Key for 2004 Calculus Challenge Exam
and Comments on How the Questions Were Answered

[3] **1.** Evaluate $\displaystyle\lim_{t\to 0} \frac{4-(t+2)^2}{t}$.

ANSWER:
-4

JUSTIFY YOUR ANSWER

METHOD 1. Note that $\dfrac{4-(t+2)^2}{t} = -4-t$ for all $t \neq 0$. Therefore

$$\lim_{t\to 0} \frac{4-(t+2)^2}{t} = \lim_{t\to 0} \; (-4-t) = -(\lim_{t\to 0} 4) - (\lim_{t\to 0} t) = -4-0.$$

Here we have used the following basic rules concerning limits: $\lim_{t\to a} -f(t) = -\lim_{t\to a} f(t)$, $\lim_{t\to a} (f(t)+g(t)) = \lim_{t\to a} f(t) + \lim_{t\to a} g(t)$, $\lim_{t\to a} t = a$, and $\lim_{t\to a} c = c$. ■

METHOD 2. One version of l'Hospital's Rule says that, if $\lim_{t\to a} f(t) = \lim_{t\to a} g(t) = 0$, then

$$\lim_{t\to a} \frac{f(t)}{g(t)} = \lim_{t\to a} \frac{f'(t)}{g'(t)}$$

provided that the limit on the right of the equation exists. Further, polynomials are continuous. Now

$$\frac{d}{dt}\, t = 1, \qquad \frac{d}{dt}\left(4-(t+2)^2\right) = -2(t+2)\,.$$

Therefore

$$\lim_{t\to 0} \frac{4-(t+2)^2}{t} = \lim_{t\to 0} \frac{-2(t+2)}{1} = \lim_{t\to 0} \; [-2(t+2)] = -2(0+2) = -4$$

where the first step uses L'Hospital's Rule and the second the continuity of polynomials. ■

COMMENTS

(a) Extrapolating from a table of values computed using a calculator is not a valid justification.

(b) Students had little difficulty with this problem. Most who lost marks made simple arithmetical errors such as:

$$4-(t+2)^2 = 4-(t^2+2t+4)$$
$$4-(t+2)^2 = 4-t^2+4t+4$$

[4] **2.** Find a constant k such that

$$y = 2x - kx^2$$

is a solution of the differential equation $xy' = y - x^2$.

ANSWER:

1

JUSTIFY YOUR ANSWER

Substituting $y = 2x - kx^2$ in the two sides of the differential equation we get LHS $= x(2 - 2kx) = 2x - 2kx^2$, and RHS $= 2x - kx^2 - x^2 = 2x - (k+1)x^2$. Note that the LHS and the RHS are the same function of x if and only if $2k = k + 1$, that is, if and only if $k = 1$. ■

COMMENTS

(a) It is not necessary to explain how the value of k is found. One might guess that $k = 1$ is the correct value. The justification should verify, implicitly or explicitly, that $y = 2x - x^2$ is a solution of $xy' = y - x^2$.

(b) Many students substituted either $y = 2x - kx^2$ or $y' = 2 - 2kx$ into the differential equation $xy' = y - x^2$, but not both. Why not?

(c) Many students seemed unaware that polynomials $p(x)$, $q(x)$ represent the same function from $\mathbb{R}$ into $\mathbb{R}$ if and only if for each integer $n \geq 0$ the coefficient of x^n in $p(x)$ is equal to the coefficient of x^n in $q(x)$.

[6] **3.** Let $f(x) = 1 - x^2$.

Working directly from the definition of the derivative as a limit, verify the formula:

$$f'(a) = -2a$$

SHOW YOUR WORK

The two definitions of derivative commonly used are:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Working with the first definion, note that $\dfrac{(1 - x^2) - (1 - a^2)}{x - a} = -a - x$ for all $x \neq a$. Therefore

$$\lim_{x \to a} \frac{(1 - x^2) - (1 - a^2)}{x - a} = \lim_{x \to a} (-a - x) = -(\lim_{x \to a} a) - (\lim_{x \to a} x) = -2a\,.$$

Working with the second definion, note that $\dfrac{(1 - (a+h)^2) - (1 - a^2)}{h} = -2a - h$ for all $h \neq 0$. Therefore

$$\lim_{h \to 0} \frac{(1 - (a+h)^2) - (1 - a^2)}{h} = \lim_{h \to 0} (-2a - h) = -2(\lim_{h \to 0} a) - (\lim_{h \to 0} h) = -2a\,. \qquad \blacksquare$$

COMMENTS

(a) Using L'Hospital's Rule to evaluate the limit is not acceptable, because, in essence, it requires taking the derivative of the function whose derivative one is trying to compute from first principles.

(b) Generally, this question was very well done.

[4] **4.** Let $f(x)$ denote the function defined by $f(x) = \begin{cases} \dfrac{1-e^x}{x} & \text{if } x \neq 0 \\ -1 & \text{if } x = 0. \end{cases}$

Show that $f(x)$ is continuous at $x = 0$.

EXPLANATION

Half of the battle is stating that what one has to show is that $\lim_{x\to 0} \dfrac{1-e^x}{x} = -1$ because this is the same as $\lim_{x\to 0} f(x) = f(0)$.

What remains is to verify that $\lim_{x\to 0} \dfrac{1-e^x}{x} = -1$. There are three ways of proceeding:

METHOD 1. Using l'Hospital's rule and the continuity of e^x, we have

$$\lim_{x\to 0} \frac{1-e^x}{x} = \lim_{x\to 0} \frac{-e^x}{1} = -e^{\lim_{x\to 0} x} = -e^0 = -1.$$

METHOD 2. Let $g(x) = -e^x$. Then $g(0) = -e^0 = -1$. We have

$$\lim_{x\to 0} \frac{1-e^x}{x} = \lim_{x\to 0} \frac{g(x)-g(0)}{x} = g'(0) = -1.$$

METHOD 3. The theory of linear approximation says that $e^x = 1 + x + o(x)$. Therefore

$$\lim_{x\to 0} \frac{1-e^x}{x} = \lim_{x\to 0} \frac{1-(1+x+o(x))}{x} = \lim_{x\to 0} \left(-1 - \frac{o(x)}{x}\right) = (-1) - \lim_{x\to 0} \frac{o(x)}{x} = -1 - 0. \quad \blacksquare$$

COMMENTS

(a) The question was generally well done. A common mistake was to attempt to show that $\dfrac{1-e^x}{x} = -1$ when $x = 0$.

(b) Estimating the value of the limit by computing a series of values of $\dfrac{1-e^x}{x}$ for small values of x received only partial credit.

[3] **5.** (a) Evaluate $\dfrac{d}{dx}\left(\dfrac{x^2-1}{x^2+1}\right)$ and simplify your answer.

ANSWER:
$$\frac{4x}{(x^2+1)^2}$$

SHOW YOUR WORK: One can either use the quotient rule

$$\frac{d}{dx}\left(\frac{x^2-1}{x^2+1}\right) = \frac{(x^2+1)(2x)-(x^2-1)(2x)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2},$$

or the product rule

$$\begin{aligned}\frac{d}{dx}\left[(x^2-1)(x^2+1)^{-1}\right] &= (x^2-1)\left[(-1)(x^2+1)^{-2}(2x)\right] + (2x)(x^2+1)^{-1} \\ &= \frac{(-2x)(x^2-1)+(2x)(x^2+1)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}\ .\end{aligned}$$

■

[3] (b) Evaluate $\dfrac{d}{dx}\left[\tan\left(e^x+1\right)\right]$.

ANSWER:
$$e^x\sec^2\left(e^x+1\right)$$

SHOW YOUR WORK: Using the rule for differentiating composite functions, we have

$$\frac{d}{dx}\tan\left(e^x+1\right) = \sec^2\left(e^x+1\right)\frac{d}{dx}\left(e^x+1\right) = e^x\sec^2\left(e^x+1\right).$$

■

[3] (c) Given that

$$\sqrt{x^2+y^2}+\sqrt{xy} = 3\left(1+\sqrt{2}\right)$$

find an expression for dy/dx in terms of x, y.

No simplification is necessary.

ANSWER:
$$\frac{-\left(\frac{x}{\sqrt{x^2+y^2}}+\frac{y}{2\sqrt{xy}}\right)}{\left(\frac{y}{\sqrt{x^2+y^2}}+\frac{x}{2\sqrt{xy}}\right)}$$

SHOW YOUR WORK: Differentiating implicitly, we get

$$\frac{x+y(dy/dx)}{\sqrt{x^2+y^2}}+\frac{y+x(dy/dx)}{2\sqrt{xy}} = 0. \tag{1}$$

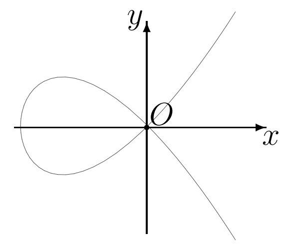
Solving for dy/dx we get the answer in the answer box. ■

COMMENT All three parts were well done. Some students had difficulty differentiating $\sqrt{x^2+y^2}$ and $\sqrt{xy}$. For example,

$$\frac{d}{dx}\sqrt{xy} = \frac{1}{2\sqrt{xy}}\frac{dy}{dx}\ .$$

6. Consider the curve $\mathcal{C}$ described by the equation:

$$y^2 = x^3 + x^2$$



[4] (a) Find the coordinates of the points of $\mathcal{C}$ at which the tangent lines are parallel to the x-axis.

ANSWER:
$\left(-2/3, \pm 2/(3\sqrt{3})\right)$

SHOW YOUR WORK

METHOD 1. Differentiating we get $2y(dy/dx) = 3x^2 + 2x$. So for $dy/dx = 0$ we need either $x = 0$ or $x = -2/3$. Now, as we see in part (b), $x = 0$ does not imply $dy/dx = 0$. However, $x = -2/3$ *does* because the corresponding values of y, $y = \pm 2/(3\sqrt{3})$, are nonzero. ∎

METHOD 2. Taking square roots in the given equation we see that the curve $\mathcal{C}$ is the union of two curves $\mathcal{C}^+$, $\mathcal{C}^-$ described by the respective equations:

$$y = x\sqrt{x+1}, \qquad y = -x\sqrt{x+1}\,. \tag{1}$$

A tangent to $\mathcal{C}$ is a tangent to at least one of $\mathcal{C}^+$, $\mathcal{C}^-$. Conversely, a tangent to one of $\mathcal{C}^+$, $\mathcal{C}^-$ is tangent to $\mathcal{C}$. Differentiating the equations (1), we get

$$\frac{dy}{dx} = \pm\left[\sqrt{x+1} + \frac{x}{2\sqrt{x+1}}\right] = \pm\left[\frac{x + (2/3)}{(2/3)\sqrt{x+1}}\right].$$

For a tangent parallel to Ox we need $dy/dx = 0$. Clearly, $dy/dx = 0$ if and only if $x = -2/3$. It is easy to see that the corresponding y-coordinates are $y = \pm 2/(3\sqrt{3})$. ∎

COMMENT

This part of the question was done well.

[4] (b) Find the equations of the tangents to $\mathcal{C}$ at the origin $(0,0)$ and justify your answer.

ANSWER:

$y = \pm x$

JUSTIFICATION:

METHOD 1. Taking square roots see that near $x = 0$ the curve consists of two parts $y = x\sqrt{x+1}$ and $y = -x\sqrt{x+1}$. The first branch of the curve has tangent $y = x$ at $x = 0$. The second branch of the curve has tangent $y = -x$ at $x = 0$. ■

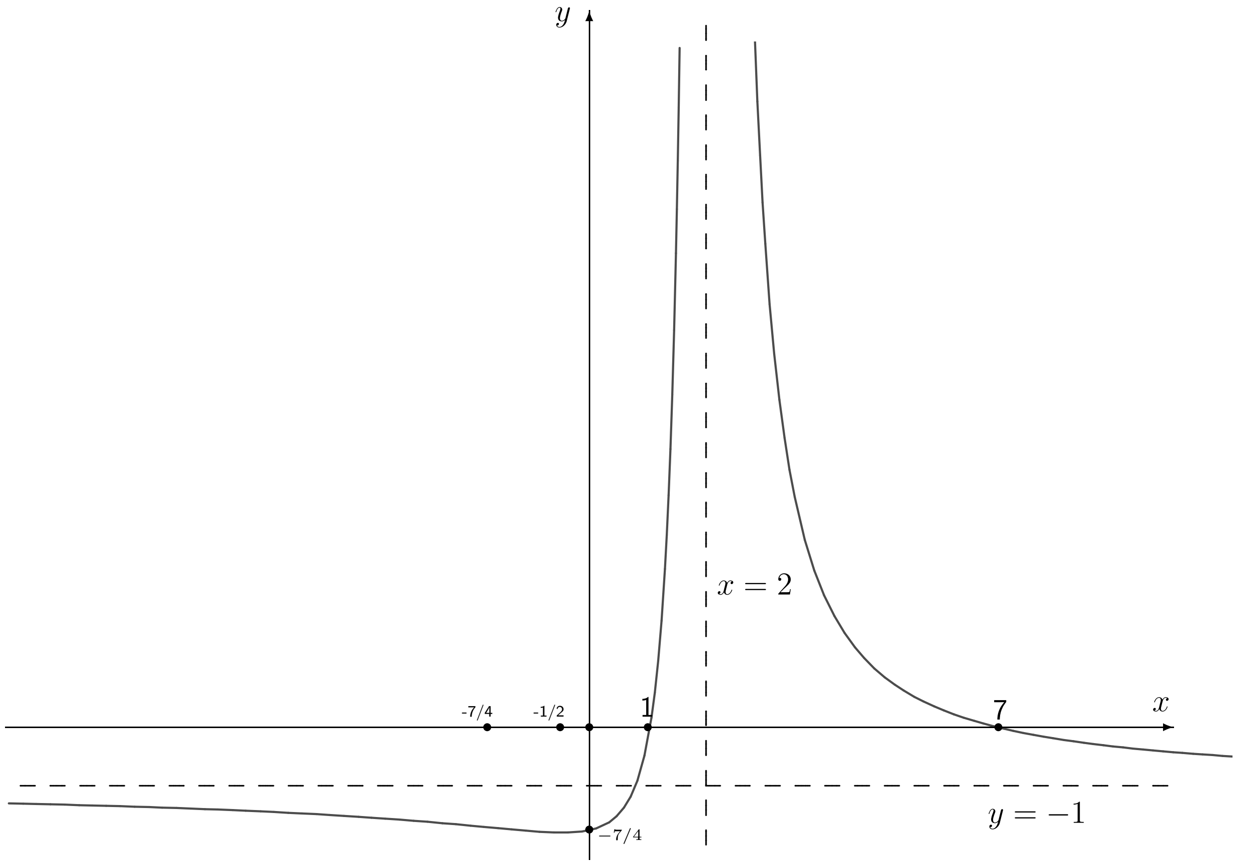
METHOD 2. For $|x|$ small, in the defining equation of the curve x^3 may be ignored in comparison with x^2 yielding the equation $y^2 = x^2$. Now $y^2 = x^2$ represents the pair of lines $y = x$, $y = -x$.

Thus the tangents to $\mathcal{C}$ at the origin $(0,0)$ are $y = x$, $y = -x$. ■

COMMENT

Although it was not intended as a trick question, this problem floored almost all who wrote the examination because it focused on a point at which the curve crosses itself, a point at which the slope has two distinct values.

The graph of the curve was shown. It was surprising that the existence of the two tangents at $(0,0)$ was not clear from the picture.



[6] **7.** The following information is given about the function f:

$$f(x),\ f'(x),\ f''(x) \text{ are defined and continuous for all } x \neq 2$$

$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = -1, \quad \lim_{x\to 2} f(x) = \infty$$

$$f(1) = f(7) = 0 \quad f'(-1/2) = 0 \quad f(0) = -7/4$$

$$1,\ 7 \text{ are the only zeros of } f(x);\ -1/2 \text{ is the only zero of } f'(x)$$

$$f''(x) < 0 \text{ for all } x \in (-\infty, -7/4),\ f''(-7/4) = 0$$

$$f''(x) > 0 \text{ for all } x \text{ in } (-7/4, 2) \text{ and all } x \text{ in } (2, \infty)$$

Using the axes provided above sketch the graph of $y = f(x)$ in a manner consistent with all the information.

No justification is required, but you may add comments if you wish.

COMMENTS

This question was done very well. Some students had difficulty with the shape of the curve on the interval $(-2, 1.5)$. The rest of the curve seemed to give no problem.

[6] **8.** (a) The sun is directly overhead at noon and sets at 6 p.m. Assuming that θ, the angle of elevation of the sun above the horizon, changes at a constant rate, show that, at 4 p.m., the length of the shadow cast by a 6 metre post is increasing at a rate of $\pi/30$ metres per minute.

EXPLANATION: The angle of θ of elevation is $\pi/2$ at noon and 0 at 6 p.m. Therefore $d\theta/dt = -(\pi/2)/360 = -\pi/720$, and at 4 p.m. $\theta = \pi/6$.

The length of the shadow is $L = 6\cot\theta$ metres. So its rate of increase is $dL/dt = -6\,\text{cosec}^2\theta(d\theta/dt) = -6(2^2)(-\pi/720) = \pi/30$ metres per minute. ■

COMMENTS

Common errors:

(a) Many students thought that $\theta = \pi/3$ at 4 p.m.

(b) Many wrote $dx/dt = \pi/30$ without specifying explicitly that this is what has to be shown. (No marks were deducted for this.)

In some cases students got the question right, but made heavy weather of what should be a very brief argument.

[6] (b) The pressure P, volume V, and temperature T of the gas in a spherical balloon of radius r are related by the universal gas equation

$$PV = nRT$$

where n is the number of moles of gas, and R is a constant. Here the temperature T is measured in Kelvins.

Let t be the elapsed time in hours. A variable x is said to be *increasing at a% per hour* if $\frac{1}{x}\frac{dx}{dt} = \frac{a}{100}$.

At the instant under consideration n is not changing, the temperature of the gas is increasing at 4% per hour, and the pressure of the gas is increasing at 1% per hour. Show that r, the radius of the balloon is also increasing at 1% per hour.

EXPLANATION:

There is a fair amount of variation possible in answering this question. One can take logarithms, or not, before differentiating. One can replace V by a constant times r^3 at the outset, or first compute the percentage rate of change of V and then that of r.

METHOD 1. Without logarithmic differentiation. We have

$$Pr^3 = CT \tag{1}$$

where C is constant with respect to t. Differentiating with respect to t, we get

$$r^3\frac{dP}{dt} + 3Pr^2\frac{dr}{dt} = C\frac{dT}{dt}\,.$$

Dividing this equation by (1) yields

$$\frac{1}{P}\frac{dP}{dt} + 3\frac{1}{r}\frac{dr}{dt} = \frac{1}{T}\frac{dT}{dt}\,. \tag{2}$$

Substituting $1/100$ for $(1/P)dP/dt$ and $4/100$ for $(1/T)dT/dt$, we see that $(1/r)dr/dt = 1/100$, which is enough. ■

METHOD 2. Starting with $Pr^3 = CT$, taking natural logarithms, and then differentiating with respect to t , we obtain (2). The rest is the same. ■

COMMENTS

This part was less well done than (a). Many students got partial credit for writing down something like

$$P'V + PV' = (nR)T'$$

but could not find the next step.

[4] **9.** (a) Find the linear approximation of the function $\sqrt[3]{x}$ at $x = 1000$ and use it to approximate $\sqrt[3]{1002}$.

ANSWER:
linearization: 10 + (1/300)(x-1000)

$\sqrt[3]{1002} \approx 10 + (2/300)$

SHOW YOUR WORK: The linearization of $f(x)$ at $x = a$ is $f(a) + (x - a)f'(a)$. Here $a = 1000$ and $f'(a) = (1/3)a^{-2/3} = 1/300$.

So $f(1002) \approx f(1000) + (2)f'(1000) = 10 + (2/300)$. ■

[4] (b) Beginning with the initial estimate $x_0 = 10$ apply one step of Newton's Method to find an estimate for a root of $x^3 - 1002 = 0$.

ANSWER:
$x_1 = 10 + (2/300)$

SHOW YOUR WORK: Given initial estimate $x_0 = a$, the next estimate to a root of $f(x) = 0$ is $x_1 = a - \dfrac{f(a)}{f'(a)}$. Take $f(x) = x^3 - 1002$ and $x_0 = a = 10$. Then $f(a) = -2$ and $f'(a) = 3a^2 = 300$. So $x_1 = 10 + (2/300)$. ■

COMMENT
Both parts were well done. On an earlier draft there was an additional part of this question asking students to explain why the answers to parts (a) and (b) are the same. The equality is part of a pattern rather than being just happenstance.

10. Let $f(x) = \frac{1}{2}x^4 - x^3 - 6x^2 + 4x$.

$x = a$ is called a *critical point of* $f(x)$ if $f'(x) = 0$.

[2] (a) Find all the critical points of $f(x)$ given that one of them is $x = -2$.

ANSWER:

$$-2,\ \frac{7 \pm \sqrt{33}}{4}$$

SHOW YOUR WORK: Note that $f'(x) = 2x^3 - 3x^2 - 12x + 4$. Since -2 is given as a root we see that $f'(x) = (x+2)(2x^2 - 7x + 2)$. Solving the quadratic we get the roots:

$$x = -2,\ \frac{7 \pm \sqrt{33}}{4}$$

■

[3] (b) At which values of x, if any, does $f(x)$ have a local maximum?

At which values of x, if any, does $f(x)$ have a local minimum?

ANSWER:

local maximum(s): $[7 - \sqrt{33}]/4$

local minimum(s): $-2,\ [7 + \sqrt{33}]/4$

EXPLAIN:

METHOD 1. First-derivative test. Let $\alpha_1 = -2$, $\alpha_2 = (7 - \sqrt{33})/4$, and $\alpha_3 = (7 + \sqrt{33})/4$. Then α_1, α_2, α_3 are the roots of $f'(x)$ in increasing order, and $f(x) = 2(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)$. As x increases through α_1 and α_3, $f'(x)$ changes from negative to positive. So $f(x)$ has local minimums at $x = \alpha_1, \alpha_3$. As x increases through α_2, $f'(x)$ changes from positive to negative. So $f(x)$ has a local maximum at $x = \alpha_2$. ■

METHOD 2. Differentiating again, we get $f''(x) = 6x^2 - 6x - 12 = 6(x-2)(x+1)$. Clearly, $f''(x) > 0$ for all $x < -1$ and all $x > 2$. Also, $f''(x) < 0$ for all x in $(-1, 2)$.

Now the roots $[7 \pm \sqrt{33}]/4$ are approximately $1/4$ and $13/4$ since $\sqrt{33} \approx 6$. Therefore

$$f''(-2) > 0, \quad f''\left(\frac{7 - \sqrt{33}}{4}\right) < 0, \quad f''\left(\frac{7 + \sqrt{33}}{4}\right) > 0.$$

So $x = -2,\ [7 + \sqrt{33}]/4$ give local maxima, and $x = [7 - \sqrt{33}]/4$ gives a local minimum. ■

[3] (c) What is the largest interval on which $f(x)$ is concave down?

ANSWER: $[-1, 2]$

EXPLAIN: An easy calculation shows that $f''(x) = 6(x+1)(x-2)$. "Concave down" means that $f'(x)$ is decreasing. Now $f'(x)$ is decreasing on $[-1, 2]$ because $f''(x)$ is negative on $(-1, 2)$. Also, any interval which contains a point not in $[-1, 2]$ will contain a subinterval on which $f'(x)$ is increasing. So $[-1, 2]$ contains *all* intervals on which $f(x)$ is concave down. ■

COMMENTS

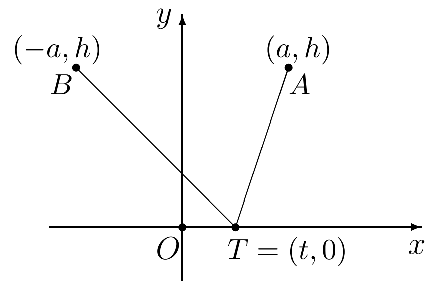
On the whole students did a good job of finding and classifying the critical points. For part (c), depending on one's definition of "concave down", one can say that the correct answer is $(-1, 2)$ or $[-1, 2]$. Both answers were awarded full marks if appropriately justified. For a discussion of the distinction which comes into play here look at:

`http://mathforum.org/library/drmath/view/53566.html`

A few students confused critical points with inflection points.

11. The point $T = (t, 0)$ varies on the x-axis. The points $A = (a, h)$ and $B = (-a, h)$ are fixed with a, $h > 0$. Define the function L by

$$L(t) = \text{length}(AT) + \text{length}(BT).$$



[4] (a) Express $L(t)$ in terms of t and the constants a, h.

ANSWER:
$$L(t) = \sqrt{h^2 + (t-a)^2} + \sqrt{h^2 + (t+a)^2}$$

EXPLANATION: Using the formula for the distance between two points in the plane, the first term is the length of AT and the second is the length of BT. ■

COMMENT

Some students lost points for offering no explanation whatever for the formula they gave as the answer.

[5] (b) Use calculus to show that $L(t)$ has an absolute minimum when $t = 0$.

EXPLANATION:

Note that

$$\frac{dL}{dt} = \frac{t-a}{\sqrt{h^2 + (t-a)^2}} + \frac{t+a}{\sqrt{h^2 + (t+a)^2}}. \tag{1}$$

Therefore $dL/dt = 0$ when $t = 0$. This is a *necessary condition* for $L(t)$ to have an absolute minimum at $t = 0$. The tricky part is completing the argument.

METHOD 1. We show that $t = 0$ is the *only critical point of* $L(t)$, $dL/dt < 0$ *for some* $t < 0$, and $dL/dt > 0$ *for some* $t > 0$. dL/dt is a continuous function of t because it is constructed by arithmetical operations and square-root from the identity function and constant functions. By the Intermediate Value Theorem $dL/dt < 0$ for all $t < 0$ and $dL/dt > 0$ for all $t > 0$. Thus $L(t)$ is decreasng on $(-\infty, 0]$, and increasing on $[0, \infty)$. Instead of appealing to the IVT we could use the Darboux continuity of dL/dt.

By inspection, for $|t|$ large, dL/dt has the sign of t. Also, for $dL/dt = 0$, a necessary condition is

$$\frac{(t-a)^2}{h^2 + (t-a)^2} = \frac{(t+a)^2}{h^2 + (t+a)^2}$$

which is easily seen to be equivalent to $t = 0$. This is enough. ■

METHOD 2. First Derivative Test. It is enough to show that $dL/dt > 0$ for all $t > 0$ and $dL/dt < 0$ for all $t < 0$. Since $L(t)$ is an even function, it is enough to consider $t > 0$. The second term on the right of (1) is clearly positive. So to show $dL/dt > 0$ it is enough to show that $\left|\dfrac{t+a}{\sqrt{h^2+(t+a)^2}}\right| > \left|\dfrac{t-a}{\sqrt{h^2+(t-a)^2}}\right|$. This is equivalent to $\dfrac{(t+a)^2}{h^2+(t+a)^2} > \dfrac{(t-a)^2}{h^2+(t-a)^2}$ which is equivalent to

$$(t+a)^2\left[h^2+(t-a)^2\right] > (t-a)^2\left[h^2+(t+a)^2\right].$$

In turn, this simplifies to $(t+a)^2h^2 > (t-a)^2h^2$ which is always true. So $dL/dt > 0$ for all $t > 0$ which is enough. ■

METHOD 3. Second Derivative Test. It is enough to show that $d^2L/dt^2 > 0$ for all t. Differentiating again we get

$$\frac{d^2L}{dt^2} = \frac{h^2}{\left(h^2+(t-a)^2\right)^{3/2}} + \frac{h^2}{\left(h^2+(t+a)^2\right)^{3/2}} > 0 \qquad (-\infty < t < \infty).$$

So dL/dt is increasing on $(-\infty, \infty)$. It follows that $dL/dt < 0$ for all $t < 0$ and $dL/dt > 0$ for all $t > 0$. Thus $L(t)$ is decreasng on $(-\infty, 0]$, and increasing on $[0, \infty)$. This is enough. ■

COMMENT

This part was intended to be a little more of a challenge. The students who came close to getting full marks knew that it would be sufficient to show that

$$L'(-1) < 0,\ L'(1) > 0, \text{ and } t = 0 \text{ is the only critical point of } L(t).$$

No one verified that $L'(-1) < 0$ and $L'(1) > 0$. Of course, the inequalities $L'(-a) < 0$ and $L'(a) > 0$ will do just as well and these two inequalities are true by inspection of the formula (1) for $L'(t)$.

[4] **12.** (a) Find the general antiderivative of $\sin 2x + \dfrac{2}{x^2}$.

ANSWER:
$$-\frac{1}{2}\cos(2x) - \frac{2}{x} + C$$

SHOW YOUR WORK: Here we are exploiting the basic differentiation rules: $(d/dx)(\cos x) = -\sin x$ and $(d/dx)(1/x) = -1/x^2$. ■

[3] (b) Find a function y defined on $(0, \infty)$ such that

$$\frac{dy}{dx} = \frac{2x^2+1}{x}, \;\; y(1) = 0.$$

ANSWER:
$$y = x^2 + \ln x - 1$$

SHOW YOUR WORK: Taking antiderivatives in the differential equation we get $y = x^2 + \ln x + C$. To satisfy the condition $y(1) = 0$ we take $C = -1$. ■

COMMENTS

Part (a) was well done except that a substantial number of students forgot about the constant. In part (b) many students got stuck because they did not recognize that one needs the identity

$$\frac{2x^2+1}{x} = 2x + \frac{1}{x}\,.$$

A substantial number of examinees forgot to compute the constant.

13. A tank of brine has 1000 litre capacity and initially contains 50 kilograms of salt dissolved in water.

Brine is drawn from the tank at rate of 5 litres per minute and water is added to the tank at the same rate to maintain the volume of solution at 1000 litres.

The tank is well-stirred so that the concentration of salt is uniform at all times.

Let S denote the amount of salt (in kilograms) in the tank after t minutes.

[2] (a) What is the approximate net change ΔS in the amount of salt in the tank in the time interval $[t, t + \Delta t]$ if Δt is small?

Write your answer as a constant multiple of $S\Delta t$.

ANSWER: $\approx -(1/200)S\Delta t$

SHOW YOUR WORK: In Δt minutes the proportion of the solution which is drawn from the tank is $(5\Delta t)/1000$. So the net change in the amount of salt is approximately $-(5S\Delta t)/1000$. Note that the exact change in the amount of salt is $-(5\overline{S}\Delta t)/1000$, where $\overline{S}$ is the (time-) average of S over the interval $[t, t + \Delta t]$. ■

[2] (b) Write down an equation relating dS/dt and S, where t is the elapsed time in minutes.

ANSWER: $$\frac{dS}{dt} = -S/200$$

SHOW YOUR WORK: From (a), $\Delta S \approx -(1/200)S\Delta t$. Dividing by Δt, we have

$$\frac{\Delta S}{\Delta t} \approx -(1/200)S.$$

Taking the limit as $\Delta t \to 0$, we have $dS/dt = -S/200$. ■

[4] (c) How many minutes pass before there are only 25 kilograms of salt in the tank?

ANSWER: $t = 200 \ln 2$

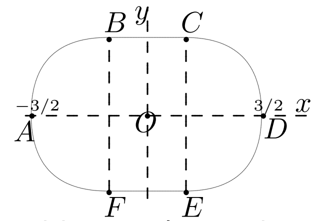
SHOW YOUR WORK: Writing the equation as $(1/S)dS/dt = -1/200$ and taking antiderivatives with respect to t we get $\ln S = -t/200 + C$. Rewriting this we get $S = ke^{-t/200}$, where k is constant. At $t = 0$, $S = 50$. So $k = 50$. For $S = 25$ we need $e^{-t/200} = 1/2$. Solving we get $t = 200 \ln 2$. ■

COMMENTS Very poorly done. Many students did not know where to start.

In part (c), students tried to fit the problem to the equation $S = S_0 e^{-kt}$. This was a good idea, of course. But many believed that

$$S(1) = 50 - 50\left(\frac{5}{1000}\right)$$

which assumes that the rate of loss of salt is constant over the first minute.



[6] **14.** An oval plate is symmetric about its axes, which are shown as Ox, Oy in the figure. The midsection of the plate is a rectangle $BCEF$ of width 1 and height 2.

The arc AB of the bounding curve has the same shape as the arc $y = 2\sqrt{x} - x \ \ (0 \le x \le 1)$. Indeed, the arc AB is obtained by translating the arc $y = 2\sqrt{x} - x \ \ (0 \le x \le 1)$ horizontally $3/2$ units to the left.

Show the area of the plate is $16/3$.

EXPLANATION: The basic principle is that, if $a < b$ and $f(x)$ is continuous on $[a, b]$, then the area 'under $y = f(x)$ from $x = a$ to $x = b$' is equal to $F(b) - F(a)$, where $F(x)$ is any antiderivative of $f(x)$.

Note that $\int (2\sqrt{x} - x)\, dx = (4/3)x^{3/2} - (1/2)x^2 + C$. Hence the area under arc AB and above the x-axis is $(4/3) - (1/2) = 5/6$. So total area of the plate is $4 \cdot (5/6) + \text{area(BCEF)} = 16/3$. ■

COMMENT

This question was done almost perfectly. Most students used calculator to show that

$$\int_0^1 \left(2\sqrt{x} - x\right)\, dx = 5/6\,.$$