On (k, d)-Colorings and Fractional Nowhere Zero Flows

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Abstract

The concepts of (k, d)-coloring and the star chromatic number, studied by Vince, by Bondy and Hell, and by Zhu are shown to reflect the cographic instance of a wider concept, that of fractional nowhere-zero flows in regular matroids.

1 Introduction

Vince [12] introduced the following generalization of chromatic number.

Definition 1.1 *A* (k, d)-coloring *of a graph G is a function $c : V(G) \to Z_k$ such that for every $xy \in E(G)$, $|c(x) - c(y)| \geq d$. (Here, Z_k denotes the cyclic group of residues mod k, and $|a|$ is the smaller of the two integers a and $k - a$.) The* star chromatic number, *$\chi^*(G)$, is the infimum of k/d over all (k, d)-colorings of G.*

Vince proved, by means of analytical arguments, that this infimum is a minimum (and hence rational). He also proved that for every k, d such that $k/d \geq \chi^*(G)$, there exists a (k, d)-coloring of G. Setting $d = 1$ we have that the chromatic number of G is $\chi(G) = \lceil \chi^*(G) \rceil$. Later, Bondy and Hell [1] improved Vince's result by giving a purely combinatorial proof. A further study and an alternate definition of $\chi^*(G)$ in terms of homomorphisms into intervals of a unit circle appear in [14]. The purpose of this note is to show that (k, d)-colorings are an instance of the more general concept of fractional nowhere-zero flows in regular matroids.

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2 Fractional Flows in Graphs

It is helpful to introduce the notion of fractional flows in graphs before considering the general matroidal case. Let k be a positive integer. A *k-flow* in a graph G is an orientation $\omega(G)$ together with a function $f : E(G) \to \{0, \pm 1, \pm 2, \ldots, \pm(k-1)\}$ such that the net flow $\sum_{vu \in \delta^+(v)} f(vu) - \sum_{uv \in \delta^-(v)} f(uv)$ is zero for each $v \in V(G)$. The *flow index*, $\xi(G)$ is the least k for which G has a *nowhere-zero* k-flow (that is, $f(e) \neq 0$, for all $e \in E(G)$). This parameter has been studied by many authors (see [8] for a thorough review). We generalize this notion with the following.

Definition 2.1 *A (k,d)-*flow *in a graph G is a k-flow $(\omega(G), f)$ such that the range of f is contained in $\{\pm d, \pm(d+1), \ldots, \pm(k-d)\}$. The* star flow index *$\xi^*(G)$ is the infimum of k/d over all (k,d)-flows in G.*

Thus, a $(k,1)$-flow is the same as a nowhere-zero k-flow. We shall see that, analogously to (k,d)-colorings, the infimum in Definition 2.1 is a minimum, and G has a (k,d)-flow whenever $k/d \geq \xi^*(G)$, and thus that $\xi(G) = \lceil \xi^*(G) \rceil$.

It is well known that, in the setting of matroids, vertex colorings and nowhere-zero flows are dual concepts. In particular, if G is a plane graph and H its planar dual, then $\chi(G) = \xi(H)$. We shall see that a similar correspondence holds between the concepts of star chromatic number and star flow index.

3 Flows in Matroids

The proper setting for the study of flows and colorings is that of regular matroids. We assume familiarity with the circuit/cocircuit axioms of basic matroid theory such as in [13]. Let $\mathcal{C}$ ($\mathcal{B}$) denote the {0,1}-valued circuit-element (cocircuit-element) incidence matrix of a matroid M. If M is binary then, over GF(2), we have $\mathcal{C}\mathcal{B}^\top = 0$. An *orientation* $\omega(M)$ of M is a signing ($1 \mapsto \pm 1$) of the elements of $\mathcal{C}$ and $\mathcal{B}$ such that $\mathcal{C}\mathcal{B}^\top = 0$ as rational matrices. It is well known that a binary matroid is orientable if and only if it is *regular*. (See [13] for terminology and a proof.) It is a good exercise to find the relationship between orientations of a graph G and of the graphic matroid $M(G)$. For any circuit C in $\omega(M)$, let C^+ (C^-) denote the set of elements in C which are positively (negatively) oriented with respect to $\omega(M)$. For any cocircuit B in $\omega(M)$, we define B^+ and B^- similarly.

Let Γ be an abelian group. A *Γ-flow* in a regular matroid M is an orientation $\omega(M)$ and a function $f : M \to \Gamma$ such that for every cocircuit B, $\sum_{e \in B^+} f(e) = \sum_{e \in B^-} f(e)$. A flow f is said to be *nowhere-zero* if $f(e) \neq 0$, for all $e \in M$. An *integer flow* is a Γ-flow where $\Gamma = \mathbb{Z}$, the ring of integers. For integers $0 < d < k$, a *(k,d)-flow* is an integer flow with values in the set $\{\pm d, \pm(d+1), \ldots, \pm(k-d)\}$, and a *nowhere-zero k-flow* is a $(k,1)$-flow. As with graphs, the *star flow index* $\xi^*(M)$ is the infimum of k/d over all (k,d)-flows in M, and the *flow index* $\xi(M)$ is the minimum k for which M has a nowhere-zero k-flow.

The following facts about nowhere-zero flows are well known and can be found in [11].

Proposition 3.1 *Let $\omega(M)$ be an oriented regular matroid.*

1. *If M has no coloops (one-element cocircuits) then M has a nowhere-zero k-flow for some integer k, and hence $\xi(M)$ and $\xi^*(M)$ are bounded.*

2. *For any abelian group Γ of order k, M has a nowhere-zero Γ-flow if and only if M has a nowhere-zero k-flow. Furthermore, if f is a Z_k-flow in M, then M has a k-flow f' such that $f'(e) \equiv f(e)$ (mod k), for all $e \in E$.*

Our starting point is the following lemma, due to Hoffman [7].

Lemma 3.2 (Hoffman's Lemma) *Let M be an oriented regular matroid. Given a pair of non-negative rational functions $l, u : M \to \mathbb{Q}$ such that $0 \leq l(e) \leq u(e)$ for $e \in M$, there exists a rational flow $f : M \to \mathbb{Q}$ such that $l(e) \leq f(e) \leq u(e)$ for every $e \in M$ if and only if, for every cocircuit B,*

$$\sum_{e \in B^+} l(e) \leq \sum_{e \in B^-} u(e) \qquad \text{and} \qquad \sum_{e \in B^-} l(e) \leq \sum_{e \in B^+} u(e). \tag{1}$$

Additionally, f can be chosen to be integer valued provided that l and u are integer valued.

In case M is graphic, Hoffman's Lemma is just the Ford-Fulkerson flow theorem [3]. If M is cographic then this is the Potential Differences Existence Theorem of Ghouila-Houri [5]. If $l(e) \equiv l$ and $u(e) \equiv u$ are constant, then (1) becomes:

$$\frac{l}{u} \leq \frac{|B^+|}{|B^-|} \leq \frac{u}{l}.$$

Thus by Hoffman's Lemma with $l \equiv d$ and $u \equiv k - d$, we obtain the following.

Theorem 3.3 *A regular matroid M has a (k, d)-flow if and only if there exists some orientation $\omega(M)$ such that, for any cocircuit B, $d/(k-d) \leq |B^+|/|B^-| \leq (k-d)/d$.*

Corollary 3.4 *The star flow index $\xi^*(M)$ of a regular matroid M is the minimum over all orientations $\omega(M)$ of*

$$1 + \max\{\frac{|B^+|}{|B^-|}, \frac{|B^-|}{|B^+|} : B \text{ is a cocircuit in } \omega(M)\}.$$

$$= \max\{\frac{|B|}{|B^-|}, \frac{|B|}{|B^+|} : B \text{ is a cocircuit in } \omega(M)\}.$$

This maximum is unbounded (and hence $\xi^*(M) := \infty$) if and only if M has a coloop. Putting $d = 1$, we have that for any regular matroid M,

$$\xi(M) = \lceil \xi^*(M) \rceil.$$

A (k, d)-coloring $c : V(G) \to Z_k$ of an (arbitrarily oriented) graph G induces a Z_k-nowhere-zero flow f in the cographic matroid $M^*(G)$ by letting $f(xy) = c(x) - c(y)$ for every arc $xy \in G$. By *2.* of Proposition 3.1, this is equivalent to the existence of an integer flow in $M^*(G)$ whose values range in absolute value between d and $k - d$, that is, a (k, d)-flow in $M^*(G)$. This process can be reversed to obtain a (k, d)-coloring of G from a (k, d)-flow of $M^*(G)$. Thus from Theorem 3.3 we have the following.

Corollary 3.5 *The star chromatic number $\chi^*(G) = \xi^*(M^*(G))$ of a graph G equals*

$$\min_{\omega(G)} \max_{C} \{\frac{|C|}{|C^+|}, \frac{|C|}{|C^-|}\}$$

where the minimum is over all orientations of G and the maximum is over all circuits of G.

We note that the characterization of the (integer) chromatic number $\chi = \lceil \chi^* \rceil$ of a graph via the formula of Corollary 3.5 was proved independently of Hoffman's Lemma by Minty [9].

4 Some Observations Regarding χ^* and ξ^*

(1) Vince's results [12] regarding the star-chromatic number of a graph immediately follow from Corollary 3.5. For example, in the case of the odd circuit C_{2k+1}, at least $k+1$ edges must be similarly oriented in any orientation and hence $\chi^*(C_{2k+1}) = (2k+1)/k = 2 + 1/k$.

(2) Let $c : V \rightarrow Z_k$ be a (k,d)-coloring of a graph $G = (V, E)$. For each $a \in Z_k$ let $I(a)$ denote the independent set $\{v \in V : c(v) \in \{a, a+1, \ldots, a+d-1\} \pmod k\}$. The k independent sets $\{I(a) : a \in Z_k\}$ together cover every vertex exactly d times. Let us call such a collection a (k,d)-*independent cover.* Since any graph with a (k,d)-independent cover has an independent set of size at least $|V|d/k$, it follows that $\alpha(G) \geq |V|/\chi^*(G)$, an improvement on the well-known bound $|V|/\chi(G)$.

Although a (k,d)-coloring always provides a (k,d)-independent cover, the two concepts are not equivalent. Take, for example, the graph G_{10} on 10 vertices and 35 edges obtained by adding all edges joining two disjoint circuits of length five. Each 'side' of G_{10} induces a C_5 subgraph and hence has a $(5,2)$-independent cover. Two such covers, one from each 'side', form a $(10,2)$-independent cover of G_{10}. On the other hand, G_{10} does not admit a $(10,2)$-coloring as $\chi(G_{10}) = 6$.

(3) A *weighted independent cover* is a collection of independent sets, each of which is assigned a positive rational weight, such that the total weight of the sets containing each vertex is at least 1. The *fractional chromatic number* $\chi^f(G)$ is defined to be the least total weight of any weighted independent cover of G. This parameter has been studied in several papers (see [4],[6] for example). As the existence of a (k,d)-independent cover of G implies $\chi^f(G) \leq k/d$ we have the following.

Observation 4.1 *For any graph G, $\chi^f(G) \leq \chi^*(G)$.*

Equality does not always hold here; for instance, $\chi^f(G_{10}) = 5$ while $\chi^*(G_{10}) = 6$. (We leave these for the reader to check!)

(4) Let the graph $G = (V, E_1 \cup E_2)$ be the union two subgraphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$. Obviously, $\chi(G) \leq \chi(G_1)\chi(G_2)$. Such a product formula also holds for the flow index — a fact utilized in Seymour's proof [10] that $\xi(G) \leq 6 = 2 \times 3$ for any 2-edge connected graph G. Unfortunately analogous statements, where χ and ξ are replaced by χ^* and ξ^*, are false. A counterexample for χ^* is provided again by the graph G_{10}; the star chromatic number of the disjoint union of two C_5's is 2.5 and $\chi^*(K_{5,5}) = 2$, whereas $\chi^*(G_{10}) = 6$. Using a similar construction one can find, for any pair of rational numbers $a, b \geq 2$, a graph G consisting of two subgraphs G_1 and G_2, such that $\chi^*(G_1) = a$, $\chi^*(G_2) = b$ and $\chi^*(G) = \lceil a \rceil \lceil b \rceil$. Analogous examples exist for ξ^*.

(5) We finish with an extension of the notion of chromatic number to (general) orientable matroids. As explained in [2], orientable matroids need not be binary (as is tacitly assumed in some works such as [13]). The following definition is more general than — but consistent with — that given in in Section 3. An *orientation* of an arbitrary matroid is a signing $1 \rightarrow \pm 1$ of $\mathcal{C}$ and $\mathcal{B}$ such that, for any row C of $\mathcal{C}$ and any row B of $\mathcal{B}$, if $C_e, B_e \neq 0$ for some $e \in E$, then there exists $f \in E \backslash \{e\}$ such that one of C_eB_e, C_fB_f equals $+1$ and the other equals -1. A matroid is *orientable* if it has at least one orientation. One can use Corollaries 3.4 and 3.5 to *define* $\xi^*(M)$ and $\chi^*(M)$ (and hence $\xi(M)$ and $\chi(M)$) for an arbitrary orientable matroid M. There are several natural questions one might ask. For example, the chromatic number of a (loop-free) orientable matroid of rank r is bounded by the size of its largest circuit, which is at most $r+1$. However, we do not know whether the flow index of a (coloop-free) orientable matroid of bounded rank is bounded. (This is true for regular matroids since their underlying simple matroids have bounded size.)

Two orientations of M are said to belong to the same *reorientation class* if one is obtained from the other by multiplying a corresponding set of columns of $\mathcal{B}$ and $\mathcal{C}$ by -1. Although regular matroids have only one reorientation class, orientable matroids can have many reorientation classes. Winfried Hochstättler has pointed out that it may be more sensible to define ξ^* (and χ^*) for each reorientation class $\psi(M)$ of M by appropriately restricting the minimum in Corollary 3.4.

Definition 4.2 *The star flow index of a reorientation class $\psi(M)$ of an orientable matroid M is given by*

$$\xi^*(\psi(M)) = \min_{\omega\in\psi(M)} \max_B \{\frac{|B|}{|B^+|}, \frac{|B|}{|B^-|}\}.$$

where the maximum is taken over the cocircuits B of M.

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