**This examination has 15 pages including this cover.**

# UBC-SFU-UVic-UNBC Calculus Exam Solutions

# 7 June 2007

**Name:** ____________________ **Signature:** ____________________

**School:** ____________________ **Candidate Number:** ____________________

**Rules and Instructions**

1. *Show all your work!* Full marks are given only when the answer is correct, and is supported with a written derivation that is orderly, logical, and complete. Part marks are available in every question.
2. Calculators are optional, not required. Correct answers that are "calculator ready," like $3 + \ln 7$ or $e^{\sqrt{2}}$, are fully acceptable.
3. Any calculator acceptable for the Provincial Examination in Principles of Mathematics 12 may be used.
4. A basic formula sheet has been provided. No other notes, books, or aids are allowed. In particular, *all calculator memories must be empty when the exam begins.*
5. If you need more space to solve a problem on page $n$, work on the back of page $n - 1$.
6. CAUTION - Candidates guilty of any of the following or similar practices shall be dismissed from the examination immediately and assigned a grade of 0:
   (a) Using any books, papers or memoranda.
   (b) Speaking or communicating with other candidates.
   (c) Exposing written papers to the view of other candidates.
7. Do not write in the grade box shown to the right.

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[4] **1.** Find an equation for the line tangent to the following curve at the point where $x = 1$:

$$y = \frac{6x - 2/x}{x^2 + \sqrt{x}}.$$

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At the point where $x = 1$, one has $y = 2$. By the quotient rule

$$y' = \frac{\left[6 + 2/x^2\right](x^2 + \sqrt{x}) - (6x - 2/x)\left[2x + \frac{1}{2}x^{-1/2}\right]}{(x^2 + \sqrt{x})^2}.$$

Substituting $x = 1$ gives the slope of the desired line:

$$m = y'(2) = \frac{[6 + 2](2) - (6 - 2)\left[2 + \frac{1}{2}\right]}{4} = \frac{3}{2}.$$

Using the point $(1, 2)$ and the slope $3/2$, we have the line's equation:

$$y = 2 + \frac{3}{2}(x - 1) = \frac{3}{2}x + \frac{1}{2}.$$

[6] **2.** Find an equation for the line tangent to the following curve at the point $(2, 0)$:

$$y = 2 + x - x^2 - \sin(xy).$$

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Taking the derivative of both sides with respect to $x$ gives

$$\frac{dy}{dx} = 0 + 1 - 2x - \cos(xy)\left[y + x\frac{dy}{dx}\right].$$

Plugging in $(x, y) = (2, 0)$ and writing $m$ for the corresponding value of $dy/dx$, we get

$$m = 1 - 4 - [0 + 2m], \qquad \text{i.e.,} \qquad 3m = -3, \qquad \text{i.e.,} \qquad m = -1.$$

The desired tangent line has equation

$$y = 0 - (x - 2), \qquad \text{i.e.,} \qquad y = -x + 2.$$

Continued on page 4

[6] **3.** Find the derivatives of the three functions below. Do not simplify.

$$a(x) = \tan\left(x + e^{-x}\right)$$

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$$a'(x) = \sec^2(x + e^{-x})[1 - e^{-x}] = \frac{1 - e^{-x}}{\cos^2(x + e^{-x})}.$$

$$b(x) = e^{x^2} \tan^{-1}\left(x^3 - x\right)$$

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$$b'(x) = \left[2xe^{x^2}\right] \tan^{-1}\left(x^3 - x\right) + e^{x^2} \left[\frac{3x^2 - 1}{1 + (x^3 - x)^2}\right].$$

$$c(x) = \frac{\sin^{-1}(x)}{\sin^{-1}(2x)}$$

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$$c'(x) = \frac{\frac{1}{\sqrt{1 - x^2}} \left[\sin^{-1}(2x)\right] - \left[\sin^{-1}(x)\right] \frac{2}{\sqrt{1 - 4x^2}}}{\left(\sin^{-1}(2x)\right)^2}.$$

Continued on page 5

[6] **4.** For each limit below, find the exact value (with justification) or explain why the limit does not exist.

(a) $\displaystyle\lim_{x\to-\infty}\left(x+\sqrt{x^2-4x}\right)$.

---

When $x<0$, $\sqrt{x^2}=|x|=-x$. Thus

$$\begin{aligned}A &= \lim_{x\to-\infty}\left(x+\sqrt{x^2-4x}\right)\left(\frac{x-\sqrt{x^2-4x}}{x-\sqrt{x^2-4x}}\right) = \lim_{x\to-\infty}\left(\frac{x^2-[x^2-4x]}{x-\sqrt{x^2-4x}}\right)\\ &= \lim_{x\to-\infty}\left(\frac{4x}{x-(-x)\sqrt{1-4/x}}\right) = \lim_{x\to-\infty}\left(\frac{4}{1+\sqrt{1-4/x}}\right) = \frac{4}{2} = 2.\end{aligned}$$

(b) $\displaystyle\lim_{x\to3}\left(\frac{1}{x-3}-\frac{6}{x^2-9}\right)$.

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$$B = \lim_{x\to3}\left(\frac{1}{x-3}\frac{(x+3)}{(x+3)}-\frac{6}{x^2-9}\right) = \lim_{x\to3}\frac{(x+3)-6}{(x-3)(x+3)} = \lim_{x\to3}\left(\frac{1}{x+3}\right) = \frac{1}{6}.$$

[6] **5.** Use the definition of the derivative as a limit to find $f'(x)$, given

$$f(x)=\frac{x}{1+x}.$$

(Finding $f'(x)$ using differentiation rules will earn no marks, but it could help you check your work with limits.)

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$$\begin{aligned}f'(x) &= \lim_{h\to0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to0}\frac{1}{h}\left\{\frac{x+h}{1+(x+h)}-\frac{x}{1+x}\right\}\\ &= \lim_{h\to0}\frac{1}{h}\left\{\frac{x+h}{1+x+h}\left(\frac{1+x}{1+x}\right)-\left[\frac{1+x+h}{1+x+h}\right]\frac{x}{1+x}\right\}\\ &= \lim_{h\to0}\frac{1}{h}\left\{\frac{x+h+x^2+hx-x-x^2-hx}{[1+x+h](1+x)}\right\}\\ &= \lim_{h\to0}\frac{1}{h}\left\{\frac{h}{[1+x+h](1+x)}\right\} = \frac{1}{(1+x)^2}.\end{aligned}$$

Continued on page 6

[6] **6.** Let $f(t) = \dfrac{e^{rt} - e^{-rt}}{e^{rt} + e^{-rt}}$ where $r > 0$ is a constant.

(a) Derive the "small-$t$ approximation"

$$f(t) \approx rt \qquad \text{for } t \approx 0.$$

(b) Explain why the approximate value $rt$ is greater than the exact value $f(t)$ whenever $t > 0$. What happens when $t < 0$?

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(a) The basic idea of differential calculus is the tangent-line approximation:

$$f(t) \approx f(0) + f'(0)t \qquad \text{for } t \approx 0. \tag{$**$}$$

For the given function $f$, we have $f(0) = 0$ and (by the quotient rule)

$$f'(t) = \frac{[re^{rt} + re^{-rt}](e^{rt} + e^{-rt}) - (e^{rt} - e^{-rt})[re^{rt} - re^{-rt}]}{(e^{rt} + e^{-rt})^2}.$$

Plugging in $t = 0$ reveals $f'(0) = r$, so the desired statement is an instance of $(**)$.

(b) Expanding the numerator above allows massive cancellation:

$$\begin{aligned} f'(t) &= r\frac{(e^{rt} + e^{-rt})^2 - (e^{rt} - e^{-rt})^2}{(e^{rt} + e^{-rt})^2} \\ &= r\frac{\left(e^{2rt} + 2 + e^{-2rt}\right) - \left(e^{2rt} - 2 + e^{-2rt}\right)}{(e^{rt} + e^{-rt})^2} \\ &= 4r(e^{rt} + e^{-rt})^{-2}. \end{aligned}$$

This makes it rather easy to find

$$f''(t) = -8r(e^{rt} + e^{-rt})^{-3}\left[re^{rt} - re^{-rt}\right] = -8r^2\left[\frac{e^{rt} - e^{-rt}}{(e^{rt} + e^{-rt})^3}\right].$$

When $r > 0$ and $t > 0$, we have $rt > 0$ so $e^{rt} > 1 > e^{-rt}$. Thus the bracketed ratio above is positive valued, and we have $f''(t) < 0$ whenever $t > 0$. Thus $f$ is concave down on the interval $[0, +\infty)$, and therefore the exact value $f(t)$ is less than its tangent-line approximation throughout this interval. Similarly, $f$ is concave up on the interval $(-\infty, 0]$, so its exact values are larger than the values predicted by its tangent lines in that interval.

Continued on page 7

[8] **7.** A conical reservoir with an open top and vertex down holds water for a desert community. The reservoir is 6 metres deep at its centre; its diameter at the top is 10 metres. The rate at which water evaporates from the reservoir is proportional to the area of the water's top surface (a circular disk). When the water is 5 metres deep, its depth is decreasing at the instantaneous rate of 2 cm per day. Find, with suitable units, ...

(a) the rate of change of depth when there is 3 metres of water in the reservoir, and

(b) the rate of change of volume when there is 3 metres of water in the reservoir.

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Draw a picture (please!) and name some unknowns:

$h$ = depth of water in the reservoir, metres,

$r$ = radius of the water's top surface, metres,

$A$ = area of the water's top surface, square metres,

$V$ = volume of water in the reservoir, cubic metres,

$H$ = total depth of reservoir as built, metres,

$R$ = radius of reservoir's top surface as built, metres.

Famous geometric formulas, and an argument with similar triangles, give three useful facts:

$$(1)\quad V = \frac{1}{3}\pi r^2 h, \qquad (2)\quad A = \pi r^2, \qquad (3)\quad \frac{r}{h} = \frac{R}{H} = \frac{5}{6}.$$

Since the radius is not mentioned anywhere in the question, we use (3) to eliminate $r$:

$$r = \frac{5}{6}h \quad \Longrightarrow \quad (1')\quad V = \frac{1}{3}\left(\frac{25\pi}{36}\right)h^3, \qquad (2')\quad A = \frac{25\pi}{36}h^2.$$

(a) Evaporation causes a loss of volume. The proportionality statement means that, for some constant $c$,

$$\frac{dV}{dt} = -cA, \qquad \text{i.e.,} \qquad \left(\frac{25\pi}{36}\right)h^2\frac{dh}{dt} = -c\frac{25\pi}{36}h^2. \qquad (*)$$

This identity is valid for all times: cancellation gives

$$\frac{dh}{dt} = -c.$$

That is, the water level in the reservoir decreases as a constant rate. This rate is 2 cm/day no matter what the depth is, so this value applies in particular when the depth is 3 metres.

(b) Convert $c$ to 0.02 m/day and use it in line $(*)$ above:

$$\frac{dV}{dt} = -cA = -\left(\frac{2}{100}\ \frac{\text{m}}{\text{day}}\right)\left(\frac{25\pi}{36}\ (3\text{m})^2\right) = -\frac{\pi}{8}\ \frac{\text{m}^3}{\text{day}}.$$

At the given instant, the volume is decreasing at $\pi/8$ m$^3$ per day.

Continued on page 8

[7] **8.** The position of a moving particle at time $t$ is $x = s(t)$, where

$$s(t) = \ln(1+t) - \frac{t}{1+t}.$$

(a) Express the particle's velocity as a function of $t$. Simplify your answer.

(b) Explain why $s(t) > 0$ whenever $t > 0$.

(c) Find the particle's maximum velocity, and the time when it occurs.

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(a) The particle's velocity at time $t$ is the rate of change of its position:

$$v(t) = \frac{ds}{dt} = \frac{1}{1+t} - \frac{(1+t)-t}{(1+t)^2} = \frac{1}{1+t} - \frac{1}{(1+t)^2} = \frac{1+t}{(1+t)^2} - \frac{1}{(1+t)^2} = \frac{t}{(1+t)^2}.$$

(b) Clearly $s'(t) = v(t) > 0$ whenever $t > 0$, so function $s$ is increasing on the interval $[0, +\infty)$. Its minimum value on that interval must occur at the left endpoint, where $s(0) = \ln(1) = 0$. Thus $s(t) > s(0) = 0$ for all $t > 0$.

(c) The instant when velocity is maximized must obey $v'(t) = 0$. So we calculate

$$v'(t) = \frac{(1+t)^2 - t \cdot 2(1+t)}{(1+t)^4} = \frac{(1+t)-2t}{(1+t)^3} = \frac{1-t}{(1+t)^3}.$$

The critical point $t = 1$ provides an absolute maximum for the function $v$, because $v'(t) > 0$ when $-1 < t < 1$ and $v'(t) < 0$ when $t > 1$. The particle's maximum velocity is

$$v_{\max} = v(1) = \frac{1}{4}.$$

(Points where $t \leq -1$ can be ignored because they lie outside the domain of $s$.)

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[5] **9.** Let $f(x) = \left(1 + \dfrac{3}{x}\right)^x$.

By using your findings from the previous question, or otherwise, ...

(a) Show that $f$ is increasing on the interval $x > 0$.

(b) Find the exact value of $\lim_{x\to\infty} f(x)$.

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(a) One effective way to show that $f$ is increasing on an interval is to prove that $f'(x) > 0$ there. Here $\ln(f(x)) = x \ln\left(1 + \dfrac{3}{x}\right)$, so differentiation gives

$$\frac{f'(x)}{f(x)} = \ln\left(1 + \frac{3}{x}\right) + x\left(\frac{-3/x^2}{1+3/x}\right) = \ln\left(1 + \frac{3}{x}\right) - \left(\frac{3/x}{1+3/x}\right).$$

In terms of the function $s$ from question 8, $f'(x) = f(x)s(3/x)$. Clearly $f(x) > 0$ whenever $x > 0$, and we showed earlier that $s(t) > 0$ whenever $t > 0$. These facts together imply $f'(x) > 0$ whenever $x > 0$, so function $f$ is increasing on the interval $(0, +\infty)$.

(b) The idea behind logarithmic differentiation works here, too. Recognizing the definition of derivative completes the calculation:

$$\begin{aligned}
\lim_{x\to\infty} \ln(f(x)) &= \lim_{x\to\infty} \frac{\ln(1+3/x)}{1/x} \\
&= 3 \lim_{h\to 0} \frac{\ln(1+h) - \ln(1)}{h/3} \qquad \text{(let } h = 3/x\text{)} \\
&= 3 \left.\frac{d}{dx}\ln(x)\right|_{x=1} \\
&= 3\left[\frac{1}{x}\right]_{x=1} = 3.
\end{aligned}$$

This shows that $\ln(f(x)) \to 3$ as $x \to \infty$. Hence $f(x) \to e^3$ as $x \to \infty$.

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[8] **10.** A cup of coffee at 96°C is set on a table in an air-conditioned classroom. It cools to 60°C in 10 minutes, and then to 40°C in another 10 minutes. What is the temperature of the room?

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Let $A$ denote the ambient temperature we seek. To find $A$, let $u(t)$ be the coffee temperature at time $t$, and recall Newton's law of cooling: the temperature difference $y(t) \stackrel{\text{def}}{=} u(t) - A$ obeys

$$\frac{dy}{dt} = ky(t)$$

for some constant $k$. It follows that, for some constant $C$,

$$u(t) - A = y(t) = Ce^{kt}.$$

The problem statement gives relevant information for three different $t$-values:

$$\begin{array}{llllll} (1) & C & = & u(0) - A & = & 96 - A; \\ (2) & Ce^{10k} & = & u(10) - A & = & 60 - A; \\ (3) & Ce^{20k} & = & u(20) - A & = & 40 - A. \end{array}$$

Using (1) to eliminate $C$ reduces (2)–(3) to the pair of equations

$$(96 - A)e^{10k} = 60 - A, \qquad (96 - A)e^{20k} = 40 - A.$$

Recognizing $e^{20k} = (e^{10k})^2$ then gives

$$\frac{40 - A}{96 - A} = \left(\frac{60 - A}{96 - A}\right)^2, \qquad \text{i.e.,} \qquad (40 - A)(96 - A) = (60 - A)^2.$$

Expanding this equation and simplifying the result leads to

$$3840 - 136A + A^2 = 3600 - 120A + A^2, \qquad \text{i.e.,} \qquad 240 = 16A \qquad \text{i.e.,} \qquad 15 = A.$$

The ambient temerature is 15°C.

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[10] **11.** Let $f(x) = x^2\sqrt{24 - x^2}$.

(i) Find the domain of $f$.

(ii) Find the intervals in which $f$ is increasing and decreasing.

(iii) Find the absolute maximum and minimum values for $f$ on its domain, and all the points where these are attained.

(iv) Find the $x$-coordinates of all inflection points for $f$.

(v) Sketch the graph of $f$. (A pair of axes is supplied on the next page.)

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(i) The value $f(x)$ is defined if and only if $24 - x^2 \geq 0$, i.e., $x^2 \leq 24$. So the domain of $f$ is the closed interval $[-\sqrt{24}, \sqrt{24}]$.

(ii) The product rule gives

$$f'(x) = 2x\sqrt{24 - x^2} + x^2\left(\frac{-2x}{2\sqrt{24 - x^2}}\right) = \frac{2x(24 - x^2)}{\sqrt{24 - x^2}} - \frac{x^3}{\sqrt{24 - x^2}}$$
$$= \frac{48x - 3x^3}{\sqrt{24 - x^2}} = \frac{3x(4 - x)(4 + x)}{\sqrt{24 - x^2}}.$$

Thus $f'(x) > 0$ iff $-\sqrt{24} < x < -4$ or $0 < x < 4$. Since $f$ is continuous at the endpoints of these intervals,

$f$ is increasing on the intervals $[-\sqrt{24}, -4]$ and $[0, 4]$.

Likewise, $f'(x) < 0$ iff $-4 < x < 0$ or $4 < x < \sqrt{24}$. Since $f$ is continuous at the endpoints of these intervals,

$f$ is decreasing on the intervals $[-4, 0]$ and $[4, \sqrt{24}]$.

(iii) Absolute extrema may happen only at critical points (CP), singular points (SP), or endpoints of the domain (EP). Here there are no SP's, three CP's ($x = 0, \pm 4$), and two EP's ($x = \pm\sqrt{24}$). In this case, each one gives an absolute extremum of some kind:

abs min: $f_{\text{min}} = 0$, attained at $x = 0, \pm\sqrt{24}$,
abs max: $f_{\text{max}} = 32\sqrt{2}$, attained at $x = \pm 4$.

(One has $32\sqrt{2} \approx 45.255$.)

(iv) Using the quotient rule, we find

$$f''(x) = 3\frac{d}{dx}\frac{16x - x^3}{\sqrt{24 - x^2}} = 3\left(\frac{[16 - 3x^2]\sqrt{24 - x^2} - [16x - x^3]\frac{-2x}{2\sqrt{24 - x^2}}}{24 - x^2}\right)$$
$$= 3\left(\frac{[16 - 3x^2](24 - x^2) + [16x - x^3]x}{(24 - x^2)^{3/2}}\right)$$
$$= \frac{6\left(x^4 - 36x^2 + 192\right)}{(24 - x^2)^{3/2}}$$

This has zeros at points where

$$x^2 = \frac{36 \pm \sqrt{36^2 - 4(192)}}{2} = 18 \pm \sqrt{132} = 18 \pm 2\sqrt{33}. \qquad (*)$$

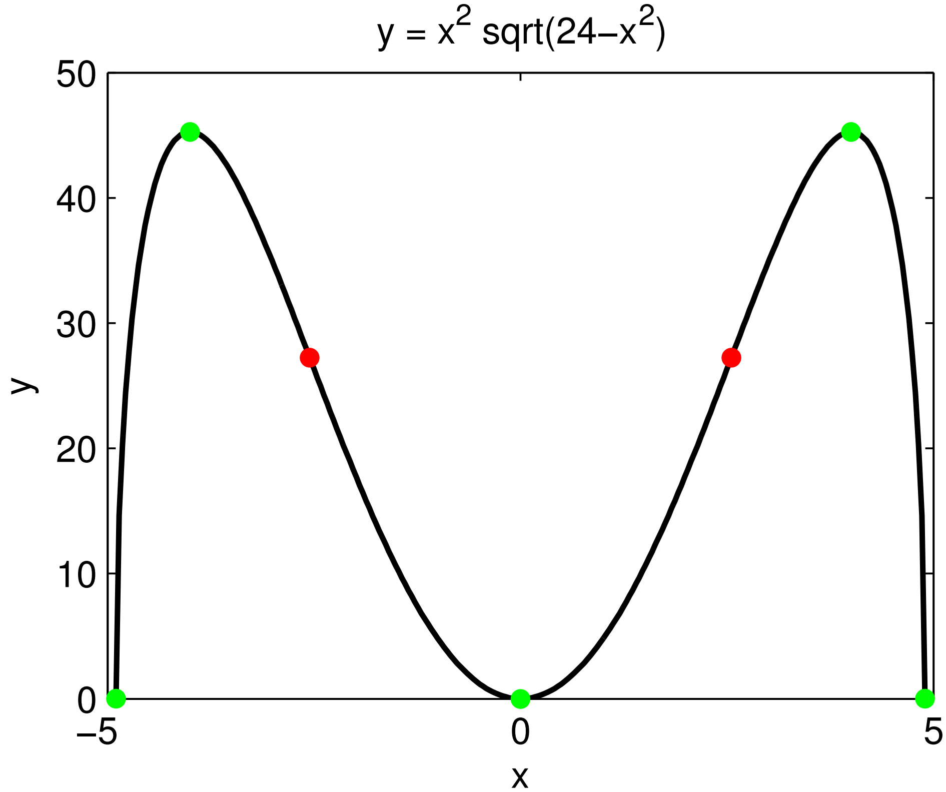
Continued on page 12

Notice that $18+2\sqrt{33} > 18+2\sqrt{25} = 28 > 24 = (\sqrt{24})^2$, so choosing the "+" sign in $(*)$ generates $x$-values outside the domain of $f$. It follows that there are just two inflection points, and their $x$-coordinates are

$$x = \pm\sqrt{18-2\sqrt{33}} \approx \pm 2.5516.$$

(The corresponding $y$-coordinates are about 27.228.)

(v) In the sketch below, a green dot highlights each local extremum, and a red dot is shown at each inflection point.

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[6] **12.** The nonlinear equation $x^{-1}\sin(x) = \cos(x)$ has a solution near the point $x_0 = 3\pi/2$. Use the tangent lines at $x = x_0$ to the two curves

$$y = x^{-1}\sin(x), \qquad y = \cos(x)$$

to find a better approximation (call it $x_1$) to the solution near $x_0$.

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Let $f(x) = \dfrac{\sin x}{x}$. The quotient rule gives

$$f'(x) = \frac{(\cos x)x - \sin x}{x^2}, \qquad \text{so} \qquad f'(x_0) = f'(3\pi/2) = \frac{0-(-1)}{(3\pi/2)^2} = \frac{4}{9\pi^2}.$$

The line tangent to the curve $y = f(x)$ at $(x_0, f(x_0))$ is

$$y = f(x_0) + f'(x_0)[x - x_0] = \frac{-1}{3\pi/2} + \frac{4}{9\pi^2}[x - 3\pi/2] = -\frac{2}{3\pi} + \frac{4}{9\pi^2}\left[x - \frac{3\pi}{2}\right]. \tag{1}$$

Next, let $g(x) = \cos x$. Then $g'(x) = -\sin x$, so the line tangent to the curve $y = g(x)$ at $(x_0, g(x_0))$ is

$$y = g(x_0) + g'(x_0)[x - x_0] = 0 + 1[x - (3\pi/2)] = x - \frac{3\pi}{2}. \tag{2}$$

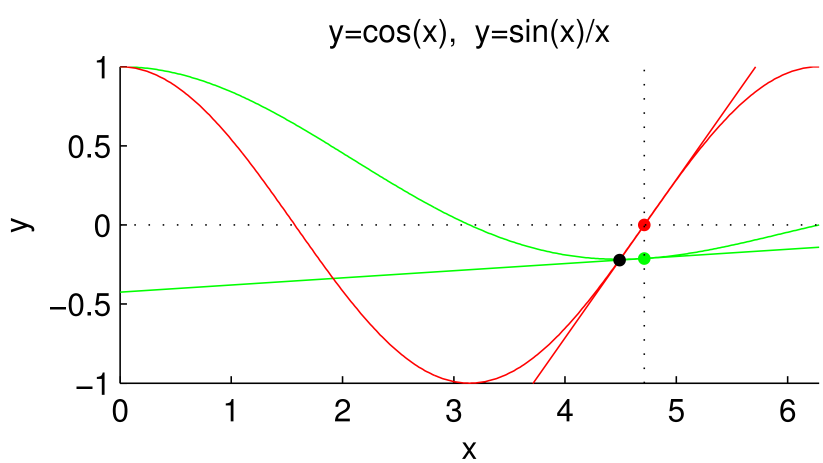
The point of intersection for the tangent lines in (1) and (2) above can be found by using (2) to eliminate $y$ from (1):

$$x - \frac{3\pi}{2} = -\frac{2}{3\pi} + \frac{4}{9\pi^2}\left[x - \frac{3\pi}{2}\right] \implies \left[x - \frac{3\pi}{2}\right] = \frac{(2/(3\pi))}{\frac{4}{9\pi^2} - 1} = \frac{6\pi}{4 - 9\pi^2}.$$

This gives the improved estimate

$$x_1 = \frac{\pi(24 - 27\pi^2)}{2(4 - 9\pi^2)}.$$

The following sketch (not required) shows the two given curves with their tangent lines at $x = 3\pi/2$ and the point $(x_1, y_1) \approx (4.49018, -0.22221)$ where the tangent lines meet. (The curves meet when $x \approx 4.49341$, accurate to six digits.)

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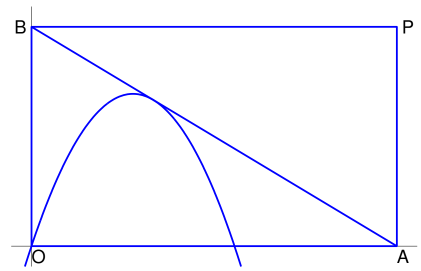
[8] **13.** Every tangent line of negative slope for the curve $y = 3x - x^2$ can be used to construct a rectangle, as shown below: put one corner at the origin ($O$), one at the line's $x$-intercept ($A$), one at the line's $y$-intercept ($B$), and one in the first quadrant to complete the figure ($P$). Find the coordinates of $P$ for the rectangle of *smallest perimeter* that can be constructed this way. (The sketch shows the construction, but not the minimizing configuration.)

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For $y = 3x - x^2$ we have $y' = 3 - 2x$. Every tangent line "of negative slope" has a point of tangency obeying $x > 3/2$.

Suppose the point of tangency is $x = z$. Then the equation of the tangent line is

$$y = (3z - z^2) + (3 - 2z)[x - z] = (3 - 2z)x + z^2.$$

This line's $y$-intercept (set $x = 0$) is $B = z^2$. Its $x$-intercept (set $y = 0$) is $A = z^2/(2z - 3)$. So the perimeter of the constructed rectangle when the point of tangency is $x = z$ is

$$f(z) = 2A + 2B = \frac{2z^2}{2z - 3} + 2z^2, \qquad z > \frac{3}{2}.$$

To minimize this, calculate

$$\tfrac{1}{2}f'(z) = \frac{2z(2z - 3) - z^2(2)}{(2z - 3)^2} - 2z = 2z\left[\frac{z - 3}{(2z - 3)^2} - \frac{(2z - 3)^2}{(2z - 3)^2}\right] = \cdots = \frac{2z(z - 2)(4z - 3)}{(2z - 3)^2}.$$

There is only one critical point in the interval of interest, namely, $z = 2$.

Clearly $f'(z) < 0$ for $3/2 < z < 2$ and $f'(z) > 0$ for $z > 2$, so the point $z = 2$ gives an absolute minimum for $f$ over the interval $(3/2, +\infty)$.

The corresponding intercepts give the coordinates of the minimizing point $P$:

$$P = (A, B), \qquad \text{where} \qquad A = \left.\frac{z^2}{2z - 3}\right|_{z=2} = 4, \quad B = \left.z^2\right|_{z=2} = 4.$$

Continued on page 15

[8] **14.** Use the three properties below to identify the function $f$ and sketch its graph:

(i) $f(0) = 0$, and

(ii) the graph of $f$ has an inflection point at which the tangent line is horizontal, and

(iii) $f''(x) = 6x + 2$ for all $x$.

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Fact (iii) gives $f'(x) = 3x^2 + 2x + C$ for some $C$.

An inflection point for $f$ must obey $f''(x) = 0$, i.e., $x = -1/3$. Fact (ii) requires $f'(x) = 0$ at this point, so $C = 1/3$. Detail:

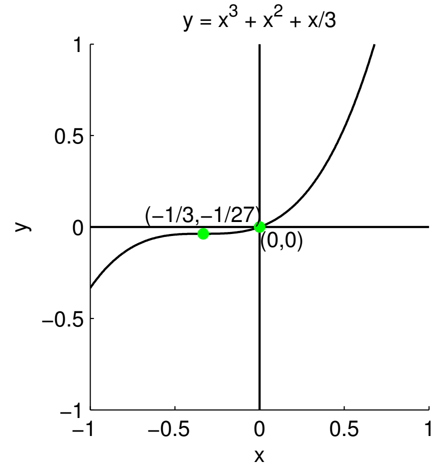
$$0 = f'(-1/3) = 3\left(\frac{-1}{3}\right)^2 + 2\left(-\frac{1}{3}\right) + C = C - \frac{1}{3}.$$

Thus $f'(x) = 3x^2 + 2x + 1/3$. This implies

$$f(x) = x^3 + x^2 + \frac{x}{3} + K$$

for some $K$. Fact (i) requires $0 = f(0) = K$, so

$$f(x) = x^3 + x^2 + \frac{x}{3}.$$




[6] **15.** Make a rough sketch of the following figures on the same set of axes:

$$C: \quad y = \frac{10}{1+x^2}, \qquad L: \quad y = 5.$$

Then find the area of the region lying below $C$ and above $L$.

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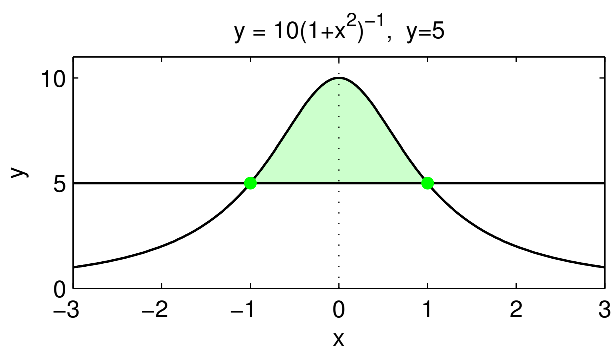
The curve $C$ and the line $L$ meet above the points where

$$\frac{10}{1+x^2} = 5, \qquad \text{i.e.,} \qquad 1 + x^2 = 2, \qquad \text{i.e.,} \qquad x = \pm 1.$$

Between these points, the curve $C$ lies above the line $L$. The desired area is

$$A = \int_{x=-1}^{1} \left(\frac{10}{1+x^2} - 5\right) dx = \left[10\tan^{-1} x - 5x\right]_{x=-1}^{1} = 2\left(10\tan^{-1}(1) - 5\right) = 5\pi - 10.$$




**The End**