15}{5^3} = 0.12$.

(d) Find the arc length for the quarter turn of the helix with $0 \le t \le \pi/2$.

$L = \int_0^{\pi/2} |\mathbf{r}'(t)|\,dt = \int_0^{\pi/2} 5\,dt = 5t\Big]_0^{\pi/2} = 5\pi/2$.

5. (a) Prove that if $\mathbf{u}(t) = \langle u_1(t), u_2(t), u_3(t)\rangle$ and $\mathbf{v}(t) = \langle v_1(t), v_2(t), v_3(t)\rangle$ are differentiable vector-valued functions and $\varphi(t) = \mathbf{u}(t) \cdot \mathbf{v}(t)$ then $\varphi'(t) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$.

$$\varphi(t) = \mathbf{u}(t) \cdot \mathbf{v}(t) = u_1(t)v_1(t) + u_2(t)v_2(t) + u_3(t)v_3(t) \quad \text{so}$$
$$\varphi'(t) = u_1'(t)v_1(t) + u_1(t)v_1'(t) + u_2'(t)v_2(t) + u_2(t)v_2'(t) + u_3'(t)v_3(t) + u_3(t)v_3'(t) =$$
$$= u_1'(t)v_1(t) + u_2'(t)v_2(t) + u_3'(t)v_3(t) + u_1(t)v_1'(t) + u_2(t)v_2'(t) + u_3(t)v_3'(t) =$$
$$= \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t).$$

(b) Use part (a) to show that if $\mathbf{u}(t) = \langle u_1(t), u_2(t), u_3(t)\rangle$ is a differentiable vector-valued function such that $|\mathbf{u}(t)| = 1$ for all t, then $\mathbf{u}(t)$ must be orthogonal to $\mathbf{u}'(t)$.

Take $\mathbf{v}(t) = \mathbf{u}(t)$ in part (a).
Then $\varphi(t) = \mathbf{u}(t) \cdot \mathbf{u}(t) = |\mathbf{u}(t)|^2 = 1$ so $\varphi'(t) = 0$.
But $\varphi'(t) = \mathbf{u}'(t) \cdot \mathbf{u}(t) + \mathbf{u}(t) \cdot \mathbf{u}'(t) = 2\mathbf{u}(t) \cdot \mathbf{u}'(t)$,
so $\mathbf{u}(t) \cdot \mathbf{u}'(t) = 0$ and $\mathbf{u}(t)$ must be orthogonal to $\mathbf{u}'(t)$.
Notice that the numerical value, 1, of the constant length is irrelevant.
This will be true for any differentiable vector-valued function whose direction may change but whose length is constant.