SFU - UBC - UNBC - UVic

Calculus Challenge Exam

First Practice

Host: SIMON FRASER UNIVERSITY

Student signature

INSTRUCTIONS

1. Show all your work. Full marks are given only when correct, and is supported with a written derivation that is orderly, logical, and complete. Part marks are available in every question.
2. Calculators are optional, not required. Correct answers that are calculator ready, like $3 + \ln 7$ or e^2, are preferred.
3. Any calculator acceptable for the Provincial Examination in Principles of Mathematics 12 may be used.
4. A basic formula sheet has been provided. No other notes, books, or aids are allowed. In particular, all calculator memories must be empty when the exam begins.
5. If you need more space to solve a problem on page n, work on the back of page n - 1.
6. CAUTION - Candidates guilty of any of the following or similar practices shall be dismissed from the examination immediately and assigned a grade of 0:
   (a) Using any books, papers or memoranda.
   (b) Speaking or communicating with other candidates.
   (c) Exposing written papers to the view of other candidates.

Question	Maximum	Score
1	8	
2	6	
3	8	
4	6	
5	6	
6	6	
7	8	
8	8	
9	8	
10	8	
11	6	
12	8	
13	8	
14	6	
Total	100	

1. For each of the following evaluate the limit if it exists and explain why it does not otherwise.

[2] (a) $\lim_{x\to 1} \dfrac{x^3-1}{x-1}$

ANSWER:

JUSTIFY YOUR ANSWER

[3] (b) $\lim_{x\to 0} \dfrac{|x|}{x}$

ANSWER:

JUSTIFY YOUR ANSWER

[3] (c) $\lim_{x\to 1-} \dfrac{1-x}{(\pi/2)-\sin^{-1} x}$

ANSWER:

JUSTIFY YOUR ANSWER

2. Let $f(x) = \dfrac{\sec x - \tan x}{\sec x + \tan x}$.

[4] (a) Find an expression for $f'(x)$

ANSWER:

[2] (b) Simplify $f'(x) \,/\, f(x)$.

ANSWER:

JUSTIFY YOUR ANSWER

3. Let l_t denote the tangent line to the parabola $y = -x^2$ at the point $(t, -t^2)$.

[4] (a) Find the equation of l_t.

ANSWER:

EXPLANATION

[2] (b) Assuming that l_t meets the hyperbola $xy = 1$ in two points find the x-coordinates of those points.

ANSWER:

EXPLANATION

[2] (c) Find a value of t such that the tangent line l_t is also tangent to $xy = 1$.

ANSWER:

EXPLANATION

[6] **4.** Consider the curve whose equation is

$$2(x^2 + y^2)^2 = 25xy.$$

Find all points of the curve at which $\dfrac{dy}{dx} = 0$.

ANSWER:

SHOW YOUR WORK

[3] **5.** (a) Find the general antiderivative of $\dfrac{1}{1+4x^2}$.

ANSWER:

SHOW YOUR WORK

[3] (b) Find a function y defined on $(0,\infty)$ such that

$$\frac{dy}{dx} = 2x + \frac{1}{x^2}, \;\; y(1) = 0.$$

ANSWER:

SHOW YOUR WORK

[6] **6.** A spherical planet has radius R and a narrow hole bored along a diameter. An object falling towards the centre through the hole satisfies the differential equation

ANSWER:

$$\frac{d^2x}{dt^2} = -c^2x$$

where c is constant, x the distance from the centre, and t the elapsed time.

Initially,i.e. at $t = 0$, the object is at rest at the surface of the planet.

The motion of the falling object is given by an equation

$$x = A\cos ct + B\sin ct\,.$$

Given that $R = 2 \times 10^7$ and $c = 4 \times 10^{7/2}$, compute the values of the constants A, B from the initial data.

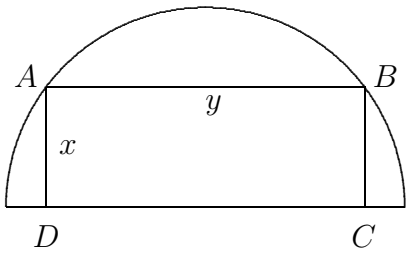
Show that the time taken for the object to fall to the centre is $\pi/2c$.

EXPLANATION

[8] **7.** A rectangle $ABCD$ with sides of length x, y is inscribed in a semicircle of radius 1 as shown in the figure.

The rectangle is chosen so that its perimeter $2x+2y$ is as large as possible.

Show that $xy = 4/5$.



EXPLANATION

8. Let $f(x) = \sqrt[4]{10^{100} + x} - 10^{25}$.

[4] (a) Give the best possible linear estimate for the function $f(x)$ for small values of x, i.e., for x close to 0.

A *linear estimate* is a function $Ax + B$, where A and B are constants to be determined.

ANSWER:

SHOW YOUR WORK

[4] (b) Is your linear estimate an understimate (meaning $Ax+B \leq f(x)$ whenever $|x| < 1$), an overestimate (meaning $Ax + B \geq f(x)$ whenever $|x| < 1$), or neither?

Explain your answer using the Mean Value Theorem or any other theorem known to you which fits the context.

ANSWER

[8] **9.** Newton's law of cooling may be stated as:

$$\frac{dT}{dt} = kT, \tag{1}$$

where T is the *difference between* the temperature of a specified object at time t and the ambient temperature (usually a constant), t is the elapsed time, and k is a constant which depends on the particular case.

A kettle initially at 100° (Celsius) cools to 80° in one minute. The ambient temperature is constant at 20°.

Show that the time taken for the kettle to cool from 80° to 60° is:

$$\ln(2/3)\,/\ln(3/4) \approx 1.4 \text{ minutes}$$

EXPLANATION

10. Let $f(x) = x^3(x-1)^2$.

[4] (a) Find the largest interval on which $f(x)$ is decreasing.

ANSWER:

SHOW YOUR WORK

[4] (b) Find all points of inflection of the function $f(x)$.

ANSWER:

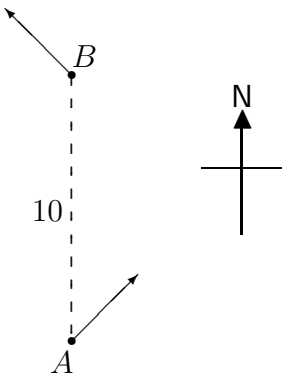
SHOW YOUR WORK

[6] **11.** At time $t = 0$ plane A is traveling NW at 500 mph while plane B, 10 miles north of plane A, is traveling NE at 500 mph.

The planes are travelling with constant speed and direction.

Let $f(t)$ denote the distance between the planes so that $f(0) = 10$.

Show that $f'(0) = 0$.



SHOW YOUR WORK

[3] **12.** (a) Let $f(x)$ be a function whose domain includes $(0, \infty)$ and $c > 0$. Define $f'(c)$ as a limit.

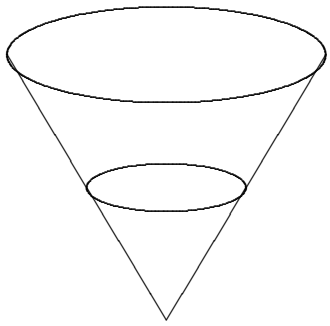
DEFINITION

[5] (b) Using only the limit laws and the definition given in part (a) show that, if $f(x) = 1/x$, then

$$f'(c) = -1/c^2 \qquad (c > 0).$$

EXPLANATION

13. A water tank is in the shape of an inverted right circular cone. It has diameter 6 feet at the top and height 4 feet. The tank is being filled with water at a rate of 10 cubic feet per minute . A leaf is floating in the centre of the tank and moving along the axis of the tank as the water level rises.



[4] (a) What is the velocity of the leaf when the water level is 2 feet?

ANSWER:

SHOW YOUR WORK

[4] (b) What is the acceleration of the leaf when the water level is 2 feet?

ANSWER:

SHOW YOUR WORK

[6] **14.** Compute the area enclosed between the parabolas $y = 1 - x^2$ and $y = x^2 - 1$.

Note that these parabolas are mirror images of each other in the x-axis.

ANSWER:

SHOW YOUR WORK