ON HILBERT BASES OF CUTS

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Abstract. A *Hilbert basis* is a set of vectors $X \subseteq \mathbb{R}^d$ such that the integer cone (semigroup) generated by X is the intersection of the lattice generated by X with the cone generated by X. Let $\mathscr{H}$ be the class of graphs whose set of cuts is a Hilbert basis in $\mathbb{R}^E$ (regarded as $\{0,1\}$-characteristic vectors indexed by edges). We show that $\mathscr{H}$ is not closed under edge deletions, subdivisions, nor 2-sums. Furthermore, no graph having $K_6 \setminus e$ as a minor belongs to $\mathscr{H}$. This corrects an error in [M. Laurent. Hilbert bases of cuts. *Discrete Math.*, 150(1-3):257-279 **(1996)**].

For positive results, we give conditions under which the 2-sum of two graphs produces a member of $\mathscr{H}$. Using these conditions we show that all $K_5^{\perp}$-minor-free graphs are in $\mathscr{H}$, where $K_5^{\perp}$ is the unique 3-connected graph obtained by uncontracting an edge of K_5. We also establish a relationship between edge deletion and subdivision. Namely, if G' is obtained from $G \in \mathscr{H}$ by subdividing e two or more times, then $G \setminus e \in \mathscr{H}$ if and only if $G' \in \mathscr{H}$.

1. Introduction

Let X be a set of vectors in $\mathbb{R}^d$. We define

$$\mathsf{Cone}(X) := \left\{ \sum_{s \in X} c_s s : c_s \in \mathbb{R}_{\geq 0} \right\},$$

$$\mathsf{Lattice}(X) := \left\{ \sum_{s \in X} c_s s : c_s \in \mathbb{Z} \right\},$$

$$\mathsf{IntCone}(X) := \left\{ \sum_{s \in X} c_s s : c_s \in \mathbb{Z}_{\geq 0} \right\}.$$

Definition 1.1. A set of vectors X in $\mathbb{R}^d$ is a *Hilbert basis* if

$$\mathsf{IntCone}(X) = \mathsf{Cone}(X) \cap \mathsf{Lattice}(X).$$

Hilbert bases were introduced by Giles and Pulleyblank [12] as a tool to study total dual integrality. They are also connected to set packing, toric ideals and perfect graphs [19]. Combinatorially defined Hilbert bases have computational consequences, since membership testing is often easier for the cone and the lattice than for the integer cone. This is the case, for example, with edge colouring and the set of perfect matchings of a regular graph [10, 17]. We are interested here in the class of finite graphs whose sets of edge cuts form Hilbert bases.

All graphs here are assumed to be finite and loopless (but may have parallel edges). Let $G = (V, E)$ be a graph. A *circuit* is the edge set of a cycle of G. For $S \subseteq V$, we denote by $\delta(S) = \delta_G(S)$ the set of edges in G having exactly one endpoint in S, and call $\delta(S)$ the *cut* in G generated by S. Regarding each cut $\delta(S)$ as a $\{0,1\}$-characteristic vector in $\mathbb{R}^E$, we define the vector set $\mathcal{B}(G) := \{\delta(S) : S \subseteq V(G)\}$. We define $\mathscr{H}$ to be the class of finite graphs G for which $\mathcal{B}(G)$ forms a Hilbert basis.

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Our aim is to study the class $\mathscr{H}$. We remark that the (matroidal) version of the dual problem was completely solved by Alspach, Goddyn, and Zhang [1], where they show that the set of circuits of a graph G is a Hilbert basis if and only if G does not contain the Peterson graph as a minor. However, the class $\mathscr{H}$ is less well-behaved; we show $\mathscr{H}$ is not closed under edge deletions, subdivisions, nor 2-sums. This contrasts sharply with the analogous problem for the *convex hull* of cuts, where Ohsugi [18] showed that normality is a minor-closed property. We also prove that no graph having $K_6 \setminus e$ as a minor belongs to $\mathscr{H}$. This corrects an error in Laurent [16].

For positive results, we give conditions under which performing 2-sums does yield a graph in $\mathscr{H}$, and use this to show that all $K_5^{\perp}$-minor-free graphs are in $\mathscr{H}$, where $K_5^{\perp}$ is the unique 3-connected single-element uncontraction of K_5.

We also establish a relationship between edge deletion and subdivision; if $G \in \mathscr{H}$ and G' is obtained from G by subdividing an edge e two or more times, then $G \setminus e \in \mathscr{H}$ if and only if $G' \in \mathscr{H}$.

2. Previous Results

In this section, we review some previous results that we will need later.

A *bond* of a graph G is an inclusionwise minimal nonempty cut of G. Every cut of G is a disjoint union of bonds, so the bonds generate the same lattice and cone as $\mathcal{B}(G)$. Indeed, there is a bijection between the bonds of G and the extreme rays of $\mathsf{Cone}(\mathcal{B}(G))$. Gordan [14] (see [20]) showed that for every finite set of vectors X such that $\mathsf{Cone}(X)$ has a vertex, there is a unique minimal set of vectors $\overline{X}$ which generate the same cone and lattice as X, such that $\overline{X}$ is a Hilbert basis. The set $\overline{X}$ is called the *minimal Hilbert basis* for X. The vectors in $\overline{X} \setminus X$ are called *quasi-Hilbert elements*. We shall give specific examples of quasi-Hilbert elements in the next section.

The lattice generated by the cuts of a graph is characterized as a special case of a general statement [13, Proposition 2.4] regarding the cocircuits of a binary matroid having no Fano-minor.

Lemma 2.1. *For every simple graph G and $x \in \mathbb{Z}^{E(G)}$, $x \in \mathsf{Lattice}(\mathcal{B}(G))$ if and only if $\sum_{e\in C} x_e$ is even for each circuit C of G.*

We use the notation $x(C) := \sum_{e\in C} x_e$. The cone generated by $\mathcal{B}(G)$ is very complicated in general. See [2, 8] for a discussion of this. Seymour [21] characterized those graphs (and matroids) for which $\mathsf{Cone}(\mathcal{B}(G))$ is described by a natural family of inequalities called *cycle constraints.*

Lemma 2.2. *For every graph G with no K_5-minor and every $x \in \mathbb{R}^{E(G)}$, $x \in \mathsf{Cone}(\mathcal{B}(G))$ if and only if $x_e \geq 0$ for all $e \in E(G)$; and*

$$x_e \leq x(C \setminus \{e\}),$$

for all circuits C of G and all edges $e \in C$.

Using this result, Fu and Goddyn [11] showed that every K_5-minor-free graph is in $\mathscr{H}$.

Lemma 2.3. *All K_5-minor-free graphs are in $\mathscr{H}$.*

Earlier, Deza [6] [7] had shown that $K_5 \in \mathscr{H}$ as well.

Lemma 2.4. $K_5 \in \mathscr{H}$.

In fact, we now give an explicit description of $\mathsf{Cone}(\mathcal{B}(K_5))$, which we will need later.

Lemma 2.5 ([16])**.** *Let $G = K_5$ with $V(G) = \{1, \dots, 5\}$. The facets of $\mathsf{Cone}(\mathcal{B}(K_5))$ correspond to the following 40 inequalities.*

- *$x_e \leq x(C \setminus \{e\})$ for every 3-circuit C in G and all $e \in C$, and*

- $\sum_{1\le i<j\le 5} b_ib_jx_{ij} \le 0$, *for all ten permutations b of the vector* $(1,1,1,-1,-1)$.

Both sets of inequalities that appear in Lemma 2.5 are examples of *hypermetric inequalities*, introduced in [6]. The constraints corresponding to the permutations of the vector $(1,1,1,-1,-1)$ are also called *pentagonal inequalities*.

The next lemma follows from the observation that $\mathsf{Cone}(\mathcal{B}(G/e))$ is a face of $\mathsf{Cone}(\mathcal{B}(G))$.

Lemma 2.6 ([16])**.** *If $G \in \mathscr{H}$, then $G/e \in \mathscr{H}$ for all $e \in E(G)$.*

Let G_1 and G_2 be two graphs both containing a clique K with n vertices. The *n-clique sum (along K)* of G_1 and G_2 is the graph obtained by gluing G_1 and G_2 along K and keeping the set of edges of exactly one copy of K. On the other hand, the *n-sum (along K)* of G_1 and G_2 is the graph obtained by gluing G_1 and G_2 along K and deleting the edges from both copies of K. To alert the reader of possible confusion, note that what we are calling the *n-clique sum* is called the *n-sum* in Laurent [16].

We denote the n-clique sum and the n-sum of G_1 and G_2 (along K), as $G_1 \oplus_K G_2$ and $G_1 +_K G_2$ respectively. We warn the reader that this notation is non-standard. However, it is important for us to differentiate between $\oplus_K$ and $+_K$.

Lemma 2.7 ([16])**.** *Let G_1 and G_2 be graphs both containing a clique K with at most 3 vertices. If G_1 and G_2 are both in $\mathscr{H}$, then $G_1 \oplus_K G_2 \in \mathscr{H}$.*

3. Negative Results

In this section we exhibit some graphs which are not in $\mathscr{H}$. For graphs with fewer than about 15 edges, membership in $\mathscr{H}$ can be (and were) tested with the aid of software such as *Normaliz* [3, 4] which can recognize Hilbert bases, and compute quasi-Hilbert elements. However, we stress that no proofs in this section rely on such 'black box' computations.

The quasi-Hilbert elements of $\mathcal{B}(K_6)$ have been explicitly computed by Deza, Laburthe, and Laurent [15]. In particular, $K_6 \notin \mathscr{H}$. However, there is an incorrect claim in Laurent [16, Theorem 1.1] that all proper subgraphs of K_6 are in $\mathscr{H}$.

Theorem 3.1. *$K_6 \setminus e \notin \mathscr{H}$. Furthermore, if G contains a $(K_6 \setminus e)$-minor, then $G \notin \mathscr{H}$.*

Proof. To show that $K_6 \setminus e \notin \mathscr{H}$, we exhibit a vector that is in $\mathsf{Cone}(\mathcal{B}(K_6 \setminus e)) \cap \mathsf{Lattice}(\mathcal{B}(K_6 \setminus e))$, but not in $\mathsf{IntCone}(\mathcal{B}(K_6 \setminus e))$. We show such a vector x in Figure 1. The fact that $x \in \mathsf{Lattice}(\mathcal{B}(K_6 \setminus e))$ follows from Lemma 2.1. To see that $x \in \mathsf{Cone}(\mathcal{B}(G))$,

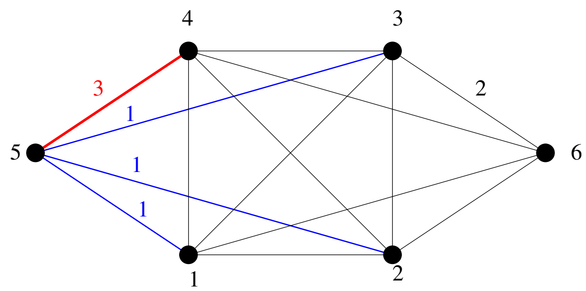

Figure 1. Unlabelled edges all have weight 2.

we observe that $x = \frac{1}{2}\sum_{s\in\mathcal{S}} s$ where

$$\mathcal{S} = \{\delta(S) : S = \{1,4\}, \{2,4\}, \{3,4\}, \{1,4,6\}, \{2,4,6\}, \{3,4,6\}, \{6\}\ \}. \tag{1}$$

Suppose, for a contradiction, that x is a positive integer combination of a set $\mathcal{T}$ of cuts of $K_6 \setminus e$. One easily checks that the seven cuts in $\mathcal{S}$ are linearly independent (for example, consider their intersections with the edge sets $\{i5 : i = 1,2,3\}$ and $\{i6 : i = 1,2,3\}$). Therefore the cuts generate a 7-dimensional subcone $\mathcal{K} \subseteq \mathsf{Cone}(\mathcal{B}(G))$ with codimension

$14-7=7$. We now verify that x lies in the intersection of seven linearly-independent facet-defining inequalities for $\mathsf{Cone}(\mathcal{B}(G))$. These facets correspond to a cycle inequality for each of the triangles $\{1,4,5\}$, $\{2,4,5\}$, $\{3,4,5\}$, $\{1,2,5\}$, $\{2,3,5\}$, $\{1,3,5\}$, and the pentagonal inequality of Lemma 2.5 (applied to K_6-6) with $b=(1,1,1,-1,-1)$. Therefore $\mathcal{K}$ is a facet of $\mathsf{Cone}(\mathcal{B}(G))$. It follows that $\mathcal{T}\subseteq\mathcal{K}$. But we have that $\mathcal{B}(G)\cap\mathcal{K}=\mathcal{S}$, so $\mathcal{T}\subseteq\mathcal{S}$. Thus x is a non-negative integer combination of vectors in $\mathcal{S}$. Since every cut in $\mathcal{S}$ has size 4 or 8, the sum of the entries in x must be a multiple of 4. However the sum of the entries of x is $26\equiv 2 \pmod 4$. This contradiction proves that $x\notin\mathsf{IntCone}(\mathcal{B}(G))$.

For the second part, suppose that $G/C\setminus D\cong K_6\setminus e$. Let $G_1:=G/C$, and G_2 be the graph obtained from G_1 be removing loops, parallel edges, and isolated vertices. Observe that $G_1\in\mathscr{H}$ if and only if $G_2\in\mathscr{H}$. However, by Lemma 2.6, if $G\in\mathscr{H}$, then $G_1\in\mathscr{H}$, so $G_2\in\mathscr{H}$. But, G_2 is either K_6 or $K_6\setminus e$, both of which are not in $\mathscr{H}$. $\square$

Remark. The exact error in [16] occurs in equation (10) on page 270 where $(5,6)$ is erroneously assumed to be an edge of $K_6\setminus(5,6)$.

Let $K_5^{\perp}$ be the graph in Figure 2, with a distinguished edge e.

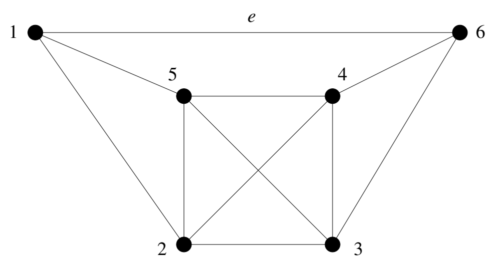

FIGURE 2. The graph $K_5^{\perp}$.

It is shown in [16] that $K_5^{\perp}\in\mathscr{H}$. On the other hand, we claim that the 2-sum $H_{10}^{-}:=K_5^{\perp}+_e K_5^{\perp}$ is not in $\mathscr{H}$.

Lemma 3.2. $H_{10}^{-}\notin\mathscr{H}$.

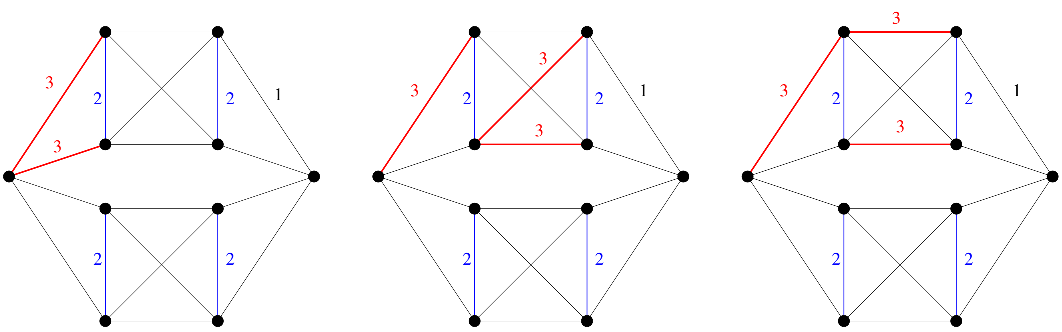

FIGURE 3. Unlabelled edges all have weight 1.

Proof. Let x be the first vector given in Figure 3. We show that $x\in\mathsf{Cone}(\mathcal{B}(H_{10}^{-}))\cap\mathsf{Lattice}(\mathcal{B}(H_{10}^{-}))$, but $x\notin\mathsf{IntCone}(\mathcal{B}(H_{10}^{-}))$. First, $x=\frac{1}{2}\sum_S(\delta(S))$ where S ranges over

$$\{1,4\},\{2,3\},\{1,3\},\{2,4\},\{5,7,9\},\{5,8,10\},\{5,7,10\},\{5,8,9\},$$

with vertices labelled as in Figure 4. Thus, $x\in\mathsf{Cone}(\mathcal{B}(H_{10}^{-}))$. It is also clear that $x\in\mathsf{Lattice}(\mathcal{B}(H_{10}^{-}))$ by Lemma 2.1.

Now x lies on the following facets of $\mathsf{Cone}(\mathcal{B}(H_{10}^{-}))$. These are the cycle inequalities determined by the triangles $\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{3,4,6\},\{5,7,8\},\{7,8,9\}$,

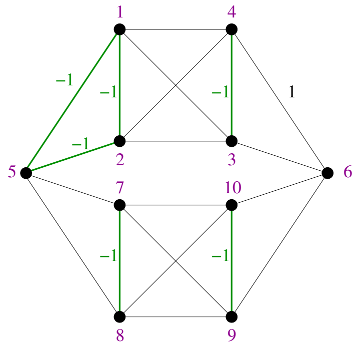

FIGURE 4. The constraint vector v for a facet of H_{10}^{-}. All unlabelled edges have weight 1.

$\{7,8,10\},\{7,9,10\},\{8,9,10\},\{6,9,10\}$ and the inequality $\sum_{e\in E} v_e x_e \leq 0$ where the constraint vector v is shown in Figure 4.

We can check by hand which cuts are tight for all of the above facets. These are precisely the cuts $\delta(S)$ where S ranges over

$$\{1,4\},\{2,3\},\{1,3\},\{2,4\},\{5,7,9\},\{5,8,10\},\{5,7,10\},\{5,8,9\}.$$

One easily checks that $x-\delta(S)\notin \mathsf{Cone}(\mathcal{B}(H_{10}^{-}))$ for all of the above S. Therefore, $x\notin \mathsf{IntCone}(\mathcal{B}(H_{10}^{-}))$, as required. $\square$

We thus have the following corollary.

Corollary 3.3. *The class $\mathscr{H}$ is not closed under 2-sums.*

We now show that $\mathscr{H}$ is also not closed under edge deletions or subdivisions. Let $H_{10} := K_5^{\perp} \oplus_e K_5^{\perp}$, so that $H_{10}^{-} = H_{10}\setminus e$. Let H_{11} be the graph obtained from H_{10} by subdividing e once.

Lemma 3.4. *$H_{10}\in\mathscr{H}$, but $H_{11}\notin\mathscr{H}$.*

Proof. The fact that $H_{10}\in\mathscr{H}$ follows from Lemma 2.7. For the second part, let y be the

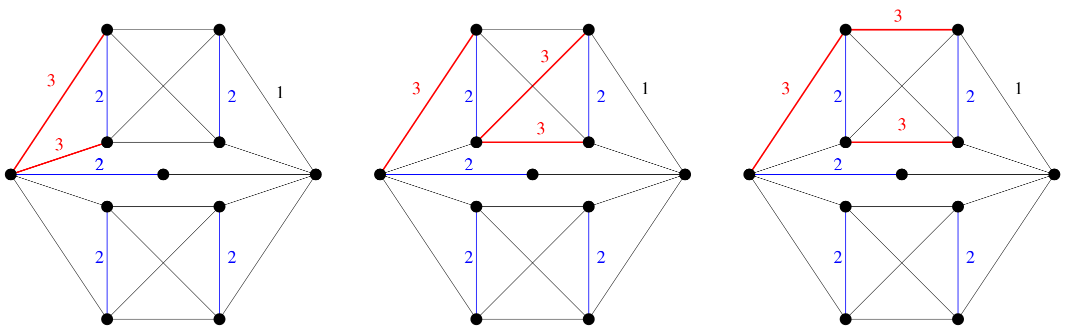

FIGURE 5. Unlabelled edges all have weight 1.

first vector given in Figure 5. Let the vertices of H_{11} be labelled as in Figure 4 with the additional vertex labeled 11. Then $y=\frac{1}{2}\sum_S(\delta(S))$ where S ranges over

$$\{1,4\},\{2,3\},\{1,3\},\{2,4\},\{5,7,9\},\{5,8,10,11\},\{5,7,10,11\},\{5,8,9,11\},\{11\}.$$

Thus, $y\in\mathsf{Cone}(\mathcal{B}(H_{11}))$. It is also clear that $y\in\mathsf{Lattice}(\mathcal{B}(H_{11}))$ by Lemma 2.1.

Consider the subgraph $H_{10}^{-}=H_{11}-11$, and the restriction y' of y to $E(H_{10}^{-})$. If $y\in\mathsf{IntCone}(\mathcal{B}(H_{11}))$, then $y'\in\mathsf{IntCone}(\mathcal{B}(H_{10}^{-}))$. But $y'=x$, where x is the vector from the proof of Lemma 3.2, a contradiction. $\square$

Since $H_{10}^- = H_{10} \setminus e$, we obtain the following corollary.

Corollary 3.5. *The class $\mathscr{H}$ is not closed under edge deletions or subdivisions.*

Remark. The vectors from Figure 3 and Figure 5 are actually the quasi-Hilbert elements of $\mathcal{B}(H_{10}^-)$ and $\mathcal{B}(H_{11})$, respectively. However, we will not need (or show) this.

4. Positive Results

By Corollary 3.3, $\mathscr{H}$ is not closed under 2-sums. On the other hand, in this section, we show that under some additional assumptions, performing a 2-sum does yield a graph in $\mathscr{H}$. We will also give two applications of this theorem.

Before starting, we require a few definitions and lemmas.

Definition 4.1. Let G be a graph with a fixed edge f. Let $x \in \mathbb{R}^{E(G)}$ and let $x(\gamma) \in \mathbb{R}^{E(G)}$ be the vector obtained from x by changing the entry x_f to γ. Define the *feasibility interval* $I(G, x, f)$ for G, x and f to be the (possibly empty) interval $[\gamma_{\mathsf{min}}, \gamma_{\mathsf{max}}]$ such that $x(\gamma) \in \mathsf{Cone}(\mathcal{B}(G))$ if and only if $\gamma \in [\gamma_{\mathsf{min}}, \gamma_{\mathsf{max}}]$.

Definition 4.2. Let G be a graph with a fixed edge f. Define a vector $x \in \mathbb{R}^{E(G)}$ to be *almost in the lattice of G with respect to f* if x restricted to $E(G)\backslash\{f\}$ is in $\mathsf{Lattice}(\mathcal{B}(G\backslash f))$.

Lemma 4.3. *Let G be a graph with a fixed edge f. If $G\backslash f \in \mathscr{H}$, then for every $x \in \mathbb{R}^{E(G)}$ such that x is almost in the lattice with respect to f and $I(G, x, f) \neq \emptyset$, there exists $\gamma \in \mathbb{Z}_{\geq 0}$ such that $x(\gamma) \in \mathsf{IntCone}(\mathcal{B}(G))$.*

Proof. Let f be an edge of G such that $G \setminus f \in \mathscr{H}$, $x \in \mathbb{R}^{E(G)}$ is almost in the lattice of G with respect to f, and $I(G, x, f)$ is non-empty. Let $G' := G \setminus f$ and $x' \in \mathbb{Z}^{E(G')}$ be the restriction of x to $E(G')$. Observe that $x' \in \mathsf{Cone}(\mathcal{B}(G')) \cap \mathsf{Lattice}(\mathcal{B}(G'))$.

Since $G' \in \mathscr{H}$, we also have $x' \in \mathsf{IntCone}(\mathcal{B}(G'))$. Therefore, there exist $\alpha_S \in \mathbb{Z}_{\geq 0}$ such that

$$x' = \sum_{S \subseteq V(G)} \alpha_S \delta_{G'}(S).$$

Define $\mathcal{F} := \{S \subseteq V(G) : f \in \delta_G(S)\}$ and set $\gamma := \sum_{S \in \mathcal{F}} \alpha_S$. The above sum shows that $x(\gamma) \in \mathsf{IntCone}(\mathcal{B}(G))$, as required. □

Definition 4.4. Let G be a graph with a fixed edge f. We say that G has the *lattice endpoint property (with respect to f)* if for all $x \in \mathbb{R}^{E(G)}$ that is almost in the lattice with respect to f and for each endpoint γ of $I(G, x, f)$, we have $\gamma = 0$ or $x(\gamma) \in \mathsf{Lattice}(\mathcal{B}(G))$.

The next lemma will be useful for verifying the lattice endpoint property.

Lemma 4.5. *Let G be a graph, $f \in E(G)$, and $x \in \mathbb{R}^{E(G)}$ be almost in the lattice with respect to f. If there is a circuit C such that $f \in C$ and $x(C)$ is even, then $x \in \mathsf{Lattice}(G)$.*

Proof. Suppose not. Then there is a circuit C' containing f such that $x(C')$ is odd. But now,

$$x(C \Delta C') = x(C) + x(C') - 2x(C \cap C') \equiv 1 \pmod 2.$$

Thus $C \Delta C'$ contains a circuit C'' such that $x(C'')$ is odd. Since $f \notin C''$, this contradicts the fact that $x \in \mathbb{R}^{E(G)}$ is almost in the lattice with respect to f. □

Lemma 4.6. *All simple K_5-minor-free graphs have the lattice endpoint property with respect to every edge.*

Proof. Let G be a simple K_5-minor-free graph, $f \in E(G)$, $x \in \mathbb{R}^{E(G)}$ be almost in the lattice with respect to f, and γ be an endpoint of $I(G, x, f)$. Since γ is an endpoint, there is a constraint involving x_f for which $x(\gamma)$ is tight for. By Lemma 2.2, such a constraint must be a non-negativity constraint or a cycle constraint. If it is a non-negatively constraint, then we have $\gamma = 0$, as required. So, we may assume that $x(\gamma)$ satisfies some cycle constraint with equality. In particular, this implies that there is a circuit C of G such that $f \in C$ and $x(C)$ is even. By Lemma 4.5, $x(\gamma) \in \mathsf{Lattice}(G)$, as required. $\square$

Lemma 4.7. *K_5 has the lattice endpoint property with respect to every edge.*

Proof. Let $f \in E(K_5)$ and $x \in \mathbb{R}^{E(K_5)}$ be almost in the lattice with respect to f, and γ be an endpoint of $I(K_5, x, f)$. As in Lemma 4.6, if $x(\gamma)$ is tight for a non-negativity constraint or a cycle constraint we are done. Thus, by Lemma 2.5 we may assume that $x(\gamma)$ is tight for a pentagonal inequality determined by a permutation of the vector $b = (1, 1, 1, -1, -1)$. In particular, $x(\gamma)$ is integer-valued and $x(\gamma)(E(K_5))$ is even. The edges of K_5 may be partitioned into two circuits C, C' and we may assume $f \in C$. Since x is almost in the lattice with respect to f, we have that $x(\gamma)(C')$ is even. It follows that

$$x(\gamma)(C) = x(\gamma)(E(K_5)) - x(\gamma)(C') \equiv 0 - 0 \pmod{2}.$$

But now $x(\gamma) \in \mathsf{Lattice}(G)$ by Lemma 4.5, and we are done. $\square$

We can now state and prove the main result of this section.

Theorem 4.8. *Let G_1 and G_2 be graphs such that $E(G_1) \cap E(G_2) = f$ and both $G_1 \setminus f$ and $G_2 \setminus f$ are connected. If the four graphs G_1, G_2, $G_1 \setminus f$, $G_2 \setminus f$ are in $\mathscr{H}$, and G_1 has the lattice endpoint property with respect to f, then $G_1 +_f G_2 \in \mathscr{H}$.*

Proof. Let $G := G_1 +_f G_2$ with $f := uv$. Suppose $x \in \mathsf{Cone}(\mathcal{B}(G)) \cap \mathsf{Lattice}(\mathcal{B}(G))$. For $\gamma \in \mathbb{R}$ and $i \in \{1, 2\}$, we define $x_i(\gamma) \in \mathbb{R}^{E(G_i)}$ as $x_i(\gamma)_e := x_e$ if $e \neq f$ and $x_i(\gamma)_f := \gamma$.

Claim. *For all γ, we have $x_1(\gamma) \in \mathsf{Lattice}(\mathcal{B}(G_1))$ if and only if $x_2(\gamma) \in \mathsf{Lattice}(\mathcal{B}(G_2))$.*

Proof. Suppose, without loss of generality, that $x_1(\gamma) \in \mathsf{Lattice}(\mathcal{B}(G_1))$ and $x_2(\gamma) \notin \mathsf{Lattice}(\mathcal{B}(G_2))$. Then γ is an integer, and G_2 contains a circuit C_2 with $x_2(\gamma)(C_2)$ odd. Since $x \in \mathsf{Lattice}(\mathcal{B}(G))$, we have $f \in C_2$. Because $G_1 \setminus f$ is connected, there is a circuit C_1 in G_1 with $f \in C_1$. We have that $x_1(\gamma)(C_1)$ is an even integer, since $x_1(\gamma) \in \mathsf{Lattice}(\mathcal{B}(G_1))$. Now $C_1 \triangle C_2$ is a circuit in G with

$$x(C_1 \triangle C_2) = x_1(\gamma)(C_1) + x_2(\gamma)(C_2) - 2\gamma \equiv 1 \pmod{2},$$

contradicting $x \in \mathsf{Lattice}(\mathcal{B}(G))$ and proving the claim. $\square$

Since $x \in \mathsf{Cone}(\mathcal{B}(G))$, there exist non-negative coefficients β_S such that

$$x = \sum_{S \subseteq V(G)} \beta_S \delta_G(S).$$

Let $\mathcal{F} := \{S \subseteq V(G) : |S \cap \{u, v\}| = 1\}$ and define $\gamma' := \sum_{S \in \mathcal{F}} \beta_S$. Consider the intervals $I_1 := I(G_1, x_1(\gamma'), f)$ and $I_2 := I(G_2, x_2(\gamma'), f)$. Note that $\gamma' \in I_1 \cap I_2 \neq \emptyset$.

Claim. *There exists $\gamma \in I_1 \cap I_2$ such that $x_1(\gamma) \in \mathsf{Lattice}(\mathcal{B}(G_1))$ and $x_2(\gamma) \in \mathsf{Lattice}(\mathcal{B}(G_2))$.*

Proof. By Lemma 4.3, there exists $\gamma_1 \in I_1$ such that $x_1(\gamma_1) \in \mathsf{IntCone}(\mathcal{B}(G_1))$. If $I_1 \subseteq I_2$, we have $\gamma_1 \in I_2$. By the previous claim, $x_2(\gamma_1) \in \mathsf{Lattice}(\mathcal{B}(G_2))$ and the claim is proved with $\gamma = \gamma_1$. Thus, we may assume that $I_1 \nsubseteq I_2$ and similarly, that $I_2 \nsubseteq I_1$. Since $I_1 \cap I_2 \neq \emptyset$, there exists a non-zero endpoint γ_2 of I_1 such that $\gamma_2 \in I_2$. As G_1 has the lattice endpoint property with respect to f, it follows that $x_1(\gamma_2) \in \mathsf{Lattice}(\mathcal{B}(G_1))$. By the previous claim, $x_2(\gamma_2) \in \mathsf{Lattice}(\mathcal{B}(G_2))$, so the claim is proved with $\gamma = \gamma_2$. $\square$

Let γ be as above. Since $G_i \in \mathscr{H}$, we have $x_i(\gamma) \in \mathsf{IntCone}(\mathcal{B}(G_i))$. Thus, there exists a sequence $A_1, \ldots, A_j$ of subsets of $V(G_1)$ and a sequence of subsets $B_1, \ldots, B_k$ of $V(G_2)$ such that $x_1(\gamma) = \sum_{i=1}^{j} \delta_{G_1}(A_i)$ and $x_2(\gamma) = \sum_{i=1}^{k} \delta_{G_2}(B_i)$. By taking complements if necessary, we may assume for each $i \in [j]$, $A_i \cap \{u, v\} = \emptyset$ or $A_i \cap \{u, v\} = \{u\}$ and that for all $i \in [k]$, $B_i \cap \{u,v\} = \emptyset$ or $B_i \cap \{u,v\} = \{u\}$. Therefore, by re-indexing, we may assume $A_i \cap \{u,v\} = \{u\}$ for $i \in \{1,\dots,\gamma\}$ and $A_i \cap \{u,v\} = \emptyset$ for $i \in \{\gamma+1,\dots,j\}$. Similarly, $B_i \cap \{u,v\} = \{u\}$ for $i \in \{1,\dots,\gamma\}$ and $B_i \cap \{u,v\} = \emptyset$ for $i \in \{\gamma+1,\dots,k\}$. Observe that $\delta_{G_1}(A_i) = \delta_G(A_i)$ for $i > \delta$ and $\delta_{G_2}(B_i) = \delta_G(B_i)$ for $i > \delta$. Therefore,

$$x = \sum_{i=1}^{\gamma} \delta_G(A_i \cup B_i) + \sum_{i=\gamma+1}^{j} \delta_G(A_i) + \sum_{i=\gamma+1}^{k} \delta_G(B_i).$$

So, $x \in \mathsf{IntCone}(\mathcal{B}(G))$, as required. $\square$

Using Theorem 4.8, we obtain the following connection between subdivision and deletion.

Theorem 4.9. *Let $G \in \mathscr{H}$ and $f \in E(G)$. Then $G \setminus f \in \mathscr{H}$ if and only if $G +_f C_n \in \mathscr{H}$ for all $n \geq 4$.*

Proof. Note that $G +_f C_4$ is just the graph obtained from G by subdividing f twice. We may clearly assume that $G \setminus f$ is connected. The forward implication then follows directly from Theorem 4.8.

For the converse, assume $G +_f C_n \in \mathscr{H}$ and let u and v be the ends of f. Now if $G \setminus f \notin \mathscr{H}$, then there exists

$$x \in \mathsf{Cone}(\mathcal{B}(G \setminus f)) \cap \mathsf{Lattice}(\mathcal{B}(G \setminus f)) \setminus \mathsf{IntCone}(\mathcal{B}(G \setminus f)).$$

Since $x \in \mathsf{Cone}(G \setminus f)$, we have that $I(G,x,f) \neq \emptyset$. Now, as $x \in \mathsf{Lattice}(\mathcal{B}(G \setminus f))$, it follows that the parity of $x(P)$ is the same for all paths P in $G \setminus f$ between u and v. Define $y \in \mathbb{Z}_{\geq 0}^{E(C_n)\setminus f}$ by setting $y_e := a$ for all e. Let y' be a vector obtained from y by changing a single entry from a to $a+1$. Note that $I(C_n, y, f) = [0,(n-1)a]$, and since $n \geq 4$, $I(C_n, y', f) = [0,(n-1)a+1]$. Let $z \in \mathbb{Z}_{\geq 0}^{E(G+_f C_n)}$ be the concatenation of x and y and let $z' \in \mathbb{Z}_{\geq 0}^{E(G+_f C_n)}$ be the concatenation x and y'.

By choosing a sufficiently large, we either have $z \in \mathsf{Cone}(\mathcal{B}(G +_f C_n)) \cap \mathsf{Lattice}(\mathcal{B}(G +_f C_n)$ or $z' \in \mathsf{Cone}(\mathcal{B}(G +_f C_n) \cap \mathsf{Lattice}(\mathcal{B}(G +_f C_n))$. However, neither z nor z' belong to $\mathsf{IntCone}(\mathcal{B}(G +_f C_n))$, since when restricted to $E(G \setminus f)$, they are both equal to x. This contradicts $G +_f C_n \in \mathscr{H}$. $\square$

Note that one direction of the above proof breaks down if f is subdivided only once, but the following conjecture may still be true.

Conjecture 4.10. *Let $G \in \mathscr{H}$ and $f \in E(G)$. Then $G \setminus f \in \mathscr{H}$ if and only if $G +_f C_3 \in \mathscr{H}$.*

For our second application of Theorem 4.8, we show that all $K_5^{\perp}$-minor-free graphs are in $\mathscr{H}$. First we require the following well-known lemma. See [9], for a proof.

Lemma 4.11. *If G is a 3-connected $K_5^{\perp}$-minor-free graph, then G is K_5-minor-free or $G \cong K_5$.*

Theorem 4.12. *All $K_5^{\perp}$-minor-free graphs are in $\mathscr{H}$.*

Proof. Let G be a counterexample with $|V(G)| + |E(G)|$ minimum. Hence, G is simple and by Lemma 2.7, G is also 2-connected. If G is 3-connected, then by Lemma 4.11, G is K_5-minor-free or $G \cong K_5$. In either case, $G \in \mathscr{H}$ by Lemma 2.4 or Lemma 2.3. Thus, $G = G_1 +_f G_2$ or $G = G_1 \oplus_f G_2$ for some G_1 and G_2 with a common edge f and $|E(G_1)|, |E(G_2)| \geq 3$. The latter is impossible by Lemma 2.7, so $G = G_1 +_f G_2$. Among all possible such choices, choose G_1 and G_2 so that $|E(G_1)|$ is minimum. Thus, G_1 is 3-connected or $G_1 \cong K_3$. By Lemma 4.11, G_1 is K_5-minor-free or $G_1 \cong K_5$. In either case, G_1 has the lattice endpoint property with respect to f by Lemma 4.7 or Lemma 4.6.

Moreover, since G is 2-connected, $G_1 \setminus f$ and $G_2 \setminus f$ are both connected. Thus, G_1 and G_2 are both minors of G, and are hence $K_5^{\perp}$-minor-free. Finally, by minimality, all four of the graphs G_1, G_2, $G_1 \setminus f$ and $G_2 \setminus f$ belong to $\mathscr{H}$. Therefore, by Theorem 4.8, $G \in \mathscr{H}$. $\square$

Remark. It is also claimed in Laurent [16] that all $K_5^{\perp}$-minor-free graphs are in $\mathscr{H}$. However, as far as we can see, the proof given (on page 260) appears to confuse k-sums and k-clique sums, and is thus incomplete. Indeed, by Corollary 3.3, we now know that the class $\mathscr{H}$ is *not* closed under 2-sums. Therefore, we believe a different approach (such as the one above) is needed.

5. Open Problems

Note that it is a bit of a curiosity that we do not know $\mathsf{Cone}(G)$ explicitly, when G is $K_5^{\perp}$-minor-free. This appears to be a rare phenomenon. Typically, it is necessary to know $\mathsf{Cone}(\mathcal{B}(G))$ to show that $G \in \mathscr{H}$. We thus have the following natural open problem.

Problem 5.1. *Give an explicit description of* $\mathsf{Cone}(\mathcal{B}(G))$*, when* G *is* $K_5^{\perp}$*-minor-free.*

Next, observe that by Theorem 4.12 and Theorem 3.1, all $K_5^{\perp}$-minor-free graphs are in $\mathscr{H}$, while all graphs with a $(K_6 \setminus e)$-minor are *not* in $\mathscr{H}$. There are still many graphs that are not covered by these two theorems. One such class of graphs are the uncontractions of K_5. In [5], it is shown that all uncontractions of K_5 are in fact in $\mathscr{H}$. However, the proof in [5] is computer assisted. Namely, there are 22 (non-isomorphic) 3-connected uncontractions of K_5, and it is verified by computer that each of these graphs is in $\mathscr{H}$. The general case then follows easily from Lemma 2.3 and Theorem 4.9.

It would be quite interesting to obtain this result without computer aid.

Problem 5.2. *Give a human proof that all uncontractions of* K_5 *are in* $\mathscr{H}$.

Indeed, it turns out that $\mathsf{Cone}(\mathcal{B}(G))$ has a very simple and beautiful description when G is a 3-connected uncontraction of K_5, see [5, Theorem 2.3.3]. Unfortunately, this characterization was also obtained by computer.

Problem 5.3. *Give a human proof that if* G *is a 3-connected uncontraction of* K_5*, then the description of* $\mathsf{Cone}(\mathcal{B}(G))$ *given in* [5] *is correct.*

Finally, by Corollary 3.5, the class $\mathscr{H}$ is not minor-closed. Thus, $\mathscr{H}$ does not have a forbidden-minor characterization, but it may still be possible to give some alternate characterization of $\mathscr{H}$.

Problem 5.4. *Characterize all graphs that are in* $\mathscr{H}$.

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