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On the Bounds of Conway's Thrackles

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Abstract A *thrackle on a surface* X is a graph of size e and order n drawn on X such that every two distinct edges of G meet exactly once either at their common endpoint, or at a proper crossing. An unsolved conjecture of Conway (1969) asserts that $e \leq n$ for every thrackle on a sphere. Until now, the best known bound is $e \leq 1.428n$. By using discharging rules we show that $e \leq 1.4n - 1.4$.

Keywords Thrackle · Generalized thrackle · Crossing number

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1 Introduction

Let $G = (V, E)$ be a finite simple connected graph with n vertices and e edges. A graph can be represented on the plane, with vertices being considered as points or shown as small circles, and edges are simple curves joining the points corresponding to their ends. Such a representing of a graph G is called a *drawing of* G *on the plane*.

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A graph is said to be *planar*, if it can be drawn on the plane so that its edges intersect only at their ends.

In late 1960s, John H. Conway defined a new kind of graph drawing: a thrackle. A *thrackle* of G is a drawing of G on the plane such that every two distinct edges of G either

- share an endpoint, and then they have no other point in common; or
- do not share an endpoint, in which case they meet exactly once at a proper crossing.

A graph that can be drawn as a thrackle is said to be *thrackable*.

Conway's Thrackle Conjecture *Every thrackable graph has at most as many edges as vertices.*

This conjecture was first mentioned by Richard Guy and Douglas R. Woodall in 1969. Woodall [9] introduced the following generalized notion of thrackle which has been used by several authors [3–7].

Generalized Thrackle A graph drawing is a *generalized thrackle* if any two distinct edges meet an odd number of times, either at a common end point, or at a proper crossing.

The Conway's Thrackle Conjecture is very difficult and still remains unresolved. In the past 40 years, many researches have worked on this problem and some progress has been made. Lovász stated that a 4-cycle is not thracklable, and claimed that no thrackable graph contains two vertex disjoint odd cycles. By various methods, Fulek and Pach determined three other similar configurations which are not thrackable, i.e., no thrackable graph contains two 6-cycles where their intersection is a path of length l, $l = 0, 1, 2, 3$. The following two results permit us to invoke properties of planar graphs when studying both bipartite and non-bipartite (generalized) thrackable graphs.

Lemma 1.1 [7] *Every thrackable bipartite graph is planar.*

Lemma 1.2 [4,7] *A bipartite graph can be drawn as a generalized thrackle on the plane if and only if it is planar; a non-bipartite graph can be drawn as a generalized thrackle on the plane if and only if it has a parity* 2*-cell embedding on* $\mathbb{N}_1$.

Lemma 1.2 was then extended by Cairns and Nikolayevsky to orientable sufaces, and they present an analogous theorem for non-bipartite graphs, as shown in the following two theorems.

Theorem 1.3 [3] *A bipartite graph G can be drawn as a generalized thrackle on a closed orientable connected surface $\mathbb{S}_g$ with genus $2g$ if and only if G can be embedded on $\mathbb{S}_g$.*

Theorem 1.4 [4] *A connected non-bipartite graph G can be drawn as a generalized thrackle on an oriented closed surface $\mathbb{S}_g$ if and only if G admits a parity embedding on a non-orientable closed surface $\mathbb{N}_{2g+1}$.*

Applying Euler's Formula [8] is a natural idea that helps us to find upper bounds on $|E|$ of a thrackle. Results stated above make it possible to reduce upper bounds on $|E|$ as in the following four theorems.

Theorem 1.5 [7] *Every thrackle with n vertices has at most $2n - 3$ edges. Every generalized thrackle with n vertices has at most $3n - 4$ edges.*

Based on the same idea, but extended to general surfaces, Cairns and Nikolayevsky [3] stated the theorem below improving the bound in Theorem 1.5.

Theorem 1.6 *Every thrackle with n vertices has at most $3(n-1)/2$ edges. Every generalized thrackle with n vertices has at most $2n - 2$ edges.*

This upper bound was further reduced to $1.428n$ by Fulek and Pach [6], where their proof involved the Conway's Doubling Method (Conway, 1969).

Theorem 1.7 *Every thrackle with n vertices has at most $1.428n$ edges.*

In this paper, we aim to show the following.

Theorem 1.8 *Let G be a thrackable graph with n vertices and e edges. Then $e \le 1.4n - 1.4$.*

2 Proof of Theorem 1.8

Definition 2.1 [6] Given three integers $c', c'' > 2, l \ge -\min(|c'|, |c''|)$, the dumbell $DB(c', c'', l)$ is a simple graph consisting of two distinct cycles of length c' and c'' such that

- $l = 0$, the two cycles share a vertex.
- $l > 0$, the two cycles are connected by a path of length l.
- $-\min(c', c'') < l < 0$, the two cycles share a path of length $-l$.

Cairns and Nikolayevsky pointed out at [4, p., 119] that the Conway's Thrackle Conjecture has been verified for every graph of order at most 11. In particular, we have the following.

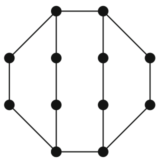
Lemma 2.2 [6] *$DB(6, 6, l)$ for $l \in \{-3, -2, -1, 0\}$ are not thrackable.*

2.1 Discharging Rules

By Lemma 1.1, every bipartite thrackable graph is planar. Let G be a plane embedding of a bipartite thrackable graph with n vertices and m edges. Let F be a face of G on the plane, and $bd(F)$ be its facial boundary. The size of $bd(F)$, that is the length of the facial walk of F, is called the degree of F, denoted by $d(F)$. A k-face is a face of degree k, and a k^+-face is a face of degree at least k. Two faces are adjacent if and only if their boundaries share some common edges.

Lemma 2.3 *Suppose G is not isomorphic to C_6. Then G has no two adjacent distinct 6-faces.*

Fig. 1 An 8-face incident with six bad edges



Proof Let H be a subgraph of G induced by two adjacent distinct 6-faces. Then $n(H) \leq 10$, and $n(H) < e(H)$. By a conclusion in [4, p., 119] that the Conway's Thrackle Conjecture is verified for every graph of order at most 11, H is not thrackable unless H has at most one circuit. But this hypothesis holds only if $H \cong C_6$, a contradiction. □

Since the Conway's Thrackle Conjecture is verified for every graph of order at most 11, every face of degree at most 10 is bounded by a circuit.

An edge $e \in E(G)$ is a *bad edge* if and only if it is incident with a 6-face. The following proposition immediately comes from Lemma 2.3.

Proposition 2.4 *Let $e \in E(G)$ be a bad edge. Then e must be incident with an 8^+-face and a 6-face.*

Lemma 2.5 *Each 8-face is incident with at most six bad edges. Furthermore, if an 8-face is incident with six bad edges, then it must be incident with two distinct 6-faces as shown in Fig. 1.*

Proof Let F be an 8-face of G , and let P be the subgraph induced by a set of edges in $bd(F)$ each of which is incident with a 6-face.

Suppose $|E(P)| \geq 7$. Since each edge in P is incident with a 6-face, by Lemmas 2.2 and 2.3, consecutive edges in P must be incident with the same 6-face. Thus, all edges in P must be incident with exactly one 6-face. Since $|E(P)| \geq 7$, this is impossible. Therefore, $|E(P)| \leq 6$.

Suppose $|E(P)| = 6$. Then the 8-face F must be adjacent with at least two distinct 6-faces of G. Let F' be one of the 6-faces adjacent with F. Let $P_{F'}$ be a subgraph induced by a set of edges in $bd(F)$, where each of them is incident with the 6-face F'. Thus, $P_{F'}$ is a union of finite vertex disjoint paths $p_1, p_2, \ldots, p_k$, where $E(p_j) \subseteq E(P_{F'})$. Let v be an endvertex of a path p_j. Then v is incident with two distinct edges e_1, e_2, where $e_1 \in E(p_j)$ and $e_2 \in E(bd(F)) \setminus E(P_{F'})$. By Lemmas 2.2 and 2.3, e_2 is not incident with any 6-face of G.

If $k \geq 3$, then F must be incident with at least three different edges each of which is not incident with any 6-face of G, implying that $|E(P)| \leq 5$, a contradiction.

Thus, $P_{F'}$ is a single path. Furthermore, there are only three possibilities when $|E(P)| = 6$, as shown below (Fig. 2). Case 2 contains a $DB(6, 6, -2)$, contradicting Lemma 2.2, and Case 3 contains a 4-cycle, which is not thrackable.

Hence, when $|E(P)| = 6$, F must be adjacent with two distinct 6-faces F_1, F_2, which is Case 1 in Fig. 2. □

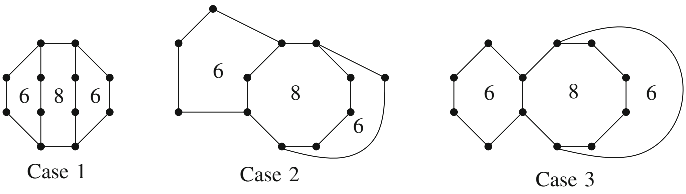


Fig. 2 Three possibilities when $|E(P)| = 6$

By Proposition 2.4, our discharging rules are as follows. Let $w(F) = d(F)$ be the original weight of each face of G, where $d(F)$ is the degree of F.

Discharging Rule Suppose G is not isomorphic to C_6. Let F be a face of G. Let e be a bad edge of G, thus e is incident with an 8^+-face and a 6-face. Transfer $1/6$ across e from the 8^+-face to the 6-face to obtain a new face weighting w^*.

Let r be the number of bad edges in G incident with F. When F is an 8^+-face, the new weight of F is $w^*(F) = w(F) - r/6$. When F is a 6-face, $w^*(F) = 6 + 6/6 = 7$.

If F is an 8-face, then by Lemma 2.5, $w^*(F) \geq 8 - 1 = 7$. If F is a 10^+-face of G, $w^*(F) \geq d(F) - d(F)/6 \geq 50/6 > 7$. Therefore, after applying the discharging rules, $w^* \geq 7$ for each face of G.

Now we are quite close to finishing the proof of Theorem 1.8. However, we need to be a little bit more careful. Notice that during the discussion above, we lost the generality by only discussing bipartite graphs. The next section will correct this.

2.2 Non-bipartite Thrackable Graph on Projective Plane

Let G be a non-bipartite graph. Let $\mathcal{T} : G \to \mathbb{S}_0$ be a thrackle drawing of G on the sphere $\mathbb{S}_0$. The next result is a statement equivalent to Theorem 1.1 in [1].

Lemma 2.6 *Let G be a non-bipartite thrackable graph on the plane. Then there exists an even-faced 2-cell embedding $\mathcal{E} : G \to \mathbb{N}_1$ of G on the projective plane.*

Since $\mathcal{E}$ is even-faced and 2-cell, and G contains no 4-cycle, the discharging rules defined above can be applied to all non-bipartite thrackable graphs after they have been embedded on the projective plane.

2.3 A New Upper Bound on $|E(G)|$

Now we are ready to complete the proof of Theorem 1.8.

Let G be a graph with n vertices and e edges. Suppose $G \cong C_6$. Apparently, $e \leq n$ without applying the discharging rules, which satisfies Theorem 1.8. Thus, we may assume that G is not isomorphic to a C_6.

Let $v \in V(G)$ be a vertex of degree 1. We apply induction to the graph $G - v$, and observe that

$$\frac{e(G-v)}{n(G-v)} = \frac{e-1}{n-1}.$$

Thus

$$\frac{e(G)}{n(G)} \le \max\left(1, \frac{e-1}{n-1}\right) \le \max\left(1, \frac{e(G-v)}{n(G-v)}\right) \le 1.4.$$

So we may also assume that each vertex of G has degree at least 2.

Now suppose G is a vertex minimal counterexample to Theorem 1.8, that is, $e > 1.4n - 1.4$. First we see that G is connected. Otherwise, let G_1 be a component of G and let $G_2 = G - V(G_1)$. Then $|E(G_1)| \le 1.4|V(G_1)| - 1.4$, and $|E(G_2)| \le 1.4|V(G_2)| - 1.4$, which implies that $e \le 1.4n - 2.8$. Suppose v is a cut-vertex of G. Let G_1 and G_2 be two parts of G that v splits G into, that is, $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = v$. Since G is a vertex minimal counterexample, we have

$$\begin{aligned} e &= |E(G_1)| + |E(G_2)| \le 1.4(n - |V(G_1)| + 1) - 1.4 + 1.4|V(G_1)| - 1.4 \\ &= 1.4n - 1.4, \end{aligned}$$

which leads to a contradiction. Hence G is 2-connected.

Then we apply the discharging rules to G. First we consider the case where G is bipartite. By Lemma 1.1, G is even-faced planar. Let F be a face of G. Since 4-cycle is not thrackable, the degree of F is at least 6. Let w^* be as defined in Sect. 2.1.

According to the above discharging rules, $w^*(F) \ge 7$. By Handshaking Lemma ([2]),

$$2e = \sum_F d(F) = \sum_F w(F) = \sum_F w^*(F) \ge 7f, \tag{1}$$

so $f \le 2e/7$. Applying Euler's Formula for the plane ([2]), we obtain

$$2 = n + f - e \le n + \frac{2}{7}e - e = n - \frac{5}{7}e \quad \Rightarrow \quad e \le \frac{7}{5}(n-2) \quad \Rightarrow \quad e \le 1.4n - 2.8.$$

This leads to a contradiction as G is a counterexample to Theorem 1.8.

Now suppose G is non-bipartite. By Lemma 2.6, G is even-faced projective planar. Let F be a face of G. Thus, $w^*(F) \ge 7$, and by Handshaking Lemma, $f \le 2e/7$. By Euler's Formula for projective plane,

$$1 = n + f - e \le n + \frac{2}{7}e - e = n - \frac{5}{7}e \quad \Rightarrow \quad e \le \frac{7}{5}(n-1) \quad \Rightarrow \quad e \le 1.4n - 1.4,$$

which leads to a contradiction as well. Hence, there exists no vertex counterexample to Theorem 1.8. Therefore, $e \le 1.4n - 1.4$. □

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