

Key for the 2002 Calculus Challenge Exam

Note: There is no attempt here to describe all possible correct answers. Students sitting the calculus challenge examination will have used a variety of texts and been exposed to a variety of teaching styles.

For the examiners, apart from the accuracy of the answers, the crucial test is whether the student has made clear the principles and/or method being used and whether those principles and/or method are sound.

Marks are not deducted for sufficiently trivial errors, e.g., inadvertently dropping a sign.

1. Compute the following limits.

[3] (a) $\lim_{t \rightarrow -2} \frac{t^2 - t - 6}{t^2 + 5t + 6}$

ANSWER:

-5

JUSTIFY YOUR ANSWER

Note that $\frac{t^2 - t - 6}{t^2 + 5t + 6} = \frac{(t+2)(t-3)}{(t+2)(t+3)} = \frac{t-3}{t+3}$ whenever $t \neq -2$. Therefore

$$\lim_{t \rightarrow -2} \frac{t^2 - t - 6}{t^2 + 5t + 6} = \lim_{t \rightarrow -2} \frac{t-3}{t+3} = \frac{(\lim_{t \rightarrow -2} t) - 3}{(\lim_{t \rightarrow -2} t) + 3} = \frac{-2 - 3}{-2 + 3} = -5.$$

Instead of using the limit laws to evaluate $\lim_{t \rightarrow -2} \frac{t-3}{t+3}$, one may use the fact that a rational function is continuous, i.e., continuous at each point of its domain. ■

[3] (b) $\lim_{x \rightarrow 0^+} \left[\frac{1}{x} (3e^{-1/x} + 5e^x) \sin x \right]$

ANSWER:

5

JUSTIFY YOUR ANSWER

From the limit laws the given limit is equal to

$$\left(\lim_{x \rightarrow 0^+} \frac{\sin x}{x} \right) \left(3 \lim_{x \rightarrow 0^+} e^{-1/x} + 5 \lim_{x \rightarrow 0^+} e^x \right)$$

provided the three limits in the line above exist. We may take $\lim_{x \rightarrow 0^+} (\sin x)/x = 1$ as known. (A second way of looking at this limit is via l'Hospital's rule. A third way is to notice that, since $(d/dx) \sin x = \cos x$, we have $\lim_{h \rightarrow 0} (\sin h)/h = \cos 0 = 1$.) Now e^x is continuous so $\lim_{x \rightarrow 0^+} e^x = e^0 = 1$. Finally, as $x \rightarrow 0^+$, $1/x \rightarrow \infty$, whence $e^{1/x} \rightarrow \infty$ and so $e^{-1/x} \rightarrow 0$. Thus the given limit evaluates to $1 \cdot (0 + 5) = 5$. ■

- [4] **2.** (a) Find the asymptotes of $y = \left(\frac{x}{x-1}\right)^2$ and justify your answer.

ANSWER:

$$y = 1$$

$$x = 1$$

EXPLANATION

The following is an acceptable explanation: $[x/(x-1)]^2$ is defined except at $x = 1$. Also,

$$\lim_{x \rightarrow \infty} \left(\frac{x}{x-1}\right)^2 = \lim_{x \rightarrow -\infty} \left(\frac{x}{x-1}\right)^2 = 1 \text{ and } \lim_{x \rightarrow 1^+} \left(\frac{x}{x-1}\right)^2 = \lim_{x \rightarrow 1^-} \left(\frac{x}{x-1}\right)^2 = \infty. \quad \blacksquare$$

- [2] (b) Where does the curve $y = \left(\frac{x}{x-1}\right)^2$ cross its horizontal asymptote?

ANSWER:

$$(1/2, 1)$$

EXPLANATION

We have to solve the equations: $y = 1$ and $y = [x/(x-1)]^2$. Eliminating y , we have $x = \pm(x-1)$. The only solution is $x = 1/2$, which gives $y = 1$. \blacksquare

- [4] **3.** (a) Find $\frac{dv}{du}$ when $v = \sqrt{\frac{\tan u}{1 + \tan u}}$.

ANSWER:

$$\frac{dv}{du} = \frac{\sec^2 u}{2(1 + \tan u)^{3/2} (\tan u)^{1/2}}$$

SHOW YOUR WORK

Using the chain rule and the quotient rule, we have:

$$\frac{dv}{du} = \frac{1}{2} \sqrt{\frac{1 + \tan u}{\tan u}} \frac{d}{du} \left(\frac{\tan u}{1 + \tan u} \right) = \frac{1}{2} \frac{(1 + \tan u)^{1/2}}{(\tan u)^{1/2}} \frac{(1 + \tan u) \sec^2 u - \tan u \sec^2 u}{(1 + \tan u)^2} \quad \blacksquare$$

- [4] (b) Let a be a constant and $f(x) = \sin(ax)$. Find the 97-th derivative, $f^{(97)}(x)$, of the function $f(x)$.

ANSWER:

$$a^{97} \cos ax$$

SHOW YOUR WORK

We have: $f(x) = \sin(ax)$, $f'(x) = a \cos(ax)$, $f''(x) = -a^2 \sin(ax)$, $f^{(3)}(x) = -a^3 \cos(ax)$, $f^{(4)}(x) = a^4 \sin(ax)$, ... There is a clear pattern from which we deduce $f^{(96)}(x) = a^{96} \sin(ax)$. \blacksquare

- [3] 4. (a) Find the general antiderivative of $(9 - 4x^2)^{-1/2}$.

ANSWER:

$$\frac{1}{2} \sin^{-1} \left(\frac{2x}{3} \right) + C$$

SHOW YOUR WORK: Using the substitution $u = (2x)/3$, we have:

$$\int (9 - 4x^2)^{-1/2} dx = \int (9 - 4(3u/2)^2)^{-1/2} (3/2) du = \frac{1}{2} \int (1 - u^2)^{1/2} du = \frac{1}{2} \sin^{-1} u + C \quad \blacksquare$$

- [3] (b) It is given that

$$f'(x) = 2^x + x^2 \quad \text{and} \quad f(0) = 0.$$

Find $f(x)$.

ANSWER:

$$f(x) = \frac{2^x - 1}{\ln 2} + \frac{x^3}{3}$$

SHOW YOUR WORK: Writing 2^x as $e^{x \ln 2}$ we see that the antiderivative of $2^x + x^2$ is $(e^{x \ln 2} / \ln 2) + (x^3)/3$. By the evaluation theorem,

$$f(x) - f(0) = \int_0^x f'(t) dt = \left[\frac{2^t}{\ln 2} + \frac{t^3}{3} \right]_0^x = \frac{2^x - 1}{\ln 2} + \frac{x^3}{3}. \quad \blacksquare$$

- [6] 5. Use the definition of derivative (and not the product rule) to show that, if $f(x)$ is differentiable at $x = c$ and $g(x) = xf(x)$, then $g'(c)$ exists and $g'(c) = f(c) + cf'(c)$.

ANSWER: Using the definition of derivative we have

$$\begin{aligned} g'(c) &= \lim_{h \rightarrow 0} \frac{g(c+h) - g(c)}{h} = \lim_{h \rightarrow 0} \frac{(c+h)f(c+h) - cf(c)}{h} \\ &= \lim_{h \rightarrow 0} \left[c \left(\frac{f(c+h) - f(c)}{h} \right) + f(c+h) \right] = c \lim_{h \rightarrow 0} \left(\frac{f(c+h) - f(c)}{h} \right) + \lim_{h \rightarrow 0} f(c+h) \\ &= cf'(c) + f(c) \end{aligned}$$

Note that $\lim_{h \rightarrow 0} f(c+h) = f(c)$ because differentiability of f at c (which is assumed) implies continuity at c . \blacksquare

- [3] 6. For what value of k is the function

$$h(x) = \begin{cases} 2x + 3 & \text{if } x \leq 1 \\ k - 1 & \text{if } x > 1 \end{cases}$$

continuous?

ANSWER:

$$k = 6$$

JUSTIFY YOUR ANSWER: Since $2x + 3$ and $k - 1$ are continuous functions, $h(x)$ is continuous at $x = c$ for all $c \in (-\infty, 1) \cup (1, \infty)$ and continuous on the left at $x = 1$. Further, $\lim_{x \rightarrow 1^+} h(x) = k - 1$ is equal to $h(1) = 5$ if and only if $k = 6$. \blacksquare

- [4] 7. (a) Express $\frac{dy}{dx}$ as a function of x , when $y = \left(\frac{x^7 \cos x}{7^x \sqrt{1+x^2}} \right)$.

ANSWER:

$$\frac{dy}{dx} = \left(\frac{x^7 \cos x}{7^x \sqrt{1+x^2}} \right) \left(\frac{7}{x} - \tan x - \ln 7 - \frac{x}{1+x^2} \right)$$

SHOW YOUR WORK: Taking natural logarithms and differentiating, we get

$$\frac{1}{y} \frac{dy}{dx} = \frac{d}{dx} \left(7 \ln x + \ln \cos x - x \ln 7 - \frac{1}{2} \ln(1+x^2) \right) = \frac{7}{x} - \tan x - \ln 7 - \frac{x}{1+x^2}.$$

Although the line above is only valid for values of x such that $x, \cos x > 0$, the resulting formula is valid for all $x \neq 0$ such that $\tan x$ is defined. ■

- [5] (b) Express $\frac{dy}{dx}$ as a function of x , when $y = x^{\ln x}$.

ANSWER:

$$\frac{dy}{dx} = 2(\ln x)x^{(\ln x)-1}$$

SHOW YOUR WORK: We have

$$\frac{dy}{dx} = \frac{d}{dx} \left((e^{\ln x})^{\ln x} \right) = \frac{d}{dx} \left(e^{(\ln x)^2} \right) = e^{(\ln x)^2} \frac{d}{dx} (\ln x)^2 = 2(\ln x)x^{(\ln x)-1}. \quad \blacksquare$$

8. A curve has the equation $\sin(x + y) = xe^y$.

[2] (a) Show that $(0, \pi)$ is on the curve.

ANSWER: We just observe that $\sin(0 + \pi) = 0 = 0 \cdot e^\pi$. ■

[4] (b) Find the equation of the line tangent to the curve at $(0, \pi)$.

ANSWER:

$$y + (1 + e^\pi)x = \pi$$

SHOW YOUR WORK: By implicit differentiation,

$$\cos(x + y) \left(1 + \frac{dy}{dx} \right) = e^y + xe^y \frac{dy}{dx}.$$

It follows that $\left(\frac{dy}{dx} \right)_{x=0, y=\pi} = -1 - e^\pi$. This allows us to write down the equation of the tangent using the point-slope form of the equation of a line. ■

[4] (c) A point moves along the curve so that at $(0, \pi)$ its x -coordinate is increasing at a rate of 3 units/sec. How fast is its y -coordinate changing at $(0, \pi)$?

ANSWER:

decreasing by $3(1 + e^\pi)$ units per sec

SHOW YOUR WORK: In general, $\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}$. Thus

$$\left(\frac{dy}{dt} \right)_{x=0, y=\pi} = (-1 - e^\pi) \left(\frac{dx}{dt} \right)_{x=0, y=\pi} = -3(1 + e^\pi). \quad \blacksquare$$

9. Let $f(x) = e^{x-2} + x^3 - 2$.

- [3] (a) Use the derivative of f to explain why the equation $f(x) = 0$ has at most one solution.

EXPLANATION: Since $f'(x) = e^{x-2} + 3x^2 > 0$ for all x , the function f is strictly increasing. ■

- [3] (b) Explain why $f(x) = 0$ has a solution in the interval $(1, 2)$.

EXPLANATION: Note that f is continuous, $f(1) = 1 - (1/e) < 0$, and $f(2) = 1 + 8 - 2 = 7 > 0$. By the intermediate value theorem, f has a zero in $(1, 2)$. ■

- [3] (c) Newton's method with an initial estimate of 2 is used to find an approximate value for the solution of $f(x) = 0$.

What is the next estimate?

ANSWER:

19/13

SHOW YOUR WORK: The next estimate is $2 - (f(2)/f'(2)) = 2 - (7/13) = 19/13$. This is the x -coordinate of the point in which the tangent to $y = f(x)$ at $(2, f(2))$ meets $y = 0$. ■

- [6] 10. A particle moves along the x -axis with velocity $\frac{1}{1+t^2}$ at time t . If it passes the point $\pi/6$ at time $t = 1$, what is its acceleration when it passes the point $\pi/4$?

ANSWER:

$-\sqrt{3}/8$

SHOW YOUR WORK: We are given $\frac{dx}{dt} = \frac{1}{1+t^2}$. Taking antiderivatives, we get $x = \tan^{-1} t + C$, where C is a constant. Since $x(1) = \pi/6$, we see that $\pi/6 = (\pi/4) + C$. So $C = -\pi/12$. Letting $x = \pi/4$, we get $\pi/4 = \tan^{-1} t - (\pi/12)$. So $\tan^{-1} t = \pi/3$, which means $t = \sqrt{3}$ when $x = \pi/4$. The acceleration is given by

$$\frac{d^2x}{dt^2} = \frac{d}{dt} \left(\frac{1}{1+t^2} \right) = \frac{-2t}{(1+t^2)^2}.$$

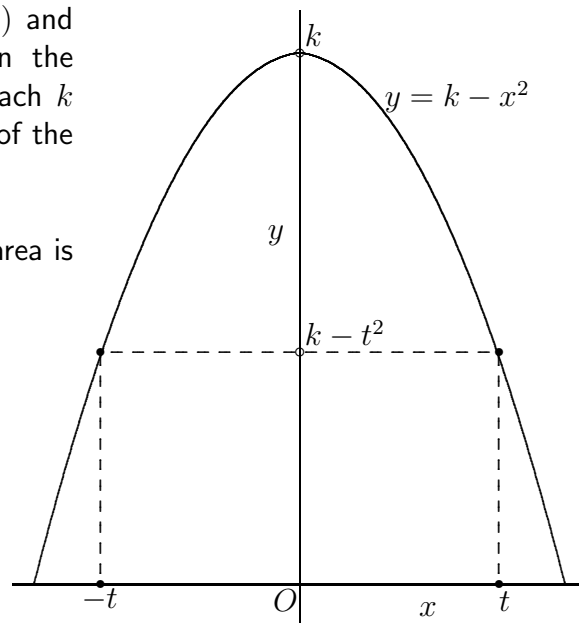
Substituting $t = \sqrt{3}$, we get $-\sqrt{3}/8$ for the acceleration. ■

- [6] **11.** A rectangle has two adjacent vertices $(-t, 0)$ and $(t, 0)$ on the x -axis and the other two on the parabola $y = k - x^2$, where $k > 0$. For each k there exists $t > 0$ which maximizes the area of the resulting rectangle.

Find k such that the rectangle of maximum area is a square.

ANSWER:

$$k = 3$$



SHOW YOUR WORK: From the figure, the area of the rectangle is given by:

$$A = 2t(k - t^2).$$

Now $dA/dt = 2k - 6t^2$ is 0 when $t = \pm\sqrt{k/3}$. Since $dA/dt > 0$ for $t \in (0, \sqrt{k/3})$ and $dA/dt < 0$ for $t \in (\sqrt{k/3}, \sqrt{k})$, $t = \sqrt{k/3}$ gives the maximum area. For the resulting rectangle to be a square, we need

$$2\sqrt{k/3} = 2t = k - t^2 = k - (\sqrt{k/3})^2 = 2k/3.$$

The only solution is $k = 3$ because k is constrained to satisfy $k > 0$. ■

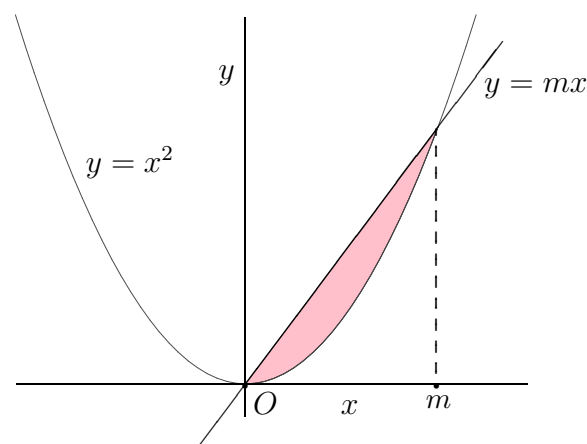
- [6] **12.** Find the line $y = mx$ through the origin, with positive slope, which together with the fragment of the parabola

$$y = x^2 \quad (0 \leq x \leq m)$$

encloses a region of area $4/3$.

ANSWER:

$$y = 2x$$



SHOW YOUR WORK: The area below $y = x^2$ from $x = 0$ to m is found to be $\int_0^m x^2 dx = m^3/3$. Thus the area between $y = m^2x$ and the parabola is $(m^3/2) - (m^3/3) = m^3/6$. For $m^3/6 = 4/3$ we need $m = 2$. ■

- [6] **13.** A bacteria-infested swimming pool was chemically treated this morning, and since then, the bacteria count has been decreasing at rate proportional to the count itself.

An hour ago, the count was a third of what it was two hours ago. For safety, the count must be $\leq 1\%$ of what it is now.

When will that be?

ANSWER:

In about 4.2 hours

SHOW YOUR WORK: Let $C(t)$ be the bacteria count at time t . It is given that $\frac{dC}{dt} = kC$, where k is a constant. This equation can be rewritten as $\frac{d}{dt} \ln C = k$. Taking antiderivatives gives $\ln C = kt + b$, where b is a constant. Hence $C = Be^{kt}$, where $B = e^b$. Let t be measured in hours from "now". Then $C(-1) = (1/3)C(-2)$ tells us that $e^k = 1/3$. Clearly, $C(0) = B$. So for the count to be $\leq B/100$ we need $e^{kt} < 100$, which is $(1/3)^t \leq 1/100$. Rearranging we get $3^t \geq 100$, which means $t \geq (\ln 100)/(\ln 3) = \log_3 100 \approx 4.2$. ■

- 14.** Let $f(x) = 2x^4 - 3x^3 + 5x^2 - 3x + 2$.

- [3] (a) Find the line tangent to $y = f(x)$ at $x = 0$.

ANSWER:

$$y + 3x = 2$$

SHOW YOUR WORK: The point on $y = f(x)$ at $x = 0$ is $(0, f(0)) = (0, 2)$. By inspection, $f'(0) = -3$. Hence the tangent at $(0, 2)$ is $y + 3x = 2$. ■

- [3] (b) Show that the line found in (a) intersects $y = f(x)$ only at $x = 0$.

EXPLANATION: The graphs of $y = f(x)$ and $y + 3x = 2$ intersect where $f(x) = 3x + 2$. This gives $2x^4 - 3x^3 + 5x^2 - 3x + 2 = -3x + 2$. Simplifying, we get

$$x^2(2x^2 - 3x + 5) = 0.$$

Since the discriminant of $2x^2 - 3x + 5$ is $(-3)^2 - 4 \cdot 2 \cdot 5 = -31 < 0$, the quadratic has no real solutions. Thus $x = 0$ is the only solution. ■

- [3] (c) Use a linear approximation to estimate $f(0.01)$.

ANSWER:

$$f(.01) \approx 1.97$$

SHOW YOUR WORK: The linear approximation is $f(0) + (.01)f'(0) = 2 - 3(.01)$. ■

- [4] (d) Let a real number a be given as well as the exact value of $f(a)$. Now suppose that a linear approximation is used to estimate $f(a + 0.01)$.

Show that the estimate will be an underestimate whatever the value of a .

EXPLAIN: The intuition is that the estimate will always be an underestimate provided $f'(x)$ is increasing on $(-\infty, \infty)$. Now $f''(x) = 24x^2 - 6x + 10$ has no zeroes because the discriminant is < 0 . Hence $f''(x)$ has the same sign for all $x \in (-\infty, \infty)$. Since $f''(0) = 10$, $f''(x) > 0$ for all x . Hence, by the Mean Value Theorem (MVT), $f'(x)$ is an increasing function.

The difference between $f(a + .01)$ and the linear estimate is

$$f(a + .01) - [f(a) + (.01)f'(a)] = [f(a + .01) - f(a)] - (.01)f'(a) = (.01)f'(c) - (.01)f'(a)$$

for some $c \in (a, a + .01)$ by the MVT. Since $c > a$, it is clear that the value $f(a + .01)$ is greater than the estimate.

Those familiar with the Lagrange form of the remainder for a Taylor series may choose to point out that there exists $c \in (a, a + .01)$ such that

$$f(a + .01) = f(a) + (.01)f'(a) + \frac{(.01)^2}{2}f''(c).$$

■