[3] **1.** Given $f(x) = \dfrac{\sin x}{\ln(x^2 + e^{\cos x})}$, find $f'(x)$. (Algebraic simplification is not required.)

Solution: By the quotient rule and the chain rule,

$$f'(x) = \frac{(\cos x)\ln\bigl(x^2 + e^{\cos x}\bigr) - (\sin x)\left[\dfrac{2x + e^{\cos x}(-\sin x)}{x^2 + e^{\cos x}}\right]}{\left[\ln\bigl(x^2 + e^{\cos x}\bigr)\right]^2}.$$

Commentary: Differentiating nested functions like this one requires a good understanding of the order of operations rules, as expressed in correct positioning of parentheses. The correct formula

$$\frac{d}{dx}\left(x^2 + e^{\cos x}\right) = \left(2x + e^{\cos x}(-\sin x)\right)$$

has a right side that is different from $(2x + e^{\cos x})(-\sin x)$. Many students wrote the latter expression instead of the right one. Another common error was thinking that $(f/g)'$ (the derivative of a quotient) could be expanded as $f'g + g'f$ (the derivative of a product).

Although algebraic simplification was explicitly discouraged, some students tried it anyway, reducing the point value of the correct expression they wrote at first by introducing errors in later steps that were not required. Trouble with algebra was a constant problem throughout the test.

Continued on page 3

[5] **2.** The curve $y = \ln(3x+1)$ has a tangent line of slope $\frac{1}{2}$ at some point. Find an equation for this line.

Solution: By the Chain Rule, $y' = \dfrac{3}{3x+1}$. The point of tangency obeys

$$\frac{1}{2} = y' = \frac{3}{3x+1}, \quad \text{i.e.,} \quad x = \frac{5}{3}.$$

When $x = 5/3$, $y = \ln(6)$. The desired line passes through this point with slope $1/2$. It is

$$y = \ln(6) + \frac{1}{2}\left(x - \frac{5}{3}\right).$$

Commentary: Any equation algebraically equivalent to the one above is equally acceptable. By contrast, the relation

$$y = \ln(6) + \left(\frac{3}{3x+1}\right)\left(x - \frac{5}{3}\right)$$

is profoundly unacceptable, mainly because it is not the equation of a line. It's another curve that happens to meet the given curve tangentially at $x = 5/3$.

A popular algebraic error here was to think $\dfrac{3}{3x+1}$ equals $\dfrac{1}{x+1}$. (Test these expressions with $x = -1$ to see how different they are.)

Continued on page 4

[7] **3.** Find all real numbers A for which each function $y = \dfrac{A}{1+ke^{-3t}}$ (where k is a constant) satisfies the differential equation $\dfrac{dy}{dt} = y(3-y)$ at each point of its domain.

Solution: By the chain rule (or the quotient rule),

$$y' = \frac{-A}{(1+ke^{-3t})^2}\left[-3ke^{-3t}\right] = \frac{3Ake^{-3t}}{(1+ke^{-3t})^2}.$$

Also,

$$y(3-y) = \frac{A}{1+ke^{-3t}}\left(\frac{3+3ke^{-3t}}{1+ke^{-3t}} - \frac{A}{1+ke^{-3t}}\right) = \frac{3A+3Ake^{-3t}-A^2}{(1+ke^{-3t})^2}.$$

Comparing these expressions reveals equality when $3A - A^2 = 0$, i.e., either $A = 0$ or $A = 3$.

Commentary: Checking when a given function satisfies a given equation is much easier than ignoring the given function and trying to solve the given equation from scratch.

Continued on page 5

[8] **4.** Find all real numbers k for which the curve $y = x^2 e^{-kx}$ has an inflection point at $x = 1$.

Solution: Combining the product rule and the chain rule, we calculate

$$\begin{aligned} f'(x) &= 2xe^{-kx} + x^2 e^{-kx}[-k] = (2x - kx^2)e^{-kx} \\ f''(x) &= (2 - 2kx)e^{-kx} + (2x - kx^2)e^{-kx}[-k] \\ &= e^{-kx}\left[k^2x^2 - 4kx + 2\right] = e^{-kx}\left[(kx-2)^2 - 2\right]. \end{aligned}$$

Thus $0 = f''(1) = e^{-k}[(k-2)^2 - 2]$ if and only if $k = 2 \pm \sqrt{2}$. These k-values really give inflection points because f'' truly changes sign at both its zeros.

Commentary: The chain rule seems to be quite well understood. However, finding d^2y/dx^2 based on dy/dx seemed to confuse many students. Errors in algebra cost many students marks: in this question, confusing $-(a-b)$ with $-a-b$ was quite common.

Continued on page 6

[8] **5.** A stalagmite has the shape of a perfect circular cone. Its height is 200 mm and is increasing at a rate of 3 mm per century. Its base radius is 40 mm and is decreasing at a rate of 0.5 mm per century. Is its volume increasing or decreasing, and at what rate?

Solution: The volume V, height h, and radius r in the relation

$$V = \frac{\pi}{3} r^2 h$$

are all changing with time, so differentiation gives

$$\frac{dV}{dt} = \frac{\pi}{3}\left[2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right].$$

At the given instant, all quantities on the right side are known, so

$$\frac{dV}{dt} = \frac{\pi}{3}\left[2(40)(200)(-\tfrac{1}{2}) + (40)^2(3)\right] = \frac{\pi}{3}\left[-8000 + 4800\right] = \frac{-3200\pi}{3}.$$

Thus the volume is decreasing at $3200\pi/3$ mm^3 per century.

Commentary: Most people knew the formula for the volume of a cone, though a few wrote $\pi r^3 h/3$. A number of people thought that the initial ratio of height to radius, namely 5, held forever. This is probably because they had experience solving a similar-sounding problem involving water in an inverted conical reservoir.

A minority calculated the *average* rate of change over one century, namely, $\frac{\pi}{3}\left[(39.5)^2(203) - (40)^2(200)\right]$. But in a Calculus test, unless otherwise specified, all rates of change are instantaneous rates, so the correct answer is quite different.

Others took the height at time t to be $200 + 3t$, and the base radius to be $40 - (0.5)t$. Then they computed the volume as a function of t and differentiated, often expanding before differentiating. This led to numerical errors, and most people who used this approach did not realize that they wanted the derivative at $t = 0$. The idea is interesting because the derivative at $t = 0$ always gives the correct answer, even though the associated formula for $V(t)$ may not be correct for any t except $t = 0$! A full explanation of why replacing every time-varying function with its linear approximation at a point leads to a composite expression with the right slope at $t = 0$ is logically equivalent to a full proof of the standard differentiation rules.

Since r is decreasing we must have $\frac{dr}{dt} < 0$, but some students used $\frac{dr}{dt} = 0.5$.

The differentiation was mostly handled correctly, though several people thought "the derivative of a product is the product of the derivatives." (It is not.) Because of incorrect use of brackets there were also problems with the numerical evaluation. Finally, a number of people wrote that the volume is "decreasing at $-3200\pi/3$ mm^3/century," which is technically wrong: a negative decrease is an increase.

Continued on page 7

[8] **6.** Find all points on the curve $2x^2 + y^2 + 2xy = 10$ where the tangent line is horizontal.

Solution: Implicit differentiation gives

$$4x + 2y\frac{dy}{dx} + 2y + 2x\frac{dy}{dx} = 0.$$

The tangent is horizontal at points where $\frac{dy}{dx} = 0$, i.e., at points on the curve satisfying

$$4x + 0 + 2y + 0 = 0, \quad \text{i.e.,} \quad y = -2x.$$

The points on the curve consistent with this relation obey

$$2x^2 + (-2x)^2 + 2x(-2x) = 10, \quad \text{i.e.,} \quad x^2 = 5, \quad \text{i.e.,} \quad x = \pm\sqrt{5}.$$

Back-substitute into the equation $y = -2x$ (not into the equation for the curve!) to find two points with the stated property:

$$\boxed{(\sqrt{5}, -2\sqrt{5}),\ (-\sqrt{5}, 2\sqrt{5})}$$

Commentary: Some writers solved for y in terms of x and then differentiated. This leads to a bit of a mess, with the attendant possibilities of error. And a small change of exponents in the given relation would lead to a problem in which this method won't work at all, because an explicit formula for y cannot be found.

Most people knew that one should use implicit differentiation. But differentiation errors were quite common. Many missed the fact that $2xy$ is a product of functions of x, and thought its derivative is simply $2x\frac{dy}{dx}$. And a number of people differentiated the left-hand side but did not differentiate the 10.

There were *many* algebra errors. Most students decided to solve for $\frac{dy}{dx}$ in general before setting the derivative equal to 0. Solving for $\frac{dy}{dx}$ was often done incorrectly, with the usual minus sign errors, wrong cancellations, and other problems. There were even (many) errors in going from an expression for $\frac{dy}{dx}$ to the equation $y = -2x$.

Students should be encouraged to think ahead a little. Having obtained the equation

$$4x + 2y\frac{dy}{dx} + 2y + 2x\frac{dy}{dx} = 0,$$

there is *no reason* to solve for $\frac{dy}{dx}$. No one has asked for a formula for $\frac{dy}{dx}$ in terms of x and y, and such a formula has no obvious advantages. We only want to find conditions under which $\frac{dy}{dx} = 0$. So do not solve, just replace $\frac{dy}{dx}$ with 0 in the above equation. We obtain trivially $4x + 2y = 0$.

A similar issue comes up in other implicit-differentiation problems. Suppose we wanted the slope of the tangent line to the above curve at some given point, say $(1, -4)$. Instead of solving for $\frac{dy}{dx}$, and then substituting, it is simpler to substitute and then solve. Also, if we want the second derivative, it is poor strategy to find an expression for $\frac{dy}{dx}$ and then differentiate. Ratios are a nuisance, flat expressions are much more pleasant to deal with. As much as possible one should work with attractive formulas (division is ugly, funny exponents are ugly).

Very often, students came up with expressions such as $y = -2x$, and thought that this was the answer. Those who substituted into the equation of the curve got the rest right or mostly right, but not enough people thought to substitute. There were the usual minor problems of missing the possibility $x = -\sqrt{5}$. And there was a real problem in computing the y-coordinate. It should have been easy, since $y = -2x$, and for many it was. But others put $x = \pm\sqrt{5}$ into the equation of the original curve, and got some spurious solutions.

Continued on page 8

[9] **7.** At what point on the parabola $y = 1 - x^2$ does the tangent line have the property that it cuts from the first quadrant a triangle of minimum area?

Solution: Let the point of tangency obey $x = u$, with $u > 0$. (If $u \le 0$, the tangent line does not cut a triangle from the first quadrant.) Then the equation of the tangent at $(u, 1 - u^2)$ is

$$y = (1 - u^2) + (-2u)(x - u) = -2ux + 1 + u^2.$$

This line has y-intercept $b = 1 + u^2$ and x-intercept $a = (1 + u^2)/(2u)$. So the area A described in the question obeys

$$A = \tfrac{1}{2}ab = \tfrac{1}{4}\tfrac{(1+u^2)^2}{u},$$

$$\text{so} \qquad \frac{dA}{du} = \frac{1}{4}\frac{4u^2(1+u^2) - (1+u^2)}{4u^2} = \frac{1}{4}\frac{(1+u^2)(3u^2-1)}{u^2}.$$

We have $dA/du = 0$ at $u = 1/\sqrt{3}$: since $dA/du < 0$ for $0 \le u < 1/\sqrt{3}$ and $dA/du > 0$ for $u > 1/\sqrt{3}$, this point gives a minimum value for A. The corresponding "point on the parabola" is $\left(\frac{1}{\sqrt{3}}, \frac{2}{3}\right)$.

Commentary: This question caused great difficulties. The question was not well understood. Many people drew pictures, but they were often not correct pictures (no tangent line shown) or the labelling was wrong.

Many used a bogus argument more or less like this. The area is xy, and $y = 1 - x^2$, so we are minimizing say $x(1 - x^2)$, which has a critical point when $x = 1/\sqrt{3}$. Amazingly this happens to be the correct x. But in fact this point gives a local *maximum* to the expression $x(1 - x^2)$. And anyway, this expression has nothing much to do with the problem: it is the area of a certain rectangle inside the triangle whose area we are trying to minimize. (Some students used a more attractive version of the same incorrect argument. It goes like this. We have $A = xy$. Differentiate with respect to x. We get $A' = xy' + y = 0$. But $y' = -2x$ and $y = 1 - x^2$ so $1 - 3x^2 = 0$.)

In order to solve the problem, we need to express the area of the triangle in terms of some quantity. Which quantity? It is reasonable (but *not* necessary) to let it be the x-coordinate of the point of tangency. But it is *essential* to label this x-coordinate as u, or w, or *something other than* x! If we let the point of tangency have x-coordinate x, then we must change the letters we use in writing equations of lines, and this makes things look more complicated.

Or else we can use the letter x instead of u, and then use the basic geometry of slopes to find an expression for area. Here a good picture is essential. Some people who tried this approach got into difficulties because the slope is $-2x$, and this is *not* the ratio of the sides of the triangle.

Once we get to an expression like $(1 + u^2)^2/(4u)$ for area, we want to differentiate. Too many people expand first. In general, one should expand only when there is good reason to expand. Since $(1 + u^2)^2$ is more attractive than $1 + 2u^2 + u^4$, the default should be not to expand. Differentiate using the quotient rule. Do not expand unless it is necessary. We get a natural $1 + u^2$ term in the numerator, and the rest of the calculation is smooth.

Continued on page 9

[8] **8.** A particle is travelling along the curve $y = x^2 - 5$. At the instant that the particle reaches the point $(3, 4)$, the particle's distance from the origin is increasing at the rate of 2 units per second. How fast is the particle's x-coordinate increasing at this instant?

Solution 1: At the instant when the particle has position $(3, 4)$, it is 5 units distant from the origin. More generally, if the particle is at (x, y) and D is its distance from the origin, then

$$D^2 = x^2 + y^2.$$

Since $y = x^2 - 5$ always, it follows that

$$D^2 = x^2 + (x^2 - 5)^2, \quad \text{so} \quad 2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2(x^2 - 5)(2x\frac{dx}{dt}).$$

This holds for all time; at the instant described in the problem it reduces to

$$5(2) = (3)\frac{dx}{dt} + (4)(6)\frac{dx}{dt}, \quad \text{i.e.,} \quad \frac{dx}{dt} = \frac{10}{27}.$$

Solution 2: Two equations are valid for all time:

$$(1) \quad D^2 = x^2 + y^2 \qquad \text{and} \qquad (2) \quad y = x^2 - 5.$$

Differentiation gives

$$(1') \quad 2D\frac{dD}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \qquad \text{and} \qquad (2') \quad \frac{dy}{dt} = 2x\frac{dx}{dt}.$$

At the instant of interest, $D = 5$ follows from (1): hence

$$5(2) = 3\frac{dx}{dt} + 4\frac{dy}{dt} \qquad \text{and} \qquad \frac{dy}{dt} = 6\frac{dx}{dt}.$$

Eliminating $\frac{dy}{dt}$ between these equations gives $\frac{dx}{dt} = \frac{10}{27}$.

Commentary: Algebraic errors were the main source of difficulty, and often struck early enough (e.g., when finding a formula for D) to cause a significant loss of marks. Many students attempting Solution 2 used the slope $dy/dx = 6$ instead of the correct rate of change $dy/dt = (dy/dx)(dx/dt) = 20/9$ in the late stages.

Continued on page 10

[6] **9.** The following facts are known about the function $f(x)$:

(a) $f(2) = 4$ and (b) $f'(x) = (x^4+1)^{-1}$ for all x.

(i) Use a linear approximation to estimate $f(2.05)$. (Call your answer α.)

(ii) Taking α from part (i), circle the correct statement below:

$\alpha < f(2.05)$ $\qquad$ $\alpha = f(2.05)$ $\qquad$ $\alpha > f(2.05)$

Clearly justify your selection, without using an antiderivative of $(x^4+1)^{-1}$.

Solution: For x near 2, the tangent-line approximation says $f(x) \approx L(x)$, where

$$L(x) = f(2) + f'(2)(x-2) = 4 + \frac{1}{2^4+1}(x-2).$$

In particular, $f(2.05) \approx L(2.05) = 4 + \dfrac{1}{340} \stackrel{\text{def}}{=} \alpha$.

Either by looking at $f'(x) = \dfrac{1}{x^4+1}$ or by calculating $f''(x) = \dfrac{-4x^3}{(x^4+1)^2}$, we see that $f'(x)$ is decreasing in the interval where $x > 0$. This means that f is concave down in this interval, and that the graph lies below its tangents. In particular, $f(x) < L(x)$ for all $x > 0$ (except $x = 2$): in particular, $f(2.05) < L(2.05) = \alpha$.

Commentary: Students who tried to approximate $f(x)$ with $y = f(2) - f'(2)(x-2)$ received no marks at all in part (i). This is no ordinary sign error: it's a fundamental misunderstanding of the most basic topic in the course, which can easily be detected by comparing the slope of the line just written (namely, $-f'(2)$) with the slope of the curve at the point of interest (namely, $f'(2)$).

More explicitly, it's clear that $f'(x) > 0$ for all real x, so function f is increasing. This will force $f(2.05) > f(2)$, so students who give an approximate value less than 4 are not taking the time to support their manipulations with common sense.

"Calculator-ready" answers deemed equally acceptable included $\dfrac{1361}{340}$, $\dfrac{6805}{1700}$, and $\dfrac{68.05}{17}$. Inexact values like 4.003 grudgingly received full marks if supported with a clear and correct derivation. Mixed fractions like $4\frac{1}{340}$ were tolerated, but the examiners wish to record their distaste for mixed fractions in general. (Who can explain why $4\frac{1}{340} = 4.0029\ldots$ but $2\frac{1}{\sqrt{2}} = \sqrt{2} = 1.414\ldots$?)

A number of students wrote $f(x) = f(2) + f'(2)(x-2)$. This statement is false: function f is not linear!

Continued on page 11

[6] **10.** Given that $f''(t) = 6e^{-3t}$ for all t, and that $f(0) = 0$ and $f'(0) = 4$, find an explicit formula for $f(t)$.

Solution: $6e^{-3t} = f''(t) = \dfrac{d}{dt}[f'(t)] \quad \Longrightarrow \quad f'(t) = \dfrac{6e^{-3t}}{-3} + C = -2e^{-3t} + C,$ for some C.

To arrange $f'(0) = 4$ requires $C = 6$, so $f'(t) = -2e^{-3t} + 6$. Repeat:

$$-2e^{-3t} + 6 = f'(t) = \frac{d}{dt}[f(t)] \quad \Longrightarrow \quad f(t) = \frac{-2e^{-3t}}{-3} + 6t + K, \quad \text{for some } K.$$

To arrange $f(0) = 0$ requires $K = -\dfrac{2}{3}$. Hence

$$f(t) = \frac{2}{3}e^{-3t} + 6t - \frac{2}{3}.$$

The desired properties of f are easy to check by differentiation.

Commentary: Almost no students checked their answers by differentiation. (There were no marks for this, but a number of students who had made sign errors or other mistakes could have found and fixed them easily if they had done so.)

There was a distressing tendency to confuse t with x, so a number of students supplied answers like,

$$\text{“}f = \frac{2}{3}e^{-3t} + 6x - \frac{2}{3}.\text{”}$$

The graders accepted such responses reluctantly, but repeat that assuming x and t are always interchangeable was a major source of confusion in Question 8.

A number of students filled a whole page with a detailed account of integration by substitution (not always correctly managed) instead of using the much simpler strategy of guess and check.

A majority of the indefinite integrals students wrote were not well-formed. Ambiguous clusters of symbols like this appeared frequently:

$$I = \int -2e^{-3t} + 6.$$

This expression has no clear meaning for two reasons. First, the integration variable is not mentioned: the integral needs a “dt”. Second, the order-of-operations precedence is unclear: one can't tell whether to integrate the exponential and then add 6 or integrate both terms. A similar-looking but correct construction that could be useful in solving this problem is

$$J = \int \left(-2e^{-3t} + 6\right)\, dt.$$

Continued on page 12

[8] **11.** Fred's body mass m satisfies the differential equation

$$\frac{dm}{dt} = \frac{C - 40m}{8000}.$$

Here m is Fred's mass in kilograms, t is the time in days, and C is Fred's energy intake, measured in Calories per day. If Fred's mass now is 100 kilograms, and he ingests 3000 Calories per day, how many days from now will his mass be 90 kilograms?

Solution 1: Rewrite the differential equation as

$$\frac{dm}{dt} = -\frac{1}{200}\left(m - \frac{C}{40}\right).$$

Since $\dfrac{dm}{dt} = \dfrac{d}{dt}\left(m - \dfrac{C}{40}\right)$, it follows that

$$\left[m(t) - \frac{C}{40}\right] = \left[m(0) - \frac{C}{40}\right] e^{-t/200}, \quad \text{i.e.,} \quad e^{-t/200} = \frac{m(t) - C/40}{m(0) - C/40} = \frac{40m(t) - C}{40m(0) - C}.$$

In particular, since $m(0) = 100$, $C = 3000$, we will have $m(t) = 90$ when

$$e^{-t/200} = \frac{3600 - 3000}{4000 - 3000} = \frac{3}{5}, \quad \text{i.e.,} \quad t = -200\ln(3/5) = 200\ln(5/3).$$

Hence Fred will have mass 90 kg in about 102 days.

Solution 2: The given differential equation is separable. In integral form,

$$\int \frac{dm}{C - 40m} = \int \frac{dt}{8000} \quad \Longleftrightarrow \quad -\frac{1}{40}\ln|C - 40m(t)| = \frac{t}{8000} + K.$$

Hence $C - 40m(t) = Qe^{-t/200}$ for some Q, and substituting $t = 0$ gives $Q = C - 40m(0)$. Thus

$$C - 40m(t) = [C - 40m(0)]e^{-t/200},$$

and one can solve for t from $m(t) = 90$ as before.

Solution 3: The differential equation gives $\dfrac{dm}{dt} = 0$ when $40m = C$, i.e., when $m = 75$. Write $\overline{m}$ for this number, which has a nice interpretation: it's Fred's long-run mass if he keeps to his 3000-Calorie per day diet forever. Define $u(t) = m(t) - \overline{m}$ as Fred's excess mass and note that

$$\frac{du}{dt} = \frac{dm}{dt} = \frac{C - 40(u + \overline{m})}{8000} = -\frac{1}{200}u.$$

Hence $u(t) = u(0)e^{-t/200}$, with $u(0) = m(0) - \overline{m} = 25$. We want to find t when $u(t) = 90 - \overline{m} = 15$, and this occurs when $t = 200\ln(25/15) = 200\ln(5/3)$.

Commentary: Fred is a pretext, not a person, so a mathematically correct handling of the differential equation was considered essential in this problem. So, for example, extrapolating the rate of change in Fred's mass from the initial instant to all future times (i.e., assuming $dm/dt = -1/8$ forever) and coming up with $t = 80$ days received no marks. Many students recognized some similarity to a differential equation and pulled a magic formula of the form $p = p_0 e^{kt}$ out of thin air, then tried inserting numbers from the question statement in various places. No marks were given for simple regurgitation of a formula whose relevance students were not subsequently able to justify. (The Ministry's Prescribed Learning Outcomes for Calculus 12 explicitly mention differential equations of the form $y' = ay + b$ under Applications of Antidifferentiation.)

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[8] **12.** (i) Find numbers A and B such that the derivative of $Ae^{-3x}\cos x + Be^{-3x}\sin x$ with respect to x is $e^{-3x}\cos x$.

(ii) Consider the triangular region T defined by $x \geq 0$, $y \geq 0$, and $x + y \leq \pi$. Find the area of the subset of T in which $y \leq e^{-3x}\cos x$.

Solution: Calculation gives

$$\begin{aligned}\frac{d}{dx}\left[Ae^{-3x}\cos x + Be^{-3x}\sin x\right] &= A\left(-3e^{-3x}\cos x - e^{-3x}\sin x\right) + B\left(-3e^{-3x}\sin x + e^{-3x}\cos x\right) \\ &= (B-3A)e^{-3x}\cos x - (A+3B)e^{-3x}\sin x.\end{aligned}$$

This matches the desired expression when both

$$(1) \quad B - 3A = 1, \quad \text{and} \quad (2) \quad A + 3B = 0.$$

Equation (2) gives $A = -3B$, so (1) says $B + 9B = 1$. Thus we get $B = 1/10$ and $A = -3/10$.

The curve $y = e^{-3x}\cos x$ lies inside the triangle T when $0 \leq x \leq \pi/2$, so the area we want is

$$A = \int_0^{\pi/2} e^{-3x}\cos x\,dx.$$

To calculate this integral we need a function whose derivative is $e^{-3x}\cos x$, and this is provided by part (i). Hence

$$A = \left[-\frac{3}{10}e^{-3x}\cos x + \frac{1}{10}e^{-3x}\sin x\right]_{x=0}^{\pi/2} = \frac{3}{10} + \frac{1}{10}e^{-3\pi/2}.$$

Commentary: Most students correctly found the derivative in (i), but then didn't know how to match it with the desired form. Part (ii) was a washout: among those who realized that an integral was required, most used the interval $0 \leq x \leq \pi$ instead of $0 \leq x \leq \pi/2$, and few realized that part (i) was relevant.

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[8] **13.** Suppose that the function $f(x)$ is defined on an interval (a, b) where $a < 3 < b$.

(i) Express $f'(3)$ as a limit.

(ii) Prove that if $f(3) = 9$ and $f'(3) = 4$ then $\lim_{x\to 3} \frac{\sqrt{f(x)} - 3}{x - 3} = \frac{2}{3}$.
Note: Assuming that $f'(x) \to 4$ as $x \to 3$ *is not allowed,* since this behaviour is not shared by all f with the given properties. Find a way to use part (i).

Solution: By definition,

$$f'(3) = \lim_{x\to 3} \frac{f(x) - f(3)}{x - 3} \qquad \text{or} \qquad f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}.$$

To evaluate the limit in (ii), note that the existence of $f'(3)$ implies that function $f(x)$ is continuous at $x = 3$. Thus $f(x) \to 9$ as $x \to 3$, and we have

$$\begin{aligned}
\lim_{x\to 3} \frac{\sqrt{f(x)} - 3}{x - 3} &= \lim_{x\to 3} \left(\frac{\sqrt{f(x)} - 3}{x - 3} \cdot \frac{\sqrt{f(x)} + 3}{\sqrt{f(x)} + 3} \right) \\
&= \lim_{x\to 3} \left[\frac{f(x) - 9}{x - 3} \cdot \frac{1}{\sqrt{f(x)} + 3} \right] \\
&= \lim_{x\to 3} \frac{f(x) - f(3)}{x - 3} \cdot \lim_{x\to 3} \frac{1}{\sqrt{f(x)} + 3} \\
&= f'(3) \cdot \left[\frac{1}{\sqrt{9} + 3} \right] = \frac{4}{6} = \frac{2}{3}.
\end{aligned}$$

Commentary: A very large number of writers could not reproduce the most basic definition in all of calculus. Many felt the need to mention all of f, a, b, 3, x, Δx, *and* h. Most statements presented made no sense, suggesting that students don't really understand the meaning and notation for limits and their relation to derivatives. (The Ministry of Education's Prescribed Learning Outcomes for Calculus 12 list five logically equivalent but cosmetically different versions of this definition, separated with "and". Any one of these would have been fully acceptable in part (i).)

The note in question (ii) implicitly disallows an appeal to L'Hospital's Rule. Many students used it anyway, but got no marks.

There were basic troubles with logic. Many students approached part (ii) by writing down the limit statement they were asked to prove, and then manipulating both sides in very strange ways.

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[8] **14.** For each positive number a, let $V(a)$ denote the maximum value of $(a/x)^x$ on the interval where $x > 0$.

(i) Find an explicit formula for $V(a)$.

(ii) Prove that $V'(a)/V(a)$ is independent of a and find its value.

Solution: Let $f(x) = \left(\dfrac{a}{x}\right)^x$ for $x > 0$, noting that $f(x) > 0$ whenever $x > 0$. Then

$$\ln f(x) = x \ln\left(\frac{a}{x}\right) = x\ln a - x\ln x \quad \Longrightarrow \quad \frac{f'(x)}{f(x)} = \ln a - \left[\ln x + x\left(\frac{1}{x}\right)\right].$$

Hence $f'(x) = 0$ if and only if $\ln x = \ln a - 1$, i.e., $x = a/e$. Since $f'(x) > 0$ for $0 < x < a/e$ and $f'(x) < 0$ for $x > a/e$, the point $x = a/e$ give a maximum value for f over the positive half-line. This maximum value is the result for part (i):

$$\boxed{V(a) = f(a/e) = e^{a/e}.}$$

Applying the chain rule gives

$$V'(a) = e^{a/e}\left[\frac{1}{e}\right] = \frac{1}{e}V(a), \quad \text{so} \quad \frac{V'(a)}{V(a)} = \frac{1}{e}.$$

The constant value requested in part (ii) is e^{-1}.

Alternative: Writing $f(x) = e^{\ln f(x)} = e^{x\ln a - x\ln x}$ leads to

$$f'(x) = e^{x\ln a - x\ln x}\left[\ln a - 1 - \ln x\right] = f(x)\left[\ln a - 1 - \ln x\right].$$

This is equivalent to the "logarithmic differentiation" approach shown above.

Commentary: Most students were weak on logarithmic differentiation. Many didn't know that the idea is to find the derivative of $(a/x)^x$ with respect to x. Those who knew more or less what to do did not prove that the point $x = a/e$ actually gives a maximum. Many gave answers containing both a and x, not understanding that $V(a)$ is to be a function of a alone.

The End

This examination has 15 pages including this cover.

UBC-SFU-UVic-UNBC Calculus Examination

5 June 2003

Name: ____________________ **Signature:** ____________________

School: ____________________ **Candidate Number:** ____________________

Rules and Instructions

1. *Show all your work!* Full marks are given only when the answer is correct, and is supported with a written derivation that is orderly, logical, and complete. Part marks are available in every question.
2. Calculators are optional, not required. Correct answers that are "calculator ready," like $3 + \ln 7$ or $e^{\sqrt{2}}$, are fully acceptable.
3. Any calculator acceptable for the Provincial Examination in Principles of Mathematics 12 may be used.
4. A basic formula sheet has been provided. No other notes, books, or aids are allowed. In particular, *all calculator memories must be empty when the exam begins.*
5. If you need more space to solve a problem on page n, work on the back of page $n-1$.
6. CAUTION - Candidates guilty of any of the following or similar practices shall be dismissed from the examination immediately and assigned a grade of 0:
   (a) Using any books, papers or memoranda.
   (b) Speaking or communicating with other candidates.
   (c) Exposing written papers to the view of other candidates.
7. Do not write in the grade box shown to the right.

1		3
2		5
3		7
4		8
5		8
6		8
7		9
8		8
9		6
10		6
11		8
12		8
13		8
14		8
Total		100