JOURNAL OF COMPUTATIONAL PHYSICS **125,** 403–414 (1996)
ARTICLE NO. 0102

Integral Equation Methods for Stokes Flow and Isotropic Elasticity in the Plane

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Received July 17, 1995

We present a class of integral equation methods for the solution of biharmonic boundary value problems, with applications to two-dimensional Stokes flow and isotropic elasticity. The domains may be multiply-connected and finite, infinite or semi-infinite in extent. Our analytic formulation is based on complex variables, and our fast multipole-based iterative solution procedure requires $O(N)$ operations, where N is the number of nodes in the discretization of the boundary. The performance of the methods is illustrated with several large-scale numerical examples. © 1996 Academic Press, Inc.

1. INTRODUCTION

During the last century, a variety of integral equation formulations have been proposed for the problems of Stokes flow and isotropic elasticity. Kolosov, Mikhlin, Muskhelishvili, and Sherman, for example, have developed an extensive theory based on complex variables [8, 15, 17, 19, 25] which provides the analytic foundation for our discussion. For an alternative approach, using fundamental solutions expressed in terms of primitive variables, we refer the reader to the literature [8, 10, 18, 22]. It is well known that two-dimensional Stokes flow and plane elasticity are closely related. Both can be expressed as biharmonic equations for a scalar potential function, although the boundary conditions differ.

In this paper, we present a collection of integral equation methods for biharmonic boundary value problems in multiply-connected domains, which may be finite, infinite, or semi-infinite (wall-bounded) in extent. Our starting point is the classical Sherman–Lauricella equation, which was derived in order to solve problems of elasticity. We have extended this theory to be able to solve problems of Stokes flow. With N points in the discretization of the boundary, direct inversion of the resulting linear systems requires $O(N^3)$ operations. Most iterative methods, on the other hand, require only $O(N^2)$ work and do not require storage of a dense matrix [13, 18, 22, 23]. Our algorithms are also based on iteration, but require only $O(N)$ operations, since they use a version of the fast multipole method [1, 5, 6, 23] to compute matrix–vector products. Earlier fast multipole-accelerated schemes for biharmonic problems [3] were limited to simply-connected interior domains.

We begin, in the next section, with the biharmonic formulation of three typical target problems. In Section 3, we briefly review the relevant complex variable theory and describe the Sherman–Lauricella integral equations. We also discuss the connection between our approach and the "completed double layer method" of Kim, Karrila, Miranda, and Power [9, 18, 21]. The latter methods, however, have not been coupled with modern fast algorithms in order to efficiently solve large-scale problems.

Section 4 contains a detailed description of the discrete algorithm, and Section 5 contains several numerical examples.

2. THE BIHARMONIC POTENTIAL

Many problems in plane strain involve a model in which one can assume that the elastic displacements of a long cylindrical body occur only in planes parallel to the (x, y) plane and that the components of displacement are independent of z. To describe the governing partial differential equations, we let u and v be the components of displacement and σ_x, σ_y, and τ_{xy} be the components of stress. To fix notation, let us consider a finite domain D with boundary Γ which is $(M + 1)$-ply connected. The outer boundary of D will be denoted by Γ_0 and the interior contours by $\Gamma_1, \Gamma_2, \ldots \Gamma_M$ (Fig. 1). In the absence of body forces [17, 19, 25], the equations of equilibrium are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0. \tag{1}$$

These are coupled with the stress–strain relations,

[1] Supported by the Applied Mathematical Sciences Program of the U.S. Department of Energy under Contract DEFGO288ER25053, by a NSF Presidential Young Investigator Award, and by a Packard Foundation Fellowship.

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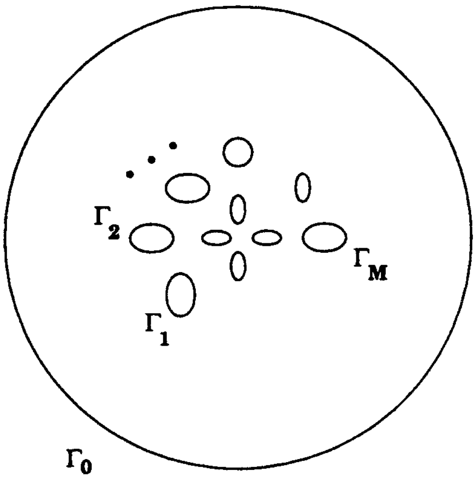

FIG. 1. A bounded multiply-connected domain D. The outer boundary is denoted by Γ_0 and the interior contours by $\Gamma_1, ..., \Gamma_M$.

$$\sigma_x = (\lambda + 2\mu)\frac{\partial u}{\partial x} + \lambda\frac{\partial v}{\partial y}, \quad \sigma_y = \lambda\frac{\partial u}{\partial x} + (\lambda + 2\mu)\frac{\partial v}{\partial y}$$
$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \tag{2}$$

where λ and μ are the Lamé coefficients and appropriate boundary conditions. The "first fundamental problem" in plane elasticity [15] is to find the state of elastic equilibrium when the displacements are specified on Γ:

$$u(\mathbf{t}) = \frac{1}{2\mu}f_1(\mathbf{t}), \quad v(\mathbf{t}) = \frac{1}{2\mu}f_2(\mathbf{t}), \quad \mathbf{t} \in \Gamma. \tag{3}$$

We will refer to this as the "displacement problem."

The "second fundamental problem" in plane elasticity is to find the state of elastic equilibrium when given external stresses are applied to Γ. We will refer to this as the "stress problem." If X, Y are the components of the applied force, then the appropriate boundary conditions are

$$\mathbf{s} \cdot (-\tau_{xy}, \sigma_x) = X(\mathbf{t}), \quad \mathbf{s} \cdot (\sigma_y, -\tau_{xy}) = -Y(\mathbf{t}), \quad \mathbf{t} \in \Gamma, \tag{4}$$

where $\mathbf{s}$ is the unit tangent vector to Γ at $\mathbf{t}$.

The governing equations (1) and (2) can be compactly expressed in terms of a single scalar function $W(x, y)$, known as Airy's stress function. Equation (1) is automatically satisfied by setting

$$\sigma_x = \frac{\partial^2 W}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 W}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 W}{\partial x \partial y},$$

and substituting the above definitions into the stress–strain relations (2) yields the biharmonic equation $\Delta^2 W = \Delta(\Delta W) = 0$. The boundary conditions for the stress problem (4) then take the form

$$\frac{\partial}{\partial s}\left(\frac{\partial W}{\partial x}\right)(\mathbf{t}) = -Y(\mathbf{t}), \quad \frac{\partial}{\partial s}\left(\frac{\partial W}{\partial y}\right)(\mathbf{t}) = X(\mathbf{t}), \quad \mathbf{t} \in \Gamma, \tag{5}$$

where $\partial/\partial s$ denotes the tangential derivative. Integrating these boundary conditions, we can write the stress problem as

$$\Delta^2 W(\mathbf{x}) = 0 \qquad \mathbf{x} \in D$$
$$\left.\begin{aligned} \frac{\partial W}{\partial x}(\mathbf{t}) &= -\int_{\Gamma_k} Y\,ds = g_1(\mathbf{t}) + A_1(k), \\ \frac{\partial W}{\partial y}(\mathbf{t}) &= \int_{\Gamma_k} X\,ds = g_2(\mathbf{t}) + A_2(k), \end{aligned}\right\} \quad \mathbf{t} \in \Gamma_k. \tag{6}$$

The constants of integration $A_1(k)$ and $A_2(k)$ may be fixed arbitrarily on one boundary, say by setting $A_1(0) = A_2(0) = 0$. For multiply-connected domains, however, the constants of integration may be different on each contour, and their values are obtained as part of the solution process. In fact, they are determined by the condition that the displacements be single-valued functions in D. The boundary conditions for the displacement problem are not easily expressed in term of the scalar stress function W. On the other hand, they are easily expressed in terms of the complex variable theory developed below.

Our third target problem is that of slow viscous flow, for which the Navier–Stokes equations can be approximated by the linear Stokes equations

$$\nu\,\Delta u = \frac{1}{\rho}\frac{\partial p}{\partial x}, \quad \nu\,\Delta v = \frac{1}{\rho}\frac{\partial p}{\partial y} \tag{7}$$

$$u_x + v_y = 0, \tag{8}$$

where u and v are the components of velocity, ρ is the density, ν is the viscosity, and p is the pressure. An additional physical quantity of interest is the vorticity

$$\zeta = u_y - v_x.$$

We will restrict our attention to problems where the velocity is given on the boundary Γ:

$$u = h_2(\mathbf{t}), \quad v = -h_1(\mathbf{t}), \quad \mathbf{t} \in \Gamma. \tag{9}$$

As in the case of the elasticity, the governing equations can be simplified by introducing a stream function $W(x, y)$ which satisfies the relations

$$u = \frac{\partial W}{\partial y}, \quad v = -\frac{\partial W}{\partial x}, \quad \zeta = \Delta W.$$

In this way, Eq. (8) is automatically satisfied, and Eq. (7), together with the boundary conditions (9), yields

$$\begin{aligned} \Delta^2 W(\mathbf{x}) &= 0, && \mathbf{x} \in D \\ \frac{\partial W}{\partial x}(\mathbf{t}) = h_1(\mathbf{t}), \quad & \frac{\partial W}{\partial y}(\mathbf{t}) = h_2(\mathbf{t}), && \mathbf{t} \in \Gamma. \end{aligned} \tag{10}$$

This is sometimes referred to as the "third fundamental problem."

Infinite and simi-infinite domain considerations for all three problems will be postponed until Sections 3.5 and 3.6.

3. THE SHERMAN–LAURICELLA REPRESENTATION

Following the discussion of Mikhlin and others [15, 17, 19], we note that any plane biharmonic function $W(x, y)$ can be expressed by Goursat's formula as

$$W(x, y) = \mathrm{Re}(\bar{z}\phi(z) + \chi(z)),$$

where ϕ and χ are analytic functions of the complex variable $z = x + iy$, and $\mathrm{Re}(f)$ denotes the real part of the complex-valued function f. The functions $\phi(z)$ and $\psi(z) = \chi'(z)$ are known as Goursat functions. A simple calculation leads to Muskhelishvili's formula

$$\frac{\partial W}{\partial x} + i\frac{\partial W}{\partial y} = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}. \tag{11}$$

This provides an expression for the integrated components of stress in elasticity and the velocity in Stokes flow. Other physical quantities of interest can be expressed in terms of the Goursat functions. In elasticity, the displacements and stresses are found to be given by

$$\begin{aligned} 2\mu(u + iv) &= \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}, \\ \sigma_x + \sigma_y &= 4\,\mathrm{Re}(\phi'(z)), \end{aligned} \tag{12}$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2(\bar{z}\phi''(z) + \psi'(z)), \tag{13}$$

where

$$\kappa = \frac{\lambda + 3\mu}{\lambda + \mu} > 1,$$

while in Stokes flow,

$$\zeta + \frac{i}{\nu}\,p = 4\phi'(z). \tag{14}$$

Muskhelishvili's formula (11) and the expression for the displacements (12) allow us to reduce the three fundamental problems into problems in analytic function theory, namely that of finding ϕ and ψ which satisfy appropriate conditions on the boundary Γ. The boundary conditions for the displacement, stress, and Stokes problems are, respectively,

$$\kappa\phi(t) - t\overline{\phi'(t)} - \overline{\psi(t)} = f_1(t) + if_2(t) \tag{15}$$

$$\phi(t) + t\overline{\phi'(t)} + \overline{\psi(t)} = g_1(t) + ig_2(t) + A(k) \tag{16}$$

$$\phi(t) + t\overline{\phi'(t)} + \overline{\psi(t)} = h_1(t) + ih_2(t), \tag{17}$$

for $t \in \Gamma_k$, $k = 0, \ldots, M$. Here, we have equated the point $\mathbf{t} \in \mathbf{R}^2$ with the complex point $t \in \mathbf{C}$, $t \in \Gamma_k$ for $k = 0, \ldots, M$, and $A(k) = A_1(k) + iA_2(k)$. As noted earlier, for simply-connected domains, the complex constant $A(0)$ in (16) can be set to zero, so that the stress and Stokes problems are indistinguishable. In multiply-connected domains, elasticity problems require that the displacements be single-valued, while Stokes flow problems require that the pressure be single valued. The corresponding representations for ϕ and ψ must be consistent with these constraints.

3.1. *The Displacement Problem*

To find analytic functions $\phi(z)$ and $\psi(z)$ which satisfy (15) on the contour Γ, Sherman [17, 19] has suggested the representations

$$\phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\omega(\xi)}{\xi - z}\,d\xi + \sum_{k=1}^{M} C_k \log(z - z_k)$$

$$\begin{aligned} \psi(z) = {} & \frac{-\kappa}{2\pi i}\int_\Gamma \frac{\overline{\omega(\xi)}}{\xi - z}\,d\xi + \frac{1}{2\pi i}\int_\Gamma \frac{\omega(\xi)}{\xi - z}\,d\bar{\xi} \\ & - \frac{1}{2\pi i}\int_\Gamma \frac{\bar{\xi}\omega(\xi)}{(\xi - z)^2}\,d\xi - \sum_{k=1}^{M} \kappa\bar{C}_k \log(z - z_k) \\ & - \sum_{k=1}^{M} \frac{C_k \bar{z}_k}{z - z_k}. \end{aligned}$$

In the preceding expressions, $\omega(\xi)$ is an unknown complex density, the z_k are arbitrarily prescribed points inside the component curves Γ_k, and the complex constants C_k are defined in terms of $\omega(\xi)$ via

$$C_k = \int_{\Gamma_k} \omega(\xi)\,ds,$$

where ds is an element of arc length.

If we let z tend to a point t on the contour Γ and use the classical formulae for the limiting values of Cauchy-

type integrals, we obtain from (15) the Sherman–Lauricella integral equation,

$$\kappa\omega(t) + \frac{\kappa}{2\pi i}\int_\Gamma \omega(\xi)\, d\ln\frac{\xi - t}{\bar{\xi} - \bar{t}} - \frac{1}{2\pi i}\int_\Gamma \overline{\omega(\xi)}\, d\frac{\xi - t}{\bar{\xi} - \bar{t}} + \sum_{k=1}^{M} \overline{C}_k \frac{t - z_k}{\bar{t} - \bar{z}_k} + 2\kappa \sum_{k=1}^{M} C_k \log|t - z_k| = f(t), \tag{18}$$

where $f(t) = f_1(t) + if_2(t)$. Letting $\xi - t = re^{i\theta}$, a straightforward calculation shows that

$$d\ln\frac{\xi - t}{\bar{\xi} - \bar{t}} = 2i\, d\theta, \quad d\frac{\xi - t}{\bar{\xi} - \bar{t}} = 2ie^{2i\theta}\, d\theta.$$

Thus, (18) can be written in the form

$$\kappa\omega(t) + \frac{\kappa}{\pi}\int_\Gamma \omega(\xi)\, d\theta - \frac{1}{\pi}\int_\Gamma \overline{\omega(\xi)} e^{2i\theta}\, d\theta + \sum_{k=1}^{M} \overline{C}_k \frac{t - z_k}{\bar{t} - \bar{z}_k} + 2\kappa \sum_{k=1}^{M} C_k \log|t - z_k| = f(t). \tag{19}$$

Assuming that the contours themselves are smooth, the latter form of the Sherman–Lauricella equation is clearly a Fredholm equation of the second kind with smooth kernel, and therefore the Fredholm alternative applies. We refer the reader to [17 or 19] for a proof of invertibility. We will simply observe here that, in the absence of the source terms,

$$\sum_{k=1}^{M} \overline{C}_k \frac{t - z_k}{\bar{t} - \bar{z}_k} + 2\kappa \sum_{k=1}^{M} C_k \log|t - z_k|,$$

the integral equation above is singular with rank deficiency $2M$.

3.2. *The Stress Problem*

To satisfy the boundary condition (16) on Γ, the Sherman–Lauricella equation is derived by letting $\phi(z)$ and $\psi(z)$ take the form

$$\phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\omega(\xi)}{\xi - z}\, d\xi,$$

$$\psi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\overline{\omega(\xi)}\, d\xi + \omega(\xi)\, d\bar{\xi}}{\xi - z} - \frac{1}{2\pi i}\int_\Gamma \frac{\bar{\xi}\omega(\xi)}{(\xi - z)^2}\, d\xi + \sum_{k=1}^{M} \frac{b_k}{z - z_k},$$

where $\omega(\xi)$ and z_k are defined as in the displacement problem, and the b_k are real constants defined by

$$b_k = i\int_{\Gamma_k} \omega(t)\, d\bar{t} - \overline{\omega(t)}\, dt.$$

Again, letting z tend to a point t on one of the contours Γ_k, we obtain from (16) the Sherman–Lauricella integral equation:

$$\omega(t) + \frac{1}{2\pi i}\int_\Gamma \omega(\xi)\, d\ln\frac{\xi - t}{\bar{\xi} - \bar{t}} - \frac{1}{2\pi i}\int_\Gamma \overline{\omega(\xi)}\, d\frac{\xi - t}{\bar{\xi} - \bar{t}} + \frac{\bar{b}_0}{\bar{t} - \bar{z}^*} + \sum_{k=1}^{M} \frac{\bar{b}_k}{\bar{t} - \bar{z}_k} - A(k) = g(t), \tag{20}$$

where $g(t) = g_1(t) + ig_2(t)$, $A(0) = 0$, and for $k > 0$, we define

$$A(k) = -\int_{\Gamma_k} \omega(\xi)\, ds.$$

Note that the left-hand side of the integral equation includes the term $\bar{b}_0/(\bar{t} - \bar{z}^*)$, where z^* is an arbitrary point in the domain D and

$$b_0 = \frac{1}{2\pi i}\int_\Gamma \left[\frac{\omega(t)}{(t - z^*)^2}\, dt + \frac{\overline{\omega(t)}}{(\bar{t} - \bar{z}^*)^2}\, d\bar{t}\right].$$

It can be shown [17, 19] that Eq. (20) is invertible and that $b_0 = 0$ as long as the data satisfies the compatibility condition

$$\operatorname{Re}\int_\Gamma \overline{g(t)}\, dt = 0,$$

which expresses the fact that the resultant moment of the external forces vanishes. In this formulation, the displacements are clearly single valued by inspection of the formula (12). Note that no logarithmic sources have been used in the integral representations.

3.3. *The Stokes Problem*

Finally, we consider the construction of functions ϕ and ψ which can be used to satisfy the boundary conditions (17) of Stokes flow. Letting $\kappa = -1$ in the displacement problem might seem like a reasonable choice, but Eq. (18) would then be singular. We propose the following:

$$\phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\omega(\xi)}{\xi - z}\, d\xi + \sum_{k=1}^{M} C_k \log(z - z_k)$$

$$\psi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\overline{\omega(\xi)}\, d\xi + \omega(\xi)\, d\bar{\xi}}{\xi - z} \tag{21}$$

$$-\frac{1}{2\pi i}\int_\Gamma \frac{\bar{\xi}\omega(\xi)}{(\xi-z)^2}\,d\xi + \sum_{k=1}^M \frac{b_k}{z-z_k} + \sum_{k=1}^M \overline{C}_k \log(z-z_k) - \sum_{k=1}^M C_k \frac{\overline{z}_k}{z-z_k}.$$

It is important to observe that, while ϕ and ψ are not single-valued functions, the physical variables of interest (velocity, pressure, and vorticity) are well defined. Letting z approach a boundary point t, we obtain

$$\omega(t) + \frac{1}{2\pi i}\int_\Gamma \omega(\xi)\,d\ln\frac{\xi-t}{\bar{\xi}-\bar{t}} - \frac{1}{2\pi i}\int_\Gamma \overline{\omega(\xi)}\,d\frac{\xi-t}{\bar{\xi}-\bar{t}} + \frac{\overline{b_0}}{\bar{t}-\bar{z}^*} + \sum_{k=1}^M \frac{\overline{b}_k}{\bar{t}-\overline{z}_k} + \sum_{k=1}^M 2C_k \log|t-z_k| + \sum_{k=1}^M \overline{C}_k \frac{t-z_k}{\bar{t}-\overline{z}_k} = h(t), \tag{22}$$

where $h(t) = h_1(t) + ih_2(t)$ and C_k, b_0, and b_k are defined as above. As in the stress problem, the integral equation includes an extra term involving b_0, which vanishes when the natural compatibility condition

$$\mathrm{Re}\int_\Gamma \overline{h(t)}\,dt = 0$$

is satisfied. In physical terms, this requires that there be zero net flux across Γ. We omit a proof of invertibility of the integral equation, since it follows the lines of the standard proof in [17, 19] for the elasticity problems.

3.4. *The Completed Double Layer Method*

In a series of papers and texts [9, 18, 20–22], Karrila, Kim, Miranda, Power, and Pozrikidis have developed what is referred to as the completed double layer method which, like the Sherman–Lauricella approach, adds singular source terms to a second-kind integral equation in order to construct nonsingular systems. These methods have been developed in both two and three dimensions and, hence, are more general than the equations described above. Remarkably, the integral equations obtained in the two-dimensional case are equivalent. In particular, Eq. (22) is equivalent to a primitive variable formulation of Stokes flow recently developed by Power. (His paper [20] includes a thorough discussion of the Fredholm alternative.) To see this equivalence, one need only rewrite the fundamental solutions expressed in primitive variables in terms of analytic functions [16]. The singular contributions

$$\sum_{k=1}^M 2C_k \log|t-z_k| + \sum_{k=1}^M \overline{C}_k \frac{t-z_k}{\bar{t}-\overline{z}_k} + \sum_{k=1}^M \frac{\overline{b}_k}{\bar{t}-\overline{z}_k}$$

can be shown to be a linear combination of a "Stokeslet" and a "Rotlet" (see the Appendix).

3.5. *Unbounded Domains*

In order for the components of stress to be bounded at infinity, it can be shown [17] that the Goursat functions must take the general form

$$\phi(z) = -\frac{(X+iY)}{2\pi(1+\kappa)}\log z + \Gamma z + \phi^*(z)$$

$$\psi(z) = \frac{(X+iY)}{2\pi(1+\kappa)}\log z + \Gamma' z + \psi^*(z),$$

where $\Gamma = B + iC$ and $\Gamma' = B' + iC'$ are complex constants, and $\phi^*(z)$ and $\psi^*(z)$ are single-valued and holomorphic at infinity. The real constants B, B', C, and C' have a direct physical interpretation. Let N_1, N_2 be the values of the principal stresses at infinity, let α be the angle made by the direction of N_1 with the x axis, and let ε_∞ be the rotation at infinity. Then $B = (N_1 + N_2)/4$, $B' + iC' = -(N_1 - N_2)e^{-2i\alpha}/2$, and $C = 2\mu\varepsilon_\infty/(1+\kappa)$. Since Γ and Γ' must be given *a priori,* we will assume that they are both zero and transform the boundary data if this is not so. With this assumption, the stresses always vanish at infinity.

The logarithmic terms in ϕ and ψ are a little more complicated. In the displacement problem, (X, Y) is the resultant vector of external forces acting on the whole boundary $\Gamma_1 \cup \cdots \cup \Gamma_M$. It is not known in advance and determined as part of the solution process. In the stress problem, (X, Y) can be determined from the boundary data. In either case, in order to have the displacement be bounded at infinity one needs the resultant vector of the external forces acting on the boundary to vanish.

For Stokes flow in unbounded domains, the nature of the behavior at infinity is slightly different. We will assume that there is no linear growth of the velocity in the far field, but we must allow for logarithmic growth (Stokes paradox) [11]. We use the Goursat representation

$$\phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\omega(\xi)}{\xi-z}\,d\xi + \int_\Gamma \omega(\xi)\,ds + \sum_{k=1}^M C_k \log(z-z_k)$$

$$\psi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\overline{\omega(\xi)}\,d\xi + \omega(\xi)\,d\bar{\xi}}{\xi-z} - \frac{1}{2\pi i}\int_\Gamma \frac{\bar{\xi}\omega(\xi)}{(\xi-z)^2}\,d\xi + \sum_{k=1}^M \frac{b_k}{z-z_k} + \sum_{k=1}^M \overline{C}_k \log(z-z_k) - \sum_{k=1}^M C_k \frac{\overline{z}_k}{z-z_k},$$

so that the leading term in the velocity field at ∞ is

$$2\left(\sum_{k=1}^M C_k\right)\log|z|.$$

In order for the problem to be well-posed,

$$\sum_{k=1}^{M} C_k = X + iY \tag{23}$$

must be given. From (17) the integral equation is

$$\omega(t) + \int_\Gamma \omega(\xi)\, ds + \frac{1}{2\pi i}\int_\Gamma \omega(\xi)\, d\ln\frac{\xi - t}{\bar{\xi} - \bar{t}}$$
$$- \frac{1}{2\pi i}\int_\Gamma \overline{\omega(\xi)}\, d\frac{\xi - t}{\bar{\xi} - \bar{t}} + \sum_{k=1}^{M} 2C_k \log|t - z_k| \tag{24}$$
$$+ \sum_{k=1}^{M} \overline{C}_k \frac{t - z_k}{\bar{t} - \bar{z}_k} + \sum_{k=1}^{M} \frac{\bar{b}_k}{\bar{t} - \bar{z}_k} = h(t),$$

where the constant C_1 is determined by the constraint equation (23) to be

$$C_1 = X + iY - \sum_{k=2}^{M} C_k, \tag{25}$$

and the remaining C_k and b_k have the same definition as in Section 3.3. Note that an additional term $\int_\Gamma \omega(\xi)\, ds$ has been added to $\phi(z)$ in order to capture the constant term in the flow at infinity. It is not known *a priori* and is computed as part of the solution process.

3.6. *Semi-infinite Domains*

A variety of problems in fluid dynamics and elasticity require the solution of the biharmonic equation when multiple obstacles are embedded in a half space S. By convention, we assume that S is the upper half plane $y > 0$. We also assume that the boundary conditions are homogeneous along $y = 0$. Inhomogeneous boundary conditions along this line can be imposed by first solving the half-space problem analytically in the absence of obstacles and then subtracting the contribution of this solution from the desired boundary conditions on all boundary components.

Suppose now that $\zeta \in S$ and that the Goursat functions ϕ and ψ are expressed as Laurent series about the point ζ:

$$\phi(z) = \sum_{k=1}^{\infty} \frac{\alpha_k}{(z-\zeta)^k}, \quad \psi(z) = \sum_{k=1}^{\infty} \frac{\beta_k}{(z-\zeta)^k}.$$

Let ϕ_I and ψ_I be reflected Laurent expansions about the point $\bar{\zeta}$, so that

$$\phi_I(z) = \sum_{k=1}^{\infty} \frac{\gamma_k}{(z-\bar{\zeta})^k}, \quad \psi_I(z) = \sum_{k=1}^{\infty} \frac{\delta_k}{(z-\bar{\zeta})^k}.$$

In order to impose zero velocity (or zero stress) conditions on ∂S we must satisfy

$$\phi(t) + t\overline{\phi'(t)} + \overline{\psi(t)} + \phi_I(t) + t\overline{\phi_I'(t)} + \overline{\psi_I(t)} = 0$$

for $t \in \partial S$. Matching appropriate terms, a straightforward calculation yields the recursion relations

$$\gamma_k = k\overline{\alpha_k} + (k-1)\bar{\zeta}\,\overline{\alpha_{k-1}} - \overline{\beta_k},$$
$$\delta_k = k\gamma_k + (k-1)\bar{\zeta}\gamma_{k-1} - \overline{\alpha_k}.$$

Logarithmic sources are handled in an analogous fashion. If

$$\phi(z) = A\log(z-\zeta), \quad \psi(z) = \overline{A}\log(z-\zeta) - \frac{A\bar{\zeta}}{(z-\zeta)},$$

then the image sources are

$$\phi_I(z) = -A\log(z-\bar{\zeta}) + \frac{\overline{A}(\zeta - \bar{\zeta})}{(z-\bar{\zeta})}$$
$$\psi_I(z) = -\overline{A}\log(z-\bar{\zeta}) + \frac{\overline{A}(\zeta-\bar{\zeta}) + A\bar{\zeta}}{(z-\bar{\zeta})} + \frac{A\zeta(\bar{\zeta}-\zeta)}{(z-\bar{\zeta})^2}.$$

Imposing zero displacement conditions yields similar formulae.

4. DESCRIPTION OF THE NUMERICAL METHOD

In order to solve the Sherman–Lauricella equations, we use a Nyström discretization based on the trapezoidal rule since it achieves superalgebraic convergence for smooth data on smooth boundaries. For this, we assume that we are given N_k points on each contour Γ_k, equispaced with respect to some parametrization $t^k : [0, L_k] \to \Gamma_k$. Associated with each such point, denoted by t_j^k, is an unknown value ω_j^k. The derivative $(t^k)'$ will be denoted by d^k and we assume that we are given the derivative values d_j^k at the discretization points. The step length in the discretization is defined by $h_k = L_k/N_k$ and the total number of points is

$$N = \sum_{k=k_0}^{M} N_k,$$

where $k_0 = 0$ for interior problems and $k_0 = 1$ for exterior or wall-bounded problems.

Consider now the stress problem, for which we need to solve the system (20). After discretization, we have

$$\omega_j^k + \sum_{m=k_0}^{M}\sum_{n=1}^{N_m} K_1(t_j^k, t_n^m)\omega_n^m + \sum_{m=k_0}^{M}\sum_{n=1}^{N_m} K_2(t_j^k, t_n^m)\overline{\omega_n^m} = g_j^k, \tag{26}$$

where $g_j^k = g(t_j^k)$ and the kernels K_1 and K_2 are given by

$$K_1(t_j^k, t_n^m) = \frac{h_m}{2\pi i}\left(\frac{-d_n^m}{t_j^k - t_n^m} + \frac{\overline{d_n^m}}{\overline{t_j^k - t_n^m}}\right) + \frac{ih_m\delta_{mm}\overline{d_n^m}}{\overline{t_j^k - z_m}} + h_m\delta_{km}$$
$$K_2(t_j^k, t_n^m) = \frac{-h_m}{2\pi i}\left(\frac{-d_n^m}{\overline{t_j^k - t_n^m}} + \frac{\overline{d_n^m}(t_j^k - t_n^m)}{(\overline{t_j^k - t_n^m})^2}\right) - \frac{ih_m\delta_{mm}d_n^m}{\overline{t_j^k - z_m}}. \tag{27}$$

In the preceding expression for K_1, δ_{km} is the usual Kronecker delta symbol, except that $\delta_{00} \equiv 0$. Furthermore, when $t_j^k = t_n^m$, K_1 and K_2 should be replaced by the appropriate limits,

$$K_1(t_j^k, t_j^k) = \frac{h_k}{2\pi}\kappa_j^k|d_j^k| + \frac{ih_k\delta_{kk}\overline{d_j^k}}{\overline{t_j^k - z_k}} + h_k\delta_{kk}$$
$$K_2(t_j^k, t_j^k) = -\frac{h_k}{2\pi}\kappa_j^k(d_j^k)^2/|d_j^k| - \frac{ih_k\delta_{kk}d_j^k}{\overline{t_j^k - z_k}}, \tag{28}$$

where κ_j^k denotes the curvature at the point t_j^k. (If the curvature data is not given, the smooth kernels K_1 and K_2 can be interpolated to high-order accuracy directly from the definition.)

Remark. We have omitted the term b_0 in Eq. (20) since its absence does not affect the behavior of the iterative solution procedure we will employ. For a detailed discussion of this point, see [13].

It should be noted that the system (26), or for that matter (20), is not simply a complex linear system for the unknowns ω_i^k, since the conjugate values $\overline{\omega_i^k}$ appear in each equation as well. We could, of course, expand (26) in terms of the real and imaginary parts of ω_i^k, but we will write the system as

$$(I + K_1 + K_2\mathscr{C})\boldsymbol{\omega} = \mathbf{g}, \tag{29}$$

where $\mathscr{C}$ denotes the conjugation operator,

$$\boldsymbol{\omega} = (\omega_1^{k_0}, ..., \omega_{N_{k_0}}^{k_0}, ..., \omega_1^M, ..., \omega_{N_M}^M)^{\mathrm{T}}, \tag{30}$$

and

$$\mathbf{g} = (g_1^{k_0}, ..., g_{N_{k_0}}^{k_0}, ..., g_1^M, ..., g_{N_M}^M)^{\mathrm{T}}, \tag{31}$$

The interior Stokes problem can be written in an analogous fashion as

$$(I + K_1' + K_2'\mathscr{C})\boldsymbol{\omega} = \mathbf{h}, \tag{32}$$

where

$$K_1'(t_j^k, t_n^m) = K_1(t_j^k, t_n^m) + 2h_m\delta_{mm}\log|t_j^k - z_m| - h_m\delta_{km}$$
$$K_2'(t_j^k, t_n^m) = K_2(t_j^k, t_n^m) + \frac{h_m\delta_{mm}(t_j^k - z_m)}{\overline{t_j^k} - \overline{z}_m}. \tag{33}$$

However, for the purposes of iterative solution, it is advantageous to separate out the influence of the singular sources involving the constants C_k. To do this, we represent the Goursat functions ϕ and ψ as in Eq. (21), but we consider the C_k to be unknowns. To compensate for the increase in the system size, we add M constraints of the form

$$h_k\sum_{l=1}^{N_k}\omega_l^k = 0. \tag{34}$$

The discrete equations may then be written as

$$\begin{pmatrix} I + \mathrm{K} & E \\ F & D \end{pmatrix}\begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} \mathbf{h} \\ 0 \end{pmatrix}, \tag{35}$$

where $K = K_1 + K_2\mathscr{C}$,

$$\mathbf{c} = (C_0, C_1, ..., C_M)^{\mathrm{T}}$$

is the vector of unknown singular source strengths, and

$$\mathbf{h} = (h_1^0, ..., h_{N_0}^0, ..., h_1^M, ..., h_{N_M}^M)^{\mathrm{T}}$$

is the vector of given boundary values of the velocity. The $2N$ by $2M$ matrix E represents the influence of the constants C_k on the velocity field and the $2M$ by $2N$ matrix F represents the discrete constraint equations in (34). The block matrix D is zero for interior problems and incorporates the constraint (23) for external flows. The other minor changes needed to treat external flows are discussed in Section 3.5.

In our implementation, the matrix equations (29) and (35) are solved iteratively using the generalized minimum residual method GMRES [24]. In the case of Stokes flow, we use a preconditioner which eliminates the influence of the logarithmic sources [4]. Thus, instead of solving the linear system in (35), we solve the system

$$\begin{pmatrix} I & E \\ F & D \end{pmatrix}^{-1} \begin{pmatrix} I+K & E \\ F & D \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{c} \end{pmatrix} = \begin{pmatrix} I & E \\ F & D \end{pmatrix}^{-1} \begin{pmatrix} \mathbf{h} \\ 0 \end{pmatrix}. \quad (36)$$

It is easy to verify that the rank of the preconditioned matrix is determined by the integral operator terms K_1 and K_2 but it is no longer affected by the block matrices E or F.

At each iteration, it is necessary to invert the preconditioning matrix. This is accomplished as follows. We first form the $2M$ by $2M$ Schur complement S of D in the preconditioner

$$\begin{pmatrix} I & E \\ F & D \end{pmatrix},$$

given by

$$S = D - FE. \quad (37)$$

We then compute and store the LU factorization of S by standard methods. To solve the linear system

$$\begin{pmatrix} I & E \\ F & D \end{pmatrix} \begin{pmatrix} \mathbf{z}_\omega \\ \mathbf{z}_c \end{pmatrix} = \begin{pmatrix} \mathbf{r}_\omega \\ \mathbf{r}_c \end{pmatrix}, \quad (38)$$

we first solve

$$S\mathbf{z}_c = \mathbf{r}_c - F\mathbf{r}_\omega \quad (39)$$

and then compute $\mathbf{z}_\omega$ from

$$\mathbf{z}_\omega = \mathbf{r}_\omega - E\mathbf{z}_c. \quad (40)$$

The initial factorization requires approximately $M^3/3$ operations, while back-solving requires M^2 operations at each GMRES iteration.

The bulk of the work at each iteration, however, lies in applying the full matrix to a vector. The product

$$\begin{pmatrix} I+K & E \\ F & D \end{pmatrix} \begin{pmatrix} \boldsymbol{\omega} \\ \mathbf{c} \end{pmatrix}$$

can be computed in $O(N + M)$ time using the adaptive fast multipole method (FMM). Instead of the approach taken in [3], our implementation works with the analytic functions ϕ and ψ separately, so that it is easy to compute derivative quantities such as ϕ', ϕ'', and ψ'. It is also easy to extend such a version of the FMM to handle wall-bounded problems using the reflection formula from Section 3.6. For further details, we refer the reader to the original papers [1, 3, 5, 6, 23].

Since the number of iterations needed to solve a Fredholm equation of the second kind to a fixed precision is bounded independent of N, we can estimate the total cost of solving the stress problem by

$$c_{\mathrm{str}}(\varepsilon)A(\varepsilon)N,$$

where $c_{\mathrm{str}}(\varepsilon)$ is the number of GMRES iterations needed to reduce the residual error to ε and $A(\varepsilon)$ is the constant of proportionality in the FMM. Similarly, the cost of solving the Stokes problem is approximately

$$c_{\mathrm{sto}}(\varepsilon)A(\varepsilon)N$$

in the original formulation and

$$M^3/3 + c'_{\mathrm{sto}}(\varepsilon)(M^2 + A(\varepsilon)N)$$

in the constrained formulation. As we shall see in the next section, the latter approach is superior for modest values of M.

5. NUMERICAL RESULTS

The algorithms described above have been implemented in Fortran. Here, we illustrate their performance on a variety of examples. All timings cited are for an SGI Onyx with a single R8000 processor.

EXAMPLE 1. $M = 10$. We first consider the problem of Stokes flow in a bounded circular domain with 10 interior elliptical contours (Fig. 1). The data is obtained by choosing a stream function of the form $W(x, y) = x^3 + xy^3$ to which are added singular solutions corresponding to

$$\phi(z) = \sum_{k=1}^{4} \log(z - z_k) + \sum_{k=5}^{10} \frac{2}{z - z_k}$$
$$\psi(z) = \sum_{k=1}^{4} \log(z - z_k) + \sum_{k=5}^{10} \frac{0.5}{z - z_k},$$

where the points z_k, $k = 1, ..., 10$, lie inside the 10 holes. These points are distinct from the auxilliary points used in the Sherman–Lauricella representation.

In Fig. 2, we illustrate the performance of the GMRES(m) iterative method on the unconstrained integral equation (32), the constrained equation (35), and the preconditioned constrained equation (36). The parameter m represents the number of direction vectors which are saved and orthogonalized against, before restarting the iteration. Each boundary component is discretized using 512 points and the right-hand side of the equation is used as

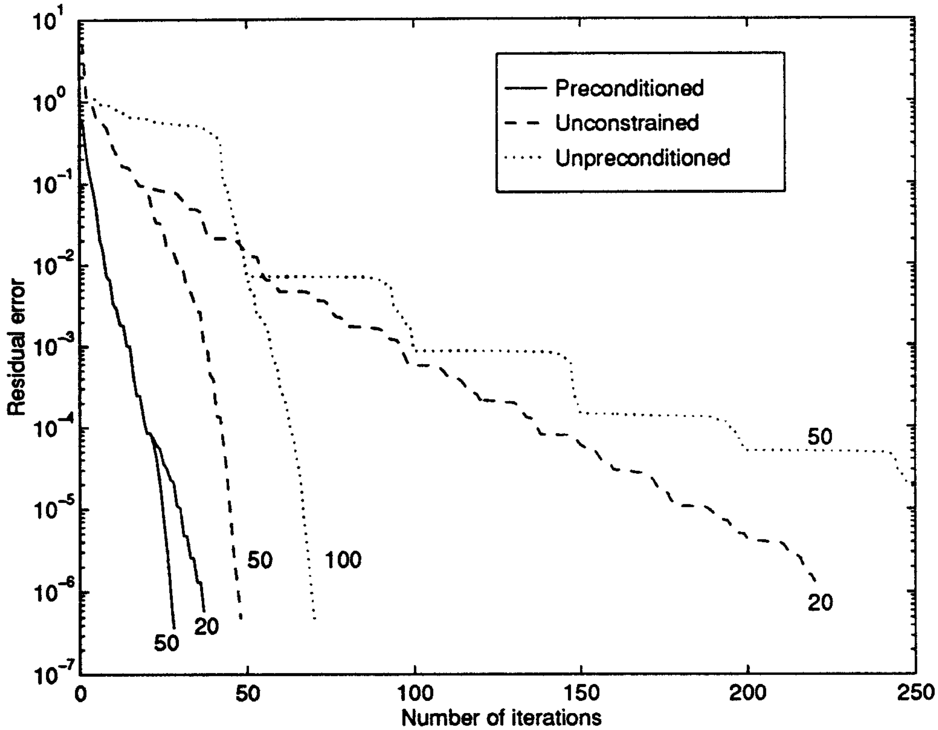

FIG. 2. Performance of the GMRES iterative method for the unconstrained, constrained, and preconditioned integral equations for an interior Stokes flow. The number next to each curve denotes the number of direction vectors saved before restarting the iteration.

the initial guess. If enough vectors are saved, then GMRES converges satisfactorily in each case. If fewer vectors are saved, the performance of the unpreconditioned constrained system is poor, the performance of the preconditioned constrained system is very good, and the performance of the unconstrained system lies somewhere in between.

Table I summarizes the performance of the preconditioned constrained integral equation method as we refine the spatial discretization. The columns indicate the number of boundary points, the number of GMRES iterations required to reduce the Euclidean norm of the residual below 10^{-10}, the number of seconds of CPU time required for the calculation, and the Euclidean norm of the error in computing the vorticity at the boundary discretization points. As an additional test, the velocity was computed at several points inside the domain; the error was found to be consistently less than the error in the vorticity.

Note that the number of GMRES iterations is independent of the number of discretization points used and that the CPU time grows linearly with the system size. The rapid convergence of the solution is consistent with the spectral accuracy of the approach. Since we halted the GMRES iteration at a residual error of 10^{-10}, the error in the vorticity is not reduced beyond approximately 10^{-9}.

TABLE I

Performance of the Algorithm in Computing the Interior Stokes Flow of Example 1

N	No. of iterations	CPU time	Error
704	37	20	0.13×10^{-1}
1408	37	41	0.37×10^{-6}
2816	37	98	0.28×10^{-8}
5632	37	199	0.18×10^{-8}

Example 2. $M = 100$. We next consider a bounded Stokes flow with 100 interior circular contours (Fig. 3). Such geometries arise in a variety of applications, including the sedimentation of particles and porous media flow. The velocity is zero on the outer contour, while each interior contour rotates with an angular velocity in the range $[-1, 1]$. Each boundary component is discretized using 100 discretization points, resulting in a matrix of order 20400. For six digits of accuracy, the preconditioned system requires 52 GMRES iterations and about 13 min CPU time. Had the GMRES method been used with a conventional matrix–vector multiply routine, rather than the FMM, about 11 h would have been required. Gaussian elimination, of course, would have required weeks (assuming sufficient memory were available).

Examples 3 and 4. $M = 80$; $M = 100$. Stokes flows in

exterior domains are difficult to handle by conventional finite difference and finite element methods since the computational domain must be artificially truncated and proper boundary conditions are not available on the artificial boundary. In our third example (Fig. 4), we give constant but distinct velocities $\mathbf{v}_i$, where $\|\mathbf{v}_i\| \leq 1$, to each of 80 particles and specify that the flow be bounded at infinity. Each boundary component is discretized using 100 points, resulting in a matrix of order 16400. For six digits of accuracy, 368 iterations are required using GMRES(100), consuming about 80 min CPU time. Note from the figure that the velocity approaches a constant away from the objects. The value of this constant is determined as part of the solution process (see Section 3.5). In our fourth example, we set the velocity to zero on each of 100 contours in a 10×10 array of disks. In order to obtain a nontrivial solution, we set the velocity to approach $(\log r, 0)$ as $r \to \infty$, consistent with Stokes paradox (Fig. 5). Each boundary component is discretized using 100 points, resulting in a matrix of order 20400. For five digits of accuracy, 40 iterations are required using GMRES(100), consuming about 8 min CPU time.

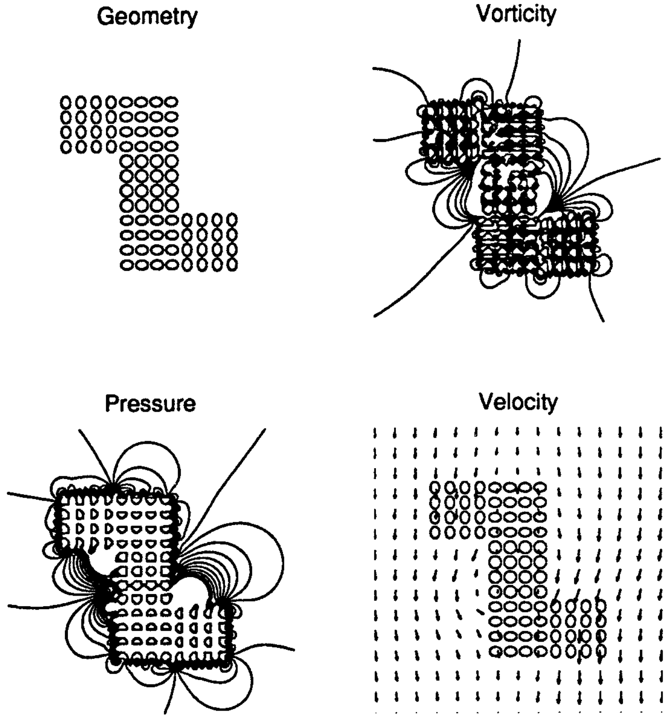

FIG. 4. The bounded domain of Example 3. The upper left-hand plot shows the geometry, the upper right-hand plot shows contours of vorticity, the lower left-hand plot shows contours of pressure, and the lower right-hand plot shows the velocity field.

EXAMPLE 5. $M = 3$. While the domains in the preceding examples are multiply connected, the individual contours consist of rather simple shapes. In the present example, we consider a wall-bounded Stokes flow around more complex contours (Fig. 6). The velocity is set to (3, 0) on the leftmost object, to (2, 0) on the middle object, to (1, 0) on the rightmost object, and to (0, 0) on the wall (x-axis). Using 1000 points on each boundary, the integral equation required 95 GMRES iterations to obtain six digits of accuracy, consuming about 4 min of CPU time.

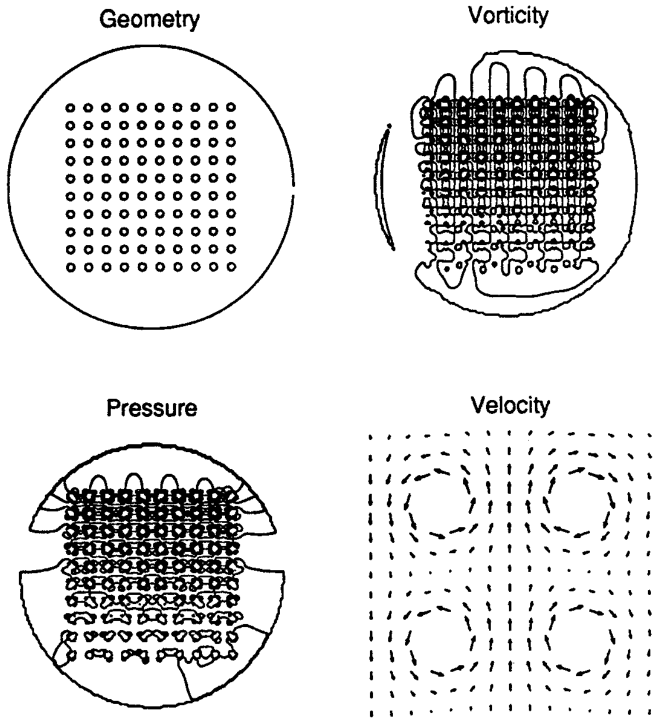

FIG. 3. The bounded domain of Example 2. The upper left-hand plot shows the geometry, the upper right-hand plot shows contours of vorticity, the lower left-hand plot shows contours of pressure, and the lower right-hand plot depicts a detail of the velocity field in the vicinity of four interior contours.

EXAMPLE 6. $M = 100$. Our final example is a calculation of plane stress in a semi-infinite region with 100 inclusions of varying shape (Fig. 7). The integrated components of stress are given by

$$W_x + iW_y = A_k(z - z_k) \quad \text{for } z \in \Gamma_k,$$

where $A_k \in [0, 1]$ and z_k is the center of the kth contour. We compute the trace of the stress tensor and the energy density $\mathscr{D}$ via Eqs. (13) and the formula

$$\mathscr{D} = \frac{1}{4\mu}\left(2\tau_{xy}^2 + \frac{\sigma_x^2 + \sigma_y^2}{(1 + \nu)}\right),$$

where the shear modulus μ and Poisson's ratio ν have been set to 1 and $\frac{1}{4}$, respectively [18]. With 128 points on each

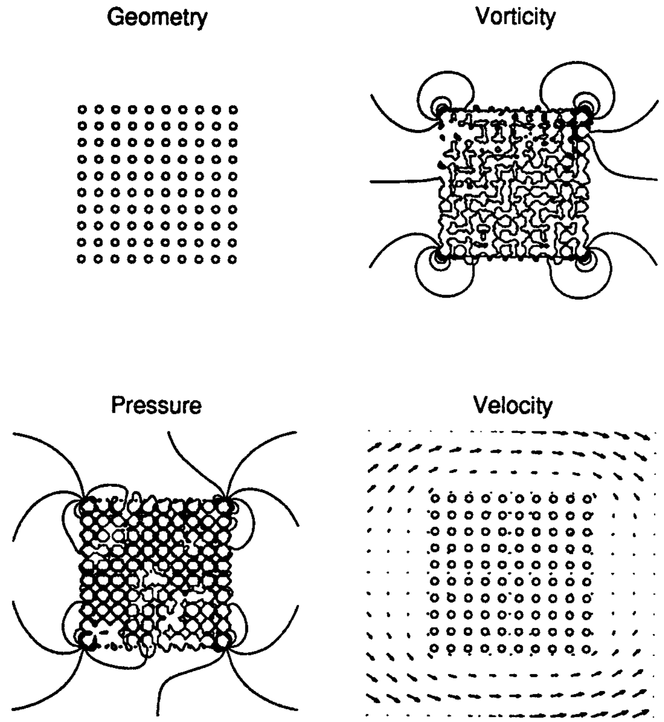

FIG. 5. The bounded domain of Example 4. The upper left-hand plot shows the geometry, the upper right-hand plot shows contours of vorticity, the lower left-hand plot shows contours of pressure, and the lower right-hand plot shows the velocity field.

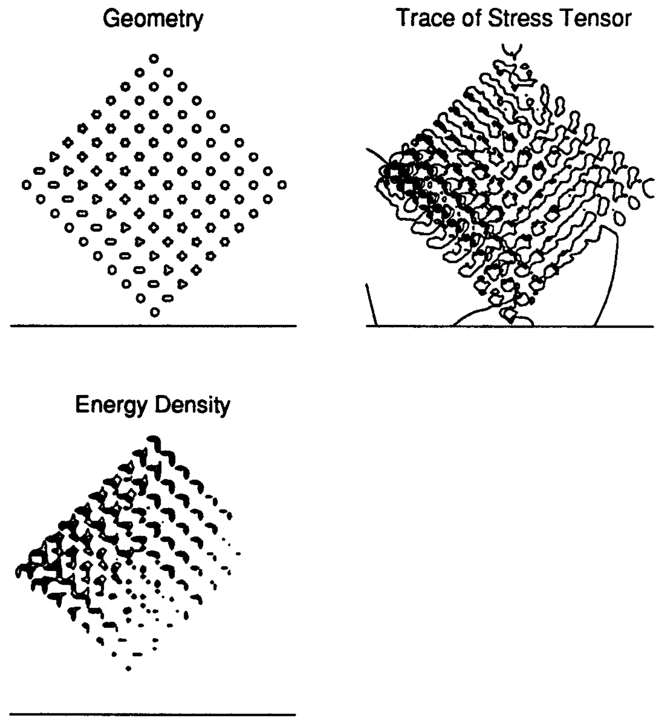

FIG. 7. Plane stress in a semi-infinite domain with one hundred inclusions (Example 6). The upper left-hand plot shows the geometry. The upper right-hand plot shows contours of the trace of the stress tensor, and the lower left-hand plot shows contours of the energy density.

contour, 36 GMRES iterations are required, consuming 1100 s CPU time.

6. CONCLUSIONS

We have presented a class of integral equation methods for the solution of biharmonic boundary value problems in bounded, unbounded, and wall-bounded domains. With the use of a fast-multipole accelerated iterative method,

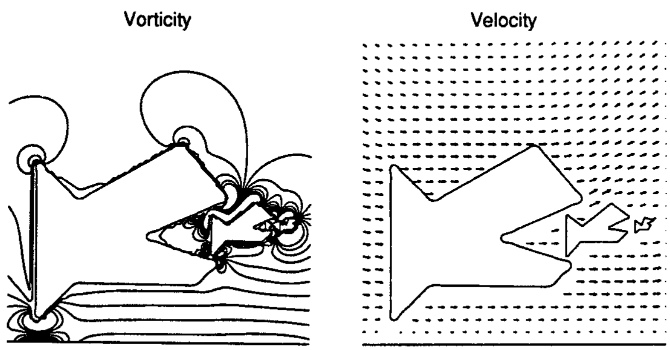

FIG. 6. The wall-bounded flow of Example 5. The left-hand plot shows contours of the vorticity and the right-hand plot shows the velocity field.

our solution procedure requires $O(N)$ operations, where N is the number of nodes in the discretization of the boundary.

An immediate extension of this work is to solve forced or inhomogeneous biharmonic problems

$$\Delta\Delta W = f.$$

One simply computes a volume integral

$$W(x, y) = \int |P - Q|^2 \log|P - Q| f(Q)\, dQ,$$

using the scheme introduced in [12], and then corrects the boundary conditions with the integral equation techniques described above. (For a discussion of this approach in the context of the Poisson equation, see [14].) Solutions to mildly nonlinear systems, such as low Reynolds number flows, can be found by solving a sequence of forced Stokes problems [7].

APPENDIX

Let us consider the singular solutions we add in the modified Sherman–Lauricella representation (21), namely

$$W_x + iW_y = 2C\log|z| + \overline{C}\frac{z}{\overline{z}} + b\frac{1}{\overline{z}},$$

where $C = C_1 + C_2$ and b is real. Letting $r = |z|$, $z = x + iy$, it is easily shown that

$$W_x + iW_y = 2C_1\log r - 2C_1\frac{y^2}{r^2} + C_1 + 2C_2\frac{xy}{r^2} + b\frac{x}{r^2} + i\left(2C_2\log r - 2C_2\frac{x^2}{r^2} + C_2 + C_1\frac{xy}{r^2} + b\frac{y}{r^2}\right),$$

which corresponds to the velocity field

$$\mathbf{u} = (W_y, -W_x) = -2C_2\mathbf{u}_x^s + 2C_1\mathbf{u}_y^s + b\mathbf{u}^r + (C_2, -C_1),$$

where

$$\mathbf{u}_x^s = \left(-\log r + \frac{x^2}{r^2}, \frac{xy}{r^2}\right), \quad \mathbf{u}_y^s = \left(\frac{xy}{r^2}, -\log r + \frac{y^2}{r^2}\right)$$

$$\mathbf{u}^r = \frac{1}{r^2}(y, -x).$$

But $\mathbf{u}_x^s$ and $\mathbf{u}_y^s$ are the velocity fields due to Stokeslets that are oriented in the x and y directions, respectively, and $\mathbf{u}^r$ is the velocity field due to a Rotlet [22]. These are precisely the singularities added by Power to "complete" the double layer representation in [20].

ACKNOWLEDGMENTS

We thank B. Braams, R. Kohn, and C. Peskin for several useful discussions.

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