Instructions for Calculus Challenge Exam
June 6, 2002, 14:00 – 17:00

1. Check that your name is on the label on the cover page of the test-booklet.
2. Sign your name in the box provided.
3. At the end of the exam only what you have written in the test-booklet will count. You may not submit answers on additional sheets of paper.
4. Write your final answer in the answer box when one is provided.

   Answers should be simplified as far as is reasonable, e.g., either $1 + \dfrac{1}{x}$ or $\dfrac{x+1}{x}$ should be given as the answer rather than $\dfrac{x}{x+1} + \dfrac{2x}{x(x+1)}$.
5. Full marks are given only when the answer is correct, and is supported with a written derivation that is orderly, logical, and complete. Part marks are available in every question.
6. Use the reverse side of the **previous page** if you need more room for your work.
7. Calculators are optional, not required. Correct answers that are "calculator ready" such as $2/\sqrt{3}$ or $3 + \pi \ln 5$ are fully acceptable, even preferred.
8. Any calculator permitted for the B.C. Provincial Examination in Principles of Mathematics 12 is allowed.
   All calculator memories must be empty when the examination begins.
   No other aid, e.g. books or notes, is permitted.
9. Points awarded to parts of questions are shown in square brackets in the left margin.
10. You may write in either pen or pencil, but answers deemed illegible will be ignored.
11. Once the exam begins, please check that you have the 14 numbered pages of the test booklet.

(Use the other side as scratch-paper. Discard at the end of the exam)

2002 Calculus Challenge Formula Sheet

Some values of the trigonometric functions

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$
$\sin\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$
$\cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0	$-1/2$	$-1/\sqrt{2}$	$-\sqrt{3}/2$

Trigonometric identities

$$\sin(\pi+\theta) = -\sin\theta = \sin(-\theta) \qquad \cos(\pi+\theta) = -\cos\theta = -\cos(-\theta)$$

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\csc\theta = 1/\sin\theta, \quad \sec\theta = 1/\cos\theta, \quad \tan\theta = \sin\theta/\cos\theta, \quad \cot\theta = \cos\theta/\sin\theta$$

Basic Integration and Differentiation formulas

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq 1) \qquad\qquad \frac{d}{dx}\,x^n = nx^{n-1}$$

$$\int \frac{1}{x}\,dx = \ln|x| + C \qquad\qquad \frac{d}{dx}\,\ln|x| = \frac{1}{x}$$

$$\int e^x\,dx = e^x + C \qquad\qquad \frac{d}{dx}\,e^x = e^x$$

$$\int \sin x\,dx = -\cos x + C \qquad\qquad \frac{d}{dx}\,\cos x = -\sin x$$

$$\int \cos x\,dx = \sin x + C \qquad\qquad \frac{d}{dx}\,\sin x = \cos x$$

$$\int \sec^2 x\,dx = \tan x + C \qquad\qquad \frac{d}{dx}\,\tan x = \sec^2 x$$

$$\int \csc^2 x\,dx = -\cot x + C \qquad\qquad \frac{d}{dx}\,\cot x = -\csc^2 x$$

$$\int \sec x\tan x\,dx = \sec x + C \qquad\qquad \frac{d}{dx}\,\sec x = \sec x\tan x$$

$$\int \csc x\cot x\,dx = -\csc x + C \qquad\qquad \frac{d}{dx}\,\csc x = -\csc x\cot x$$

$$\int \frac{1}{1+x^2}\,dx = \arctan x + C \qquad\qquad \frac{d}{dx}\,\arctan x = \frac{1}{1+x^2}$$

$$\int \left(1-x^2\right)^{-1/2}\,dx = \arcsin x + C \qquad\qquad \frac{d}{dx}\,\arcsin x = \left(1-x^2\right)^{-1/2}$$

SFU - UBC - UNBC - UVic

Calculus Challenge Exam

June 6, 2002, 14:00 – 17:00

Host: SIMON FRASER UNIVERSITY

Student signature

INSTRUCTIONS

The instructions are distributed separately. Please read them carefully.

Question	Maximum	Score
1	6	
2	6	
3	8	
4	6	
5	6	
6	3	
7	9	
8	10	
9	9	
10	6	
11	6	
12	6	
13	6	
14	13	
Total	100	

1. Compute the following limits.

[3] (a) $\lim_{t \to -2} \dfrac{t^2 - t - 6}{t^2 + 5t + 6}$

ANSWER:

JUSTIFY YOUR ANSWER

[3] (b) $\lim_{x \to 0^+} \left[\dfrac{1}{x} \left(3e^{-1/x} + 5e^{x} \right) \sin x \right]$

ANSWER:

JUSTIFY YOUR ANSWER

[4] **2.** (a) Find the asymptotes of $y = \left(\dfrac{x}{x-1}\right)^2$ and justify your answer.

ANSWER:

EXPLANATION

[2] (b) Where does the curve $y = \left(\dfrac{x}{x-1}\right)^2$ cross its horizontal aymptote?

ANSWER:

EXPLANATION

[4] **3.** (a) Find $\dfrac{dv}{du}$ when $v = \sqrt{\dfrac{\tan u}{1 + \tan u}}$.

ANSWER:

$\dfrac{dv}{du} =$

SHOW YOUR WORK

[4] (b) Let a be a constant and $f(x) = \sin(ax)$. Find the 97-th derivative, $f^{(97)}(x)$, of the function $f(x)$.

ANSWER:

SHOW YOUR WORK

[3] **4.** (a) Find the general antiderivative of $\left(9 - 4x^2\right)^{-1/2}$.

ANSWER:

SHOW YOUR WORK

[3] (b) It is given that

$$f'(x) = 2^x + x^2 \quad \text{and} \quad f(0) = 0.$$

Find $f(x)$.

ANSWER:

SHOW YOUR WORK

[6] **5.** Use the definition of derivative (and <u>not</u> the product rule) to show that, if $f(x)$ is differentiable at $x = c$ and $g(x) = xf(x)$, then $g'(c)$ exists and $g'(c) = f(c) + cf'(c)$.

ANSWER

[3] **6.** For what value of k is the function

$$h(x) = \begin{cases} 2x + 3 & \text{if } x \leq 1 \\ k - 1 & \text{if } x > 1 \end{cases}$$

continuous?

ANSWER:

JUSTIFY YOUR ANSWER

[4] **7.** (a) Express $\frac{dy}{dx}$ as a function of x, when $y = \left(\frac{x^7 \cos x}{7^x\sqrt{1+x^2}}\right)$.

ANSWER:

$\frac{dy}{dx} =$

SHOW YOUR WORK

[5] (b) Express $\frac{dy}{dx}$ as a function of x, when $y = x^{\ln x}$.

ANSWER:

$\frac{dy}{dx} =$

SHOW YOUR WORK

8. A curve has the equation $\sin(x + y) = xe^y$.

[2] (a) Show that $(0, \pi)$ is on the curve.

ANSWER

[4] (b) Find the equation of the line tangent to the curve at $(0, \pi)$.

ANSWER:

SHOW YOUR WORK

[4] (c) A point moves along the curve so that at $(0, \pi)$ its x-coordinate is increasing at a rate of 3 units/sec. How fast is its y-coordinate changing at $(0, \pi)$?

ANSWER:

SHOW YOUR WORK

9. Let $f(x) = e^{x-2} + x^3 - 2$.

[3] (a) Use the derivative of f to explain why the equation $f(x) = 0$ has at most one solution.

EXPLANATION

[3] (b) Explain why $f(x) = 0$ has a solution in the interval $(1, 2)$.

EXPLANATION

[3] (c) Newton's method with an initial estimate of 2 is used to find an approximate value for the solution of $f(x) = 0$.

What is the next estimate?

ANSWER:

SHOW YOUR WORK

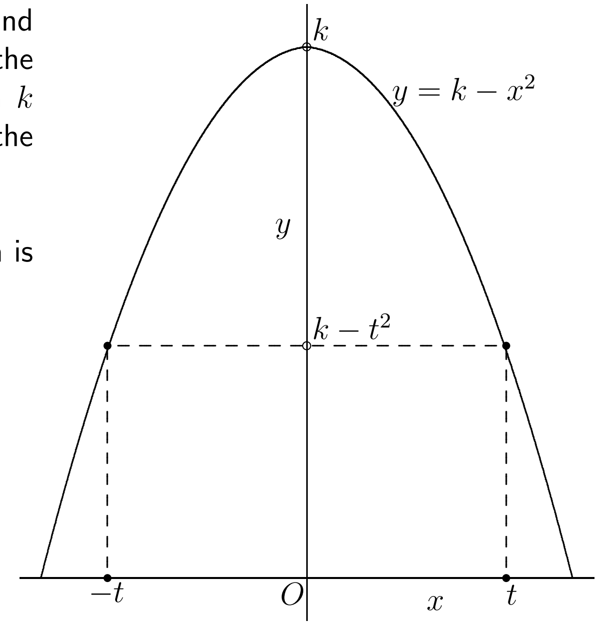
[6] **10.** A particle moves along the x-axis with velocity $\dfrac{1}{1+t^2}$ at time t. If it passes the point $\pi/6$ at time $t = 1$, what is its acceleration when it passes the point $\pi/4$?

ANSWER:

SHOW YOUR WORK

[6] **11.** A rectangle has two adjacent vertices $(-t, 0)$ and $(t, 0)$ on the x-axis and the other two on the parabola $y = k - x^2$, where $k > 0$. For each k there exists $t > 0$ which maximizes the area of the resulting rectangle.

Find k such that the rectangle of maximum area is a square.



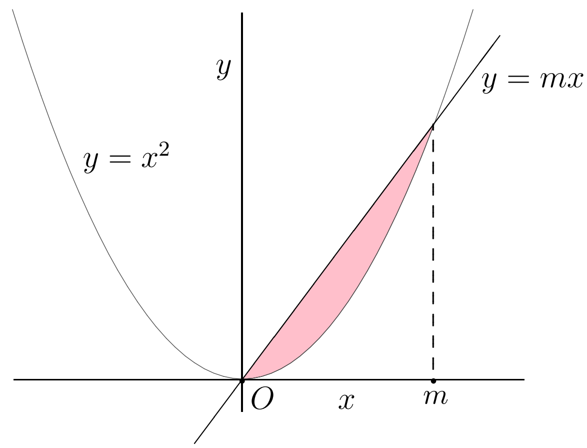
ANSWER:

SHOW YOUR WORK

[6] **12.** Find the line $y = mx$ through the origin, with positive slope, which together with the fragment of the parabola

$$y = x^2 \qquad (0 \le x \le m)$$

encloses a region of area $4/3$.



ANSWER:

SHOW YOUR WORK

[6] **13.** A bacteria-infested swimming pool was chemically treated this morning, and since then, the bacteria count has been decreasing at rate proportional to the count itself.

An hour ago, the count was a third of what it was two hours ago. For safety, the count must be $\leq 1\%$ of what it is now.

When will that be?

ANSWER:

SHOW YOUR WORK

14. Let $f(x) = 2x^4 - 3x^3 + 5x^2 - 3x + 2$.

[3] (a) Find the line tangent to $y = f(x)$ at $x = 0$.

ANSWER:

SHOW YOUR WORK

[3] (b) Show that the line found in (a) intersects $y = f(x)$ only at $x = 0$.

EXPLANATION

[3] (c) Use a linear approximation to estimate $f(0.01)$.

ANSWER:

SHOW YOUR WORK

[4] (d) Let a real number a be given as well as the exact value of $f(a)$. Now suppose that a linear approximation is used to estimate $f(a+0.01)$.

Show that the estimate will be an underestimate whatever the value of a.

EXPLAIN