# Math 251, Calculus III

J. Hebron, Fall 1999

# Final Examination

Friday, Dec 10th, 1999

This is a closed-book, 3-hour exam. No calculators are allowed. Students are allowed one 8.5 by 11 inch formula sheet which may be filled on both sides.

Fill out all the required information on the front of the exam booklet, including your name, student ID number, and signature.

Write all your answers in the exam booklet provided. You may keep the exam questions after you are done. Hand-in only your answer booklet.

This exam has 2 parts. In part A, it is not necessary to show any work. Just the answer will suffice.

In part B, please show all your work.

The total points for each question are shown in square brackets [like this]. The exam is out of a total of [140] marks.

Do not open this booklet until told to do so.

# Part A

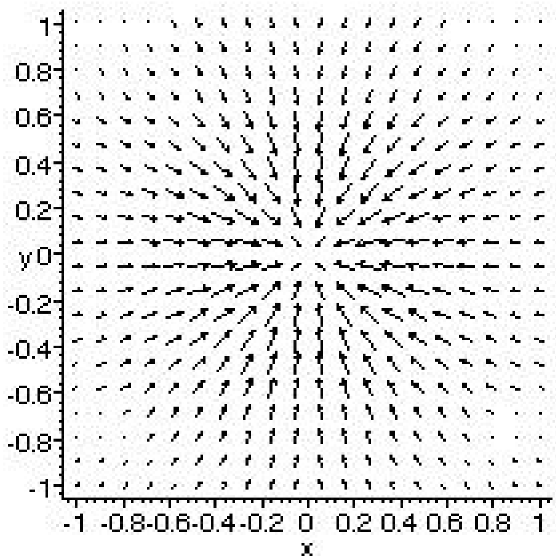
**1.** Here are 5 scalar fields, labeled f1 to f5:

$$f1 = x + y \qquad f2 = x^3 + y \qquad f3 = \frac{3}{1 + 3x^2 + 3y^2}$$

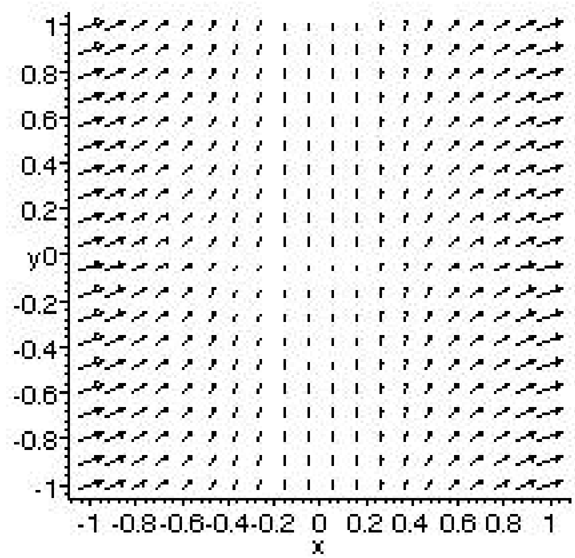
$$f4 = \sin(\pi x)\cos(\pi y) \qquad f5 = -\arctan(x / y)$$

Below, and on the next page are plotted six vector fields, labeled A to F. Five of them are gradient vector fields of the above scalar fields. Match the scalar fields with their gradient field plots. (Eg. f1-D, F2-A, etc.).

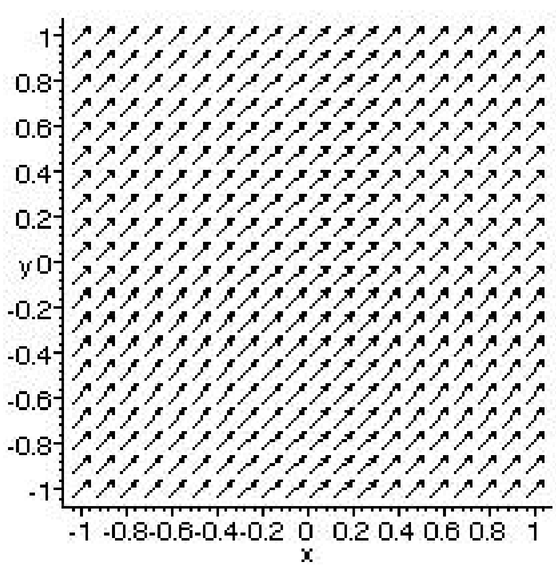
(+2 for each correct match, -2 for each incorrect match, max 10, min 0) [10]



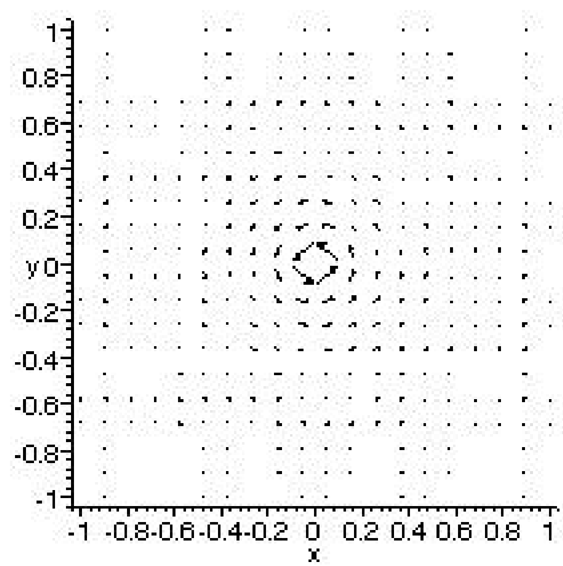
**Vector Field A**



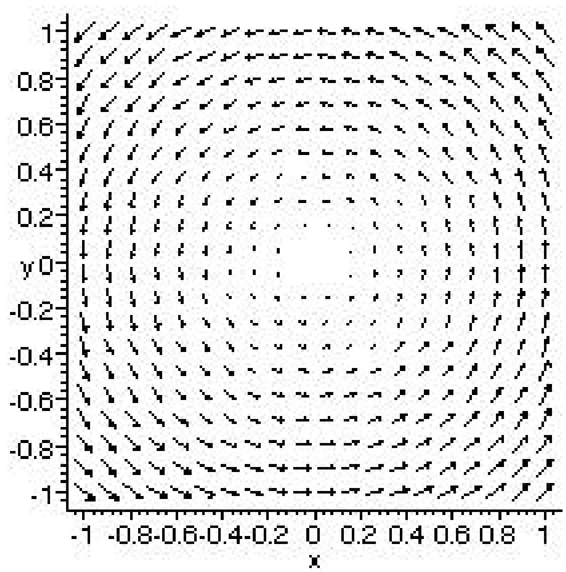
**Vector Field B**



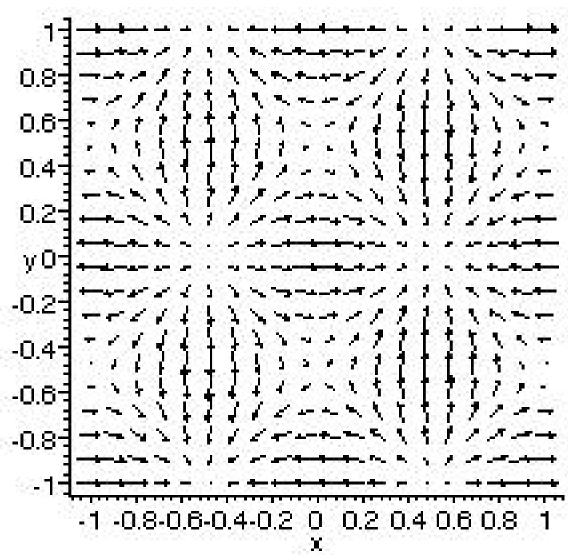
**Vector Field C**



**Vector Field D**




**Vector Field E**




**Vector Field F**

**2.** For each of the plotted vector fields A through F, state whether $\oint_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$ is positive, negative, or zero, where $C$ is a counter-clockwise circular path of radius 1 centered on the origin. [6]

# Part B

**1.** Let $f(x,y) = \dfrac{(x-1)^3 y^2}{(x-1)^4 + y^8}$. Does $\lim\limits_{(x,y)\to(1,0)} f(x,y)$ exist? If so, what is its value? If the limit exists, prove it is unique using the ε–δ method. [10]

**2.** Let a space curve be defined by $\vec{\mathbf{r}} = \langle e^t \cos(t), e^t \sin(t), e^t \rangle$.

- **(a)** Find the velocity vector and its magnitude. [2]
- **(b)** Find the acceleration vector and its magnitude [2]
- **(c)** What is the unit tangent vector $\vec{\mathbf{T}}$ ? [2]
- **(d)** What is the unit normal vector $\vec{\mathbf{N}}$ ? [2]
- **(e)** What is the binormal vector $\vec{\mathbf{B}}$ ? [2]
- **(f)** What is the curvature of the curve? [2]
- **(g)** What is the magnitude of the tangential component of acceleration vector? [2]
- **(h)** What is the magnitude of the normal component of acceleration vector? [2]
- **(i)** What is the arc length from t=0 to t=1? [2]

**3.** Find an equation of the plane that passes through the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$, and is perpendicular to the plane $x + y - 2z = 1$. [10]

**4.** Find the parametric equations for the tangent line to the curve of intersection of the paraboloid $z = x^2 + y^2$ and the ellipsoid $4x^2 + y^2 + z^2 = 9$ at the point (-1,1,2). [10]

**5.** Find and classify all critical points of $f(x,y) = -\left(x^2 - 1\right)^2 - (x^2 y - x - 1)^2$. [10]

**6.** Using the method of Lagrange Multipliers, find the volume of the largest rectangular box with edges parallel to the axes that can be inscribed in the ellipsoid $9x^2 + 36y^2 + 4z^2 = 36$. [10]

**7.** Find the area of one loop of the rose $r = \cos(n\theta)$, where $n$ is a positive integer greater than 1. [10]

**8.** Find the moment of inertia about a diameter of the base of a solid homogeneous hemisphere of radius $a$. [10]

**9.** Evaluate $\iint_R (3x + 4y)\,dA$, where $R$ is the region bounded by the lines $y = x$, $y = x - 2$, $y = -2x$, and $y = 3 - 2x$. Use the transformation $x = \frac{1}{3}(u + v)$, $y = \frac{1}{3}(v - 2u)$. [10]

**10.** Find the work done by the force field $\vec{\mathbf{F}}(x,y) = x\hat{\mathbf{i}} + (y + 2)\hat{\mathbf{j}}$ in moving an object along an arch of the cycloid $\vec{\mathbf{r}}(t) = \left(t - \sin(t)\right)\hat{\mathbf{i}} + \left(1 - \cos(t)\right)\hat{\mathbf{j}}$, $0 \le t \le 2\pi$. [10]

**11.** Evaluate $\int_C \left(y^2 - \cos(\sin(\ln(x^2 + 3x^4 + 1)))\right)dx + \left(3x + y^y \sinh(y)\right)dy$, where $C$ is the boundary of the region enclosed by the parabola $y = x^2$ and the line $y = 4$. [10]

**12.** Evaluate the following:

$$\vec{\nabla} \times \vec{\nabla}\left(\frac{x^3 y\sqrt{z} + e^{xy}\cos(yz) - \tanh^{-1}\left(\frac{x}{y}\right)}{\ln(x^2 + z^2) + 3xz^{4/3}y^{\pi\cos(x)}}\right)$$

[6]